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ON THE INCREMENTS OF WIENER AND RELATED PROCESSES

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Let $\{W(t), 0 \le t < +\infty\}$ be a standard Wiener process and $0 < b_t \le t$ be a nondecreasing function of t. The properties of the process $Y_1(t) = b_t^{-1/2}\sup_{0\le s\le t-b_t}(W(s+b_t) - W(s))$ are investigated. One of the results says that $\lim_{t\to\infty}(Y_1(t) - (2\log tb_t^{-1})^{1/2}) = 0$ a.s. if b_t is "much less" than t. Analogous properties of similar processes are studied.

1. Introduction. Let $\{X(t); t \ge 0\}$ be a stochastic process and introduce the following definitions.

DEFINITION 1.1. The function $a_1(t)$, $(t \ge 0)$, belongs to the upper-upper class of the process $X(t)$, $(a_1 \in UUC(X))$ if for almost all $\omega \in \Omega$ (the basic space) there exists a $t_0 = t_0(\omega)$ such that $X(t) < a_1(t)$ for every $t > t_0$.

DEFINITION 1.2. The function $a_2(t)$, $(t \ge 0)$, belongs to the upper-lower class of the process $X(t)$, $(a_2 \in ULC(X))$ if for almost all $\omega \in \Omega$ there exists a sequence $0 < t_1 = t_1(\omega) < t_2 = t_2(\omega) < \cdots$ with $t_i \to \infty$, $(i \to \infty)$, such that $X(t_i) \ge a_2(t_i)$, $i = 1, 2, \cdots$.

DEFINITION 1.3. The function $a_3(t)$, $t \ge 0$, belongs to the lower-upper class of the process $X(t)$, $(a_3 \in LUC(X))$ if for almost all $\omega \in \Omega$ there exists a sequence $0 < t_1 = t_1(\omega) < t_2 = t_2(\omega) < \cdots$ with $t_i \to \infty$, $(i \to \infty)$, such that $X(t_i) \le a_3(t_i)$, $i = 1, 2, \cdots$,

DEFINITION 1.4. The function $a_4(t)$, $t \ge 0$, belongs to the lower-lower class of the process $X(t)$, $(a_4 \in LLC(X_4))$ if for almost all $\omega \in \Omega$ there exists $t_0 = t_0(\omega)$ such that $X(t) > a_4(t)$ for every $t > t_0$.

The introduction of these classes appears at first in a paper of the author (Révész, 1980). However the same concepts (without using our expressions) were utilized by several authors much before.

In order to illustrate these concepts, we present the well-known results about the maximum of the Wiener process (cf. Erdös 1942, Kolmogorov-Petrovski 1933–34, Feller 1943, 1946, Chung 1948, Hirsch 1965).

Let $\{W(t); t \ge 0\}$ be a Wiener process and put

$$X_1(t) = t^{-1/2}\sup_{0\le s\le t} W(s), \qquad X_2(t) = t^{-1/2}\sup_{0\le s\le t} |W(s)|.$$

THEOREM A. *The nondecreasing function $a(t) \in UUC(X_i)$, $i = 1, 2$, if and only if*

$$I_1(a) = \int_1^\infty t^{-1}a(t)\exp\left(-\frac{a^2(t)}{2}\right) dt < \infty$$

and consequently $a(t) \in ULC(X_i)$, $i = 1, 2$, if and only if $I_1(a) = \infty$. The nondecreasing function $a(t) \in LUC(X_2)$ if and only if

$$I_2(a) = \int_1^\infty t^{-1}a^{-2}(t)\exp\left(-\frac{\pi^2}{8}a^{-2}(t)\right) dt = \infty$$

and consequently $a(t) \in LLC(X_2)$ if and only if $I_2(a) < \infty$.

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The nondecreasing function $a(t) \in LUC(X_1)$ if and only if

$$I_3(a) = \int_1^\infty t^{-1}a(t)\,dt = \infty$$

and consequently $a(t) \in LLC(X_1)$ if and only if $I_3(a) < \infty$.

Now, we give two further definitions.

DEFINITION 1.5. If we can find functions $a_1 \in UUC(X)$ and $a_4 \in LLC(X)$ such that $\lim_{t\to\infty}(a_1(t) - a_4(t)) = 0$ then we say that X is asymptotically deterministic (AD). In this case, clearly $\lim_{t\to\infty}|X(t) - a_1(t)| = \lim_{t\to\infty}|X(t) - a_4(t)| = 0$ a.s.

DEFINITION 1.6. If we can find functions $a_1 \in UUC(X)$ and $a_4 \in LLC(X)$ such that $\limsup_{t\to\infty}(a_1(t) - a_4(t)) < \infty$ then we say that X is quasi AD (QAD). In this case, clearly $\limsup_{t\to\infty}|X(t) - a_1(t)| < \infty$.

In order to present a non-trivial AD process we give the following known result on the continuity modulus of a Wiener process (cf. Chung-Erdös-Sirao 1959, Révész 1980).

THEOREM B. *Let*

$$X_3(t) = t^{1/2}\sup_{0\le s\le 1-t^{-1}}(W(s + t^{-1}) - W(s)),$$

$$X_4(t) = t^{1/2}\sup_{0\le s\le 1-t^{-1}}|W(s + t^{-1}) - W(s)|,$$

$$X_5(t) = t^{1/2}\sup_{0\le s\le 1-t^{-1}}\sup_{0\le u\le t^{-1}}(W(s + u) - W(s)),$$

$$X_6(t) = t^{1/2}\sup_{0\le s\le 1-t^{-1}}\sup_{0\le u\le t^{-1}}|W(s + u) - W(s)|.$$

Then the nondecreasing function $a(t) \in UUC(X_i)$, $i = 3, 4, 5, 6$, if and only if

$$I_4(a) = \int_1^\infty (\log t)^{3/2}\exp\left(-\frac{a^2(t)}{2}\right)dt < \infty$$

and consequently $a(t) \in ULC(X_i)$, $i = 3, 4, 5, 6$, if and only if $I_4(a) = \infty$. Further for any $\varepsilon > 0$ we have

$$(2\log t + \log\log t - 2\log\log\log t - \log(\pi - \varepsilon))^{1/2} \in LUC(X_i), \quad i = 3, 5$$

and

$$\ell_1(t) = (2\log t + \log\log t - 2\log\log\log t - \log(9\pi + \varepsilon))^{1/2} \in LLC(X_i), \quad i = 3, 5.$$

Finally for any $\varepsilon > 0$ we have

$$\left(2\log t + \log\log t - 2\log\log\log t - \log\frac{\pi - \varepsilon}{4}\right)^{1/2} \in LUC(X_i), \quad i = 4, 6$$

and

$$\ell_2(t) = \left(2\log t + \log\log t - 2\log\log\log t - \log\frac{9\pi + \varepsilon}{4}\right)^{1/2} \in LLC(X_i), \quad i = 4, 6.$$

Since

$$u(t) = (2\log t + C\log\log t)^{1/2} \in UUC(X_i), \quad i = 3, 4, 5, 6, \quad \text{if } C > 5$$

and

$$\lim_{t\to\infty}(u(t) - \ell_1(t)) = \lim_{t\to\infty}(u(t) - \ell_2(t)) = 0$$

the processes $X_i(t)$, $i = 3, 4, 5, 6$, are AD. In fact we have the following.

CONSEQUENCE.

$$\lim_{t\to\infty}(X_i(t) - (2\log t)^{1/2}) = 0 \quad \text{a.s.,} \quad i = 3, 4, 5, 6$$

and we also have the stronger version:

$$\frac{\sqrt{2}}{4} = \liminf_{t\to\infty} \frac{(\log t)^{1/2}}{\log\log t}(X_i(t) - (2\log t)^{1/2})$$

$$< \limsup_{t\to\infty} \frac{(\log t)^{1/2}}{\log\log t}(X_i(t) - (2\log t)^{1/2}) = 5\frac{\sqrt{2}}{4} \quad \text{a.s.,} \quad i = 3, 4, 5, 6.$$

REMARK 1. In the previously mentioned paper (Révész, 1980) only the case $i = 3$ was treated. However, the other cases can be handled similarly.

2. The increments of a Wiener process. Let $0 < b_t \le t$ be a nondecreasing function of t and consider the processes

$$Y_1(t; b_t) = Y_1(t) = b_t^{-1/2}\sup_{0\le s\le t-b_t}(W(s + b_t) - W(s)),$$

$$Y_2(t; b_t) = Y_2(t) = b_t^{-1/2}\sup_{0\le s\le t-b_t}|W(s + b_t) - W(s)|,$$

$$Y_3(t; b_t) = Y_3(t) = b_t^{-1/2}\sup_{0\le s\le t-b_t}\sup_{0\le u\le b_t}(W(s + u) - W(s)),$$

$$Y_4(t; b_t) = Y_4(t) = b_t^{-1/2}\sup_{0\le s\le t-b_t}\sup_{0\le u\le b_t}|W(s + u) - W(s)|.$$

Clearly in the case $b_t = t$, $Y_3 = X_1$ and $Y_4 = X_2$ (cf. Section 1). Also note that

$$Y_1(t) = \min(Y_1(t), Y_2(t), Y_3(t), Y_4(t)) \le \max(Y_1(t), Y_2(t), Y_3(t), Y_4(t)) = Y_4(t).$$

Studying the properties of the processes $Y_i(t)$, $i = 1, 2, 3, 4$, Csörgő and Révész (1979) proved the following.

THEOREM C. *Let b_t, $t \ge 0$, be a nondecreasing function of t for which*

(i) $0 < b_t \le t, \quad t \ge 0,$
(ii) *$t^{-1}b_t$ is nonincreasing.*

Then

$$\limsup_{t\to\infty}\beta_t Y_i(t) = 1 \quad \text{a.s.,} \quad i = 1, 2, 3, 4$$

where $\beta_t = (2[\log tb_t^{-1} + \log\log t])^{-1/2}$.
If we also have

(iii) $$\lim_{t\to\infty}\frac{\log tb_t^{-1}}{\log\log t} = \infty,$$

then

$$\lim_{t\to\infty}\beta_t Y_i(t) = \lim_{t\to\infty}(2\log tb_t^{-1})^{-1/2}Y_i(t) = 1 \quad \text{a.s.,} \quad i = 1, 2, 3, 4.$$

It is an interesting phenomenon that the properly normalized process $\beta_t Y_i$, $i = 1, 2, 3, 4$, has a limit if b_t is not too big (Condition (iii) holds true) but it has a lim sup only if b_t is close to t. In Csáki and Révész (1979), we were interested in whether Condition (iii) can be replaced by a weaker one. We obtained a negative answer by proving the following.

THEOREM D. *Let b_t, $t \ge 0$, be a nondecreasing function of t satisfying Conditions* (i), (ii) *and the condition*

(iv) $$\lim_{t\to\infty}\frac{\log tb_t^{-1}}{\log\log\log t}=\infty.$$

Then $$\liminf_{t\to\infty}\gamma_t Y_4(t)=1 \quad \text{a.s.}$$

where $$\gamma_t=\left(2\log\frac{\pi^2}{16}\Delta_t\right)^{-1/2} \quad \textit{and} \quad \Delta_t=\frac{tb_t^{-1}}{\log\log t}.$$

In the previously mentioned paper we also studied the case when Condition (iv) does not hold (i.e. when b_t is bigger) but we could not get precise enough results.

At present we intend to study the four classes of the processes $Y_i(t)$, $i = 1, 2, 3, 4$, in the case when Condition (iii) holds true, but first we prove some lemmas.

LEMMA 2.1. *Let k be an arbitrary positive number. Then for any $\varepsilon > 0$ there exists a $u_0 = u_0(\varepsilon) > 0$ such that*

$$\begin{aligned}(1-\varepsilon)\frac{k}{\sqrt{2\pi}}ue^{-u^2/2} &\le P\{\sup_{0\le x\le k}(W(x+1)-W(x))>u\}\\ &\le P\{\sup_{0\le x\le k}\sup_{0\le s\le 1}(W(x+s)-W(x))>u\}\le 25\frac{k}{\sqrt{2\pi}}ue^{-u^2/2}\end{aligned} \tag{2.1}$$

if $u \ge u_0$.

The constant 25 of (2.1) is certainly not the best possible but it is enough for most of our purposes.

The first inequality in (2.1) is known (see, for example, Qualls-Watanabe, 1972). Hence we prove the last inequality.

PROOF. Let

$$x_i=\frac{i}{u^2}, \quad i=1,2,\cdots,[u^2k]$$

be a partition of the interval $[0, k]$ and define the events

$$B_i=\{\sup_{0\le s\le u^{-2}}(W(x_i-s)-W(x_i))>1\}$$

$$A_i(v)=\left\{\sup_{0\le s\le 1}(W(x_i+s)-W(x_i))\ge u-\frac{v}{u},\frac{v-\Delta v}{u}\le \sup_{0\le s\le u^{-2}}(W(x_i-s)-W(x_i))\le\frac{v}{u}\right\}.$$

Then

$$P(A_i(v))\approx\frac{2}{\sqrt{2\pi}}\exp\left(-\frac{v^2}{2}\right)\cdot\frac{2}{\sqrt{2\pi}}\frac{1}{u-\frac{v}{u}}\cdot\exp\left(-\frac{1}{2}\left(u-\frac{v}{u}\right)^2\right)\Delta v, \quad \text{as } u\to\infty, \Delta v\to 0,$$

$$P(B_i)\approx\frac{2}{\sqrt{2\pi}u}\exp\left(-\frac{u^2}{2}\right)$$

and

$$\begin{aligned}&P\{\sup_{0\le x\le k}\sup_{0\le s\le 1}(W(x+s)-W(x))>u\}\\ &\le[u^2k]\left\{\frac{2}{\pi}\int_0^u\frac{1}{u-\frac{v}{u}}\exp\left(-\frac{v^2}{2}\right)\exp\left(-\frac{1}{2}\left(u-\frac{v}{u}\right)^2\right)dv+\frac{2}{\sqrt{2\pi}u}\exp\left(-\frac{u^2}{2}\right)\right\}\\ &\le\frac{25}{\sqrt{2\pi}}ku\exp\left(-\frac{u^2}{2}\right).\end{aligned}$$

REMARK 2. The exact formula for $P\{\sup_{0\le x\le k}(W(x+1)-W(x))>u\}$ was obtained by Shepp (1971) in the case when k is an integer. For our purposes these asymptotic results are more useful.

The next lemma is closely related to Lemma 1 of Révész (1980).

LEMMA 2.2 *For any* $\varepsilon>0$ *there exist a* $u_0=u_0(\varepsilon)>0$ *and a* $T_0=T_0(\varepsilon)>0$ *such that*

$$\begin{aligned}\exp\left\{-25\frac{T}{\sqrt{2\pi}}ue^{-\frac{u^2}{2}}\right\} &\le P\{\sup_{0\le x\le T}\sup_{0\le s\le 1}(W(x+s)-W(x))\le u\}\\ &\le P\{\sup_{0\le x\le T}(W(x+1)-W(x))\le u\}\\ &\le \exp\left\{-(1-\varepsilon)\frac{T}{\sqrt{2\pi}}ue^{-\frac{u^2}{2}}\right\}\end{aligned} \tag{2.2}$$

if $u\ge u_0$ *and* $T\ge T_0$.

The second inequality in (2.2) was proved in the paper (1980) of Révész. The first inequality can be proved by the proving method of Lemma 1 of the mentioned paper.
The same method is also applicable to prove our Lemma 2.3.

LEMMA 2.3. *For any* $\varepsilon>0$ *there exist a* $u_0=u_0(\varepsilon)>0$ *and a* $T_0=T_0(\varepsilon)>0$ *such that*

$$\exp\left\{-50\frac{T}{\sqrt{2\pi}}ue^{-\frac{u^2}{2}}\right\}\le P\{\sup_{0\le x\le T}\sup_{0\le s\le 1}|W(x+s)-W(x)|\le u\}$$

$$\le P\{\sup_{0\le x\le T}|W(x+1)-W(x)|\le u\}\le \exp\left\{-2(1-\varepsilon)\frac{T}{\sqrt{2\pi}}ue^{-\frac{u^2}{2}}\right\}$$

if $u\ge u_0$ *and* $T\ge T_0$.

REMARK 3. The upper bound of Lemma 2.1 and the lower bounds of Lemmas 2.2 and 2.3 do not depend on ε and these inequalities are true for all $u>0$ (resp. for all $T>0$) not only for large u's (resp. T's).
Now, we turn to the main result of this section. This gives only one representative of each class of the four classes in the case when b_t is small, i.e., when Condition (iii) of Theorem C holds.

THEOREM 2.1. *Let* b_t, $t\ge 0$, *be a nondecreasing function t satisfying Conditions* (i), (ii) *and* (iii) *of Theorem* C *and put*

$$a_1(t)=a_1(t,\varepsilon)=(2\log tb_t^{-1}+2\log\log t+(3+\varepsilon)\log\log tb_t^{-1}+(2+\varepsilon)\log\log\log t)^{1/2},$$

$$a_2(t)=(2\log tb_t^{-1}+2\log\log t+\log\log tb_t^{-1}+2\log\log\log t)^{1/2},$$

$$a_3(t)=a_3(t,\varepsilon)=\left(2\log tb_t^{-1}+\log\log tb_t^{-1}-2\log\log\log t+\log\left(\frac{51^2}{\pi}+\varepsilon\right)\right)^{1/2},$$

$$a_4(t)=a_4(t,\varepsilon)=(2\log tb_t^{-1}+\log\log tb_t^{-1}-2\log\log\log t-\log(\pi(1+\varepsilon)))^{1/2}.$$

Then for any $\varepsilon>0$ *and* $i=1,2,3,4$, *we have*

$$a_1(t)\in UUC(Y_t), \tag{2.3}$$

$$a_2(t)\in ULC(Y_t), \tag{2.4}$$

$$a_3(t)\in LUC(Y_t), \tag{2.5}$$

$$a_4(t)\in LLC(Y_t). \tag{2.6}$$

PROOF OF (2.3). It is enough to prove it for $i = 4$. Let

$$P(t) = P(t, \varepsilon) = P(Y_4(t) \geq a_1(t, \varepsilon)).$$

Then by Lemma 2.3 we have

$$P(t, \varepsilon/2) = O((\log t)^{-1} (\log tb_t^{-1})^{-1-\varepsilon/4} (\log\log t)^{-1-\varepsilon/4})$$
$$= O((\log t)^{-1}(w(t))^{-1-\varepsilon/4}(\log\log t)^{-2-\varepsilon/2})$$

where

$$w(t) = \frac{\log tb_t^{-1}}{\log\log t}.$$

Let, now t_k be the smallest real number for which

(2.7) $$(\log t_k)(w(t_k))^{1+\varepsilon/4}(\log\log t_k) = k.$$

Then

(2.8) $$P(t_k, \varepsilon/2) = O(k^{-1}(\log\log t_k)^{-1-\varepsilon/4}) = O(k^{-1}(\log k)^{-1-\varepsilon/4}).$$

(In the last equality the trivial inequality $\log w(t) < \log\log t$ should be utilized). (2.8) and the Borel-Cantelli lemma imply that among the events

$$\{Y_4(t_k) \geq a_1(t_k, \varepsilon/2)\}$$

only finitely many can occur with probability 1.

Statement (2.3) follows from the fact that the process $b_t^{1/2} Y_4(t)$ is nondecreasing and from the inequality:

$$b_{t_{k+1}} a_1(t_{k+1}, \varepsilon/2) \leq b_{t_k} a_1(t_k, \varepsilon).$$

In the proof of the last inequality the following trivial relations should be utilized:

$$\frac{b_{t_{k+1}}}{b_{t_k}} \leq \frac{t_{k+1}}{t_k} = O((\log\log t_k)^{-1}(w(t_k))^{-1-\varepsilon/4}) + 1,$$

$$\log\frac{t_{k+1}}{b_{t_{k+1}}} \leq \log\frac{t_k}{b_{t_k}} + \log\frac{t_{k+1}}{t_k} = w(t_k)\log\log t_k + O((\log\log t_k)^{-1}(w(t_k))^{-1-\varepsilon/4}).$$

PROOF OF (2.4). It is enough to prove it for $i = 1$. In fact the following stronger statement will be proved:

Among the events

$$A_k = \{\sup_{t_k \leq s \leq t_{k+1} - b_{t_{k+1}}} (W(s + b_{t_{k+1}}) - W(s)) \geq a_2(t_{k+1})\}$$

infinitely many occur with probability 1 *where the sequence* $\{t_k\}$ *is defined by* (2.7).

(Note that (2.7) implies that $t_{k+1} - t_k \geq b_{t_{k+1}}$). By Lemma 2.2 we have

$$P(A_k) = O\left(\frac{t_{k+1} - t_k}{b_{t_{k+1}}} (\log t_{k+1} b_{t_{k+1}}^{-1})^{1/2} \frac{b_{t_{k+1}}}{t_{k+1}} (\log t_{k+1})^{-1} (\log t_{k+1} b_{t_{k+1}}^{-1})^{-1/2} (\log\log t_{k+1})^{-1}\right)$$
$$= O((\log\log t_{k+1})^{-2}(w(t_{k+1}))^{-1-\varepsilon/4}(\log t_{k+1})^{-1})$$
$$= O\left(\frac{1}{k\log\log t_{k+1}}\right) = O\left(\frac{1}{k\log k}\right),$$

which proves (2.4).

PROOF OF (2.5). It is enough to prove it for $i = 4$. Let $t_k = \exp(k^{1+\rho})$, $k = 1, 2, \cdots,; \rho > 0$, and let

$$Z_4(k+1) = \sup_{t_k \leq t < t_{k+1} - b_{t_{k+1}}} \sup_{0 \leq s \leq b_{t_{k+1}}} b_{t_{k+1}}^{-1/2} | W(t+s) - W(t) |.$$

Then by Lemma 2.3 we have

$$\sum_{k=1}^{\infty} P(Z_4(k) < a_3(t_k)) = \infty,$$

and this proves that among the events

$$\{Z_4(k) < a_3(t_k)\}$$

infinitely many occur with probability 1.
Since

$$Y_4(t_{k+1}) \le Z_4(k+1) + \sup_{0\le t\le t_k} \sup_{0\le s\le b_{t_{k+1}}} b_{t_{k+1}}^{-1/2} \mid W(t+s) - W(t) \mid,$$

and by (2.3)

$$\sup_{0\le t\le t_k} \sup_{0\le s\le b_{t_{k+1}}} \mid W(t+s) - W(t) \mid = o(b_{t_{k+1}}^{-1/2} a_3(t_{k+1})),$$

we have (2.5).

Proof of (2.6). It is enough to prove it for $i = 1$. By Lemma 2.2 we have

$$P\{Y_1(t) \le a_4(t, 3\varepsilon)\} \le \exp\left\{-\frac{(1-\varepsilon)}{\sqrt{2\pi}} tb_t^{-1} a_4(t) \exp\left(-\frac{a_4^2(t)}{2}\right)\right\}$$

$$\le \exp\left\{\frac{(1-\varepsilon)}{\sqrt{\pi}} \sqrt{\pi(1+3\varepsilon)} \log\log t\right\} \le (\log t)^{-1-\delta}$$

if t is big enough, where δ is a suitable positive number. Put $t_k = \exp(k^{1-\delta})$, $k = 1, 2, \cdots$; $\rho > 0$. Then

$$\sum_{k=1}^{\infty} P\{Y_1(t_k) \le a_4(t_k)\} < \infty,$$

and the Borel-Cantelli lemma implies that among the events $\{Y_1(t_k) \le a_4(t_k)\}$ only finitely many can occur. Let $t_k \le t < t_{k+1}$. Then

$$b_t^{1/2} Y_1(t) \ge \sup_{0\le s\le t_k - b_{t_k}} (W(s+b_t) - W(s))$$

$$\ge \sup_{0\le s\le t_k - b_{t_k}} (W(s+b_{t_k}) - W(s))$$

$$- \sup_{0\le s\le t_k - b_{t_k}} \sup_{0\le u\le b_{t_{k+1}} - b_{t_k}} \mid W(s+u) - W(s) \mid.$$

Now (2.6) follows from (2.3) and Theorem 2.1 is proved.

If b_t is "small" then Theorem 2.1 gives much sharper results than Theorem C. In order to see this fact we give the following

Corollary. 2.1. *Suppose that b_t, $t \ge 0$, is a nondecreasing function of t and it satisfies the Conditions* (i), (ii) *of Theorem* C. *Also assume that instead of* (iii) *it satisfies the stronger condition*

(v) $$\lim_{t\to\infty} \frac{(\log tb_t^{-1})^{1/2}}{\log\log t} = \infty.$$

Then $Y_i(t)$, $i = 1, 2, 3, 4$, is AD *with*

(2.9) $$\lim_{t\to\infty} (Y_i(t) - (2\log tb_t^{-1})^{1/2}) = 0 \quad \text{a.s.,} \quad i = 1, 2, 3, 4.$$

If Condition (v) does not hold true then our statement (2.9) does not hold as well. In fact if

(vi) $$\lim_{t\to\infty} \frac{(\log tb_t^{-1})^{1/2}}{\log\log t} = c > 0$$

then $Y_i(t)$, $i = 1, 2, 3, 4$, is QAD with

$$(2.10)\quad \begin{aligned} 0 &= \liminf_{t\to\infty} (Y_i(t) - (2\log tb_t^{-1})^{1/2}) \\ &< \limsup_{t\to\infty} (Y_i(t) - (2\log tb_t^{-1})^{1/2}) = \frac{1}{c\sqrt{2}} \quad \text{a.s.,} \quad i = 1, 2, 3, 4. \end{aligned}$$

If b_t is even bigger so that

$$\lim_{t\to\infty} \frac{(\log tb_t^{-1})^{1/2}}{\log\log t} = 0$$

but (iii) still holds true then

$$\limsup_{t\to\infty} (Y_i(t) - (2\log tb_t^{-1})^{1/2}) = \infty \quad \text{a.s.}$$

but the processes $(2\log tb_t^{-1})^{-1/2} Y_i(t)$, $i = 1, 2, 3, 4$, will be AD (see Theorem C). If b_t is so big that even (iii) does not hold true then the correct normalizing factor of $Y_i(t)$ will be β_t (of Theorem C) or γ_t (of Theorem D) depending on whether we are interested in the problem of lim sup or lim inf respectively.

3. A continuity modulus of the Kiefer process. At first we repeat two well-known definitions.

DEFINITION 3.1. The separable Gaussian process $\{B(x); 0 \le x \le 1\}$ is a Brownian bridge if $EB(x) = 0$ and $EB(x)B(y) = \min(x, y) - xy$.

DEFINITION 3.2. The separable Gaussian process $\{K(x, y); 0 \le x \le 1, y \ge 0\}$ is a Kiefer process if $EK(x, y) = 0$ and $EK(x_1, y_1)K(x_2, y_2) = \min(y_1, y_2)[\min(x_1, x_2) - x_1x_2]$.

Applying the representation $B(x) = W(x) - xW(1)$ of a Brownian bridge, it is quite trivial to see that all results of Section 1 on the continuity modulus of a Wiener process are true if we replace the Wiener process $W(x)$ by a Brownian bridge $B(x)$ in the definition of $X_i(t)$, $i = 3, 4, 5, 6$.

It is also trivial that

$$(3.1)\quad \{B(x); 0 \le x \le 1\} =_{\mathscr{D}} \{y_0^{-1/2} K(x, y_0); 0 \le x \le 1\}$$

for any $y_0 > 0$. Looking at (3.1) it is natural to ask the continuity modulus of the process $y_0^{-1/2}K(x, y_0)$ as y_0 is varying. This question was studied by Csörgő and Révész (1981, Theorem 1.15.2) and the following was proved.

THEOREM E. *Let $\{h_n\}$ be a sequence of positive numbers for which*

$$(3.2)\quad \lim_{n\to\infty} \frac{\log h_n^{-1}}{\log\log n} = \infty.$$

Then

$$(3.3)\quad \begin{aligned} &\lim_{n\to\infty} \gamma_n \sup_{0\le t<1-h_n} | K(t + h_n, n) - K(t, n) | \\ &= \lim_{n\to\infty} \gamma_n \sup_{0\le t<1-h_n} \sup_{0\le u\le h_n} | K(t + u, n) - K(t, n) | = 1 \quad \text{a.s.} \end{aligned}$$

where $\gamma_n = (2nh_n \log h_n^{-1})^{1/2}$.

Now, we are interested in finding the analogue of Theorem B. Put

$$K_1(n) = (nh_n)^{-1/2} \sup_{0\le s<1-h_n} (K(s + h_n, n) - K(s, n)),$$

$$K_2(n) = (nh_n)^{-1/2} \sup_{0\le s\le 1-h_n} | K(s + h_n, n) - K(s, n) |,$$

$$K_3(n) = (nh_n)^{-1/2} \sup_{0\le s\le 1-h_n} \sup_{0\le u\le h_n} (K(s + u, n) - K(s, n)),$$

$$K_4(n) = (nh_n)^{-1/2} \sup_{0\le s\le 1-h_n} \sup_{0\le u\le h_n} | K(s + u, n) - K(s, n) |.$$

Then we have the following.

THEOREM 3.1. *Assume Condition* (3.2). *Then for any* $0 < \varepsilon < 1$, $i = 1, 2, 3, 4$, *we have*

$$(2 \log h_n^{-1} + 2 \log \log n + (3 + \varepsilon) \log \log h_n^{-1} + (2 + \varepsilon) \log \log \log n)^{1/2} \in UUC(K_i),$$

$$(2 \log h_n^{-1} + 2 \log \log n + \log \log h_n^{-1} + 2 \log \log \log n)^{1/2} \in ULC(K_i),$$

$$\left(2 \log h_n^{-1} + \log \log h_n^{-1} - 2 \log \log \log n + \log\left(\frac{51^2}{\pi} + \varepsilon\right)\right)^{1/2} \in LUC(K_i).$$

$$(2 \log h_n^{-1} + \log \log h_n^{-1} - 2 \log \log \log n - \log((1 + \varepsilon)\pi))^{1/2} \in LLC(K_i).$$

The proof of this theorem is going along the lines of those of Theorems B and 2.1. The details are omitted.

If instead of Condition (3.2) we assume the stronger condition

$$\lim_{n\to\infty} \frac{(\log h_n^{-1})^{1/2}}{\log \log n} = \infty \tag{3.4}$$

then instead of (3.3) we can get a stronger statement. In fact, Theorem 3.1 implies

CONSEQUENCE. *Assume condition* (3.4). *Then* $K_i(n)$, $i = 1, 2, 3, 4$, *is* AD *with*

$$\lim_{n\to\infty} (K_i(n) - (2 \log h_n^{-1})^{1/2}) = 0 \quad \text{a.s.}$$

4. Empirical density functions. Let $Z_1, Z_2, \cdots$ be a sequence of i.i.d. r.v.'s with common density function $f(x)$, let $\lambda(x)$ be an arbitrary density function and let $\{h_n\}$ be a sequence of positive numbers. Suppose that

1.a. f is vanishing outside the interval $[0, 1]$,
1.b f is twice differentiable in $(0, 1)$ and $|f''| \leq C$,
1.c. f is strictly positive in $(0, 1)$, say $f \geq \alpha > 0$,
2.a. $\lambda \leq C$,
2.b. $\lambda(-x) = \lambda(x)$,
2.c. $\lim_{x\to\infty} x^4\lambda(x) = 0$,
2.d. λ is twice differentiable in an interval $-\infty \leq -a < x < +a \leq +\infty$ vanishing outside and $|\lambda''| \leq C$ in $(-a, +a)$,
3.a. $nh_n \nearrow \infty$, $h_n \searrow 0$
3.b. $\dfrac{\log^4 n}{nh_n \log h_n^{-1}} \to 0$, $\dfrac{nh_n^5}{\log h_n^{-1}} \to 0$.

Define the empirical density function f_n of the sample $Z_1, Z_2, \cdots Z_n$ as

$$f_n(x) = (nh_n)^{-1} \sum_{k=1}^{n} \lambda((x - Z_k)h_n^{-1}) = h_n^{-1} \int_0^1 \lambda((x - y)h_n^{-1})\, dF_n(y)$$

where

$$F_n(y) = n^{-1} \sum_{k=1}^{n} I_{(-\infty,y]}(Z_k)$$

is the empirical distribution function based on the sample $Z_1, Z_2, \cdots Z_n$ and $I_{(-\infty,y]}$ is the indicator function of $(-\infty, y]$.

Studying the properties of the empirical density function, the author proved (Révész, 1978).

THEOREM F. *Suppose that Conditions* 1 *through* 3 *are satisfied. Then for any* $\varepsilon > 0$ *we have*

$$\lim_{n\to\infty} \left(\frac{nh_n}{2\Lambda^2 \log h_n^{-1}}\right)^{1/2} \sup_{\varepsilon \leq x \leq 1-\varepsilon} \left|\frac{f_n(x) - f(x)}{f^{1/2}(x)}\right| = 1 \quad \text{a.s.} \tag{4.1}$$

where $\Lambda^2 = \int_{-\infty}^{+\infty} \lambda^2(x)\, dx$.

The form and the proof of Theorem F shows that it is strongly connected to 3.3. It is natural to ask whether the analogue of Theorem 3.1 can be proved for the empirical density function. Now, we formulate a theorem giving a positive answer to the just posed question but we omit the details of the proof for they are quite close to those of Theorem 2.1 after having Theorem F.

THEOREM 4.1. *Suppose that Conditions* 1 *through* 3 *are satisfied. Then for any* $\varepsilon >$ 0 *we have*

$$\lim_{n\to\infty}\left((nh_n)^{1/2}\sup_{\varepsilon\le x\le 1-\varepsilon}\left|\frac{f_n(x)-f(x)}{f^{1/2}(x)}\right| - (2\Lambda^2\log h_n^{-1})^{1/2}\right) = 0 \quad \text{a.s.} \tag{4.2}$$

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