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ON THE INDECOMPOSABLE REPRESENTATIONS OF ALGEBRAS

by

JAMES P. JANS

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PREFACE

I wish to acknowledge my indebtedness to Professor R. M. Thrall who introduced me to the area of study covered in this dissertation. His encouragement and guidance helped to make it a reality.

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INTRODUCTION

If A is an associative algebra, any representation of A by matrices can be decomposed into a direct sum of indecomposable representations and such a decomposition is unique up to similarity of the direct summands. (Krull-Schmidt Theorem)

If the algebra A is semisimple, the indecomposable representations are actually irreducible. Counting similar representations as equal, A has only a finite number of such irreducible representations. Thus, every representation of a semisimple algebra can be formed by adding together a direct sum of representations taken from this finite set.

For nonsemisimple algebras this is no longer true. There are algebras with radical which have an infinite number of inequivalent indecomposable representations of the same degree for each of an infinite number of degrees. Such algebras are said to be of strongly unbounded representation type. The object of this paper is to classify and study algebras with unity over an algebraically closed field according to the number of inequivalent indecomsable representations they have. In Chapters III, IV, and V, four independent conditions are given which imply that an algebra over an algebraically closed field be of strongly unbounded representation type.

Although some of the results obtained can be extended to the case where the field is not algebraically closed, the assumption of algebraic closure greatly simplifies the proofs of the main theorems and gives a clearer statement of the structure of the algebras involved.

Chapter I contains a precise definition of the classes of algebras to be considered. Also included in Chapter I are several conjectures concerning these classes and a brief history of the work of others on this problem. The theorems stated in Chapter I are not proved there, because they are implied by theorems appearing later in the paper.

Chapter II is concerned with establishing fundamental concepts used in later chapters. Basic algebras are introduced and the one to one correspondence between representation theory for algebras and representation theory for their basic algebras is set forth. The assumption of algebraic closure of the field is needed for the introduction of basic algebras. A lattice isomorphism between the two-sided ideal lattice of an algebra and that of its basic algebra is established. Also in Chapter II, a method of building representations is given which is used extensively in later chapters. The only new result in Chapter II is Lemma 2.5.A which gives a criterion for a representation to contain an indecomposable direct summand of at least a certain degree.

In Chapter III the structure of the lattice L_A of two-sided ideals is investigated. It is shown that finiteness and distributivity of L_A are equivalent and imply that every two-sided ideal is principal. The main result of

Chapter III, Theorem 3.2.A, states that if L_A is infinite then the algebra A is of strongly unbounded representation type. In Theorem 3.3.B it is shown that, for a commutative algebra, finiteness of the two-sided ideal lattice L_A implies that the algebra has only a finite number of inequivalent indecomposable representations. Such algebras are shown to be a direct sum of polynomial algebras in Corollary 3.3.C. Finally, two lemmas on basic algebras show that if the basic algebra has a finite two-sided ideal lattice it is generated in the subalgebra sense by two elements. All the results in Chapter III are new.

The two-sided ideal lattice is assumed to be finite and distributive in Chapters IV and V and the algebras under consideration are assumed to be basic algebras. In Chapter IV a second condition for an algebra to be of strongly unbounded type is given. The lattice L_N of two-sided ideals contained in the radical N of A is mapped lattice homomorphically into the left and right ideal lattices in the radical. If the image of L_N has a sublattice that is a Boolean algebra with more than 2^3 elements then A is of strongly unbounded representation type.

In Chapter V a graph is associated with each two-sided ideal contained in the radical. Theorems 5.2.A, 5.4.A, and 5.3.A state that if any such graph has a cycle, a vertex of order four, or a chain which branches at each end then the algebra is of strongly unbounded representation type. The results in Chapter IV and V are extensions of previous results.

CHAPTER I

1. Initial Definitions

A precise definition of the classes of algebras under consideration is given in terms of the following function:

Definition 1.1.A: If A is an algebra and d is a positive integer, let $g_A(d)$ be the number of inequivalent indecomposable representations of A of degree d. $g_A(d)$ is integer valued or infinite.

Definition 1.1.B: A is said to be of be of bounded representation type if there exists an integer d_o such that $g_A(d) = 0$ for all $d > d_o$. If not of bounded type A is said to be of unbounded representation type.

Definition 1.1.C: A is said to be of finite representation type if $\sum_{d=1}^{\infty} g_A(d)$ is finite.

Clearly, if A is semisimple, it is of finite representation type.

The class of algebras of unbounded representation type can be further divided into subclasses according to the number of integers d for which $g_A(d) = \infty$. Of particular interest in this paper is the subclass defined as follows.

Definition 1.1.D: A is said to be of strongly unbounded representation type if $g_A(d) = \infty$ for an infinite number of integers d.

The main theorems proved in this paper are concerned with showing that algebras over an algebraically closed field are of this type.

Henceforth, where it is clear that it is the representation type that is being referred to, the terms defined above are shortened to bounded type, unbounded type, etc.

It is possible to define two additional subclasses of the class of algebras of unbounded type. One is the subclass of algebras of unbounded type for which $g_A(d)$ is finite for all integers d, and the other, the subclass for which $g_A(d)=\infty$ for a finite number of integers d. R. Brauer and R. M. Thrall have conjectured that the latter subclass is empty, and that the former is also empty provided that the underlying field is infinite.

Concerning the classes of algebras of bounded type and finite type, Brauer and Thrall conjectured that these two classes are identical.

2. History

It was first noted by Nakayama [5] that some non-semisimple algebras could have indecomposable representations of arbitrarily high degree. He remarks that if N is the radical, e a primitive idempotent and $N^{i-1}e/N^{i}e$, considered as a left A space, contains the direct sum of two isomorphic subspaces, then A is of unbounded type.

In the subsequent development of the theory, considerable attention has been given to stating sufficient conditions that an algebra be of strongly unbounded type. (The methods of showing that an algebra over an infinite field has unbounded type also show it has strongly unbounded type. This gives support to the above mentioned conjectures of Brauer and Thrall that over infinite fields algebras of unbounded type are also of strongly unbounded type.)

In a paper, as yet unpublished, R. Brauer [3] stated Nakayama's condition and the following two sufficient conditions that an algebra be of strongly unbounded type.

Theorem 1.2.A: *If N is the radical, e a primitive idempotent and Ne/N^2e considered as a left A space is the direct sum of more than three subspaces then A is of strongly unbounded type*.

The condition in the hypothesis of Theorem 1.2.A is generalized in Chapter IV and the proof of this theorem is implied by the proof of Theorem 4.1.C. in the case the underlying field is algebraically closed. The second condition given by Brauer is contained in the following theorem.

Theorem 1.2.B: *If A has irreducible representations $F_1 \ldots\ldots F_h$ distinct, and $F_1^* \ldots F_h^*$ distinct and representations*

$$W_{ij} = \begin{pmatrix} F_j & & \\ P_{ij} & Q_{ij} & \\ Y_{ij} & S_{ij} & F_i^* \end{pmatrix} \qquad \begin{array}{l} \text{for } i = j = 1 \ldots h \text{ and} \\ \text{for } i = j+1 = 1 \ldots h, F_0^* = F_h^* \end{array}$$

where F_j is the top Loewy constituent F_i^* is the bottom Loewy constituent and Y_{ij} is completely independent of P_{st} and S_{pr} then A is strongly unbounded type.

(For a discussion of Loewy series and constituents, see Artin, Nesbitt and Thrall [1].) In Chapter V a condition on the algebra is introduced, which implies the hypotheses of Theorem 1.2.B. The proof of Theorem 5.2.A is similar to the proof of Theorem 1.2.B, so the proof of the latter is omitted here.

R. M. Thrall in an unpublished paper [7] generalized Nakayama's condition that an algebra be of strongly unbounded type. A statement of the condition requires the following concepts concerning representation theory for algebras.

The left ideal Ae, where A is the algebra and e is a primitive idempotent, is a vector space over the field k. Multiplication on the left of Ae by α in A is defined by multiplication in the algebra A. In this way Ae can be considered as the space of a representation $W(\alpha)$ of A. Ae is then said to induce the representation $W(\alpha)$.

Let a representation $W(\alpha)$ of A be divided into submatrices

$$W(\alpha) = \left(W_{ij}(\alpha) \right)$$

The submatrix $W_{ij}(\alpha)$ is said to have <u>power s</u> if $W_{ij}(\alpha) = 0$ for all α in N^s, but there exists α_0 in N^{s-1} such that $W_{ij}(\alpha_0) \neq 0$.

Thrall's generalization of Nakayama's condition is now given in the following theorem.

<u>Theorem 1.2.C</u>: <u>If e is a primitive idempotent, if Ae is the left ideal it generates, if</u>

$$W = \begin{pmatrix} F_i & & & \\ P & Q & & \\ Y_1 & S_1 & F_j & \\ Y_2 & S_2 & 0 & F_j \end{pmatrix} \quad \text{is a top constituent}$$

<u>of the representation induced by the left ideal Ae, where Y_1 has power s_1 and where Y_2 has power s_2, $s_2 \geq s_1$ and where no F_i appears in the top $s_2 - s_1$ constituents of the</u>

upper Loewy series for Q, then A is of strongly unbounded type.

Since it is shown in Lemma 3.4.C that the hypotheses of Theorem 1.2.C imply that the algebra has an infinite number of two-sided ideals, the proof of 1.2.C is contained in the proof of Theorem 3.2.A (in the case the underlying field is algebraically closed).

In the same paper [7], Thrall also introduced a method for illustrating graphically the sufficient conditions that an algebra be of strongly unbounded type in the case where the square of the radical is zero.

In this paper a graph will mean a set $P_1, \ldots, P_n$ of vertices and a binary relation $\mathcal{L}$ on some pairs of vertices, $P_i \mathcal{L} P_j$. $P_i \mathcal{L} P_j$ means that the vertices P_i and P_j are connected by an (oriented) edge. A vertex P_{i_0} is said to have right order r (left order r) if there exist distinct vertices $P_{i_1}, \ldots, P_{i_r}$ such that $P_{i_0} \mathcal{L} P_{i_\nu}$ $(P_{i_\nu} \mathcal{L} P_{i_0})$ for $\nu = 1, \ldots, r$. The order of a vertex is the largest of these two orders.

A chain C is a set vertices and edges $(P_{i_1}, P_{i_1} \mathcal{L} P_{i_2}), P_{i_2}, P_{i_3} \mathcal{L} P_{i_2}, \ldots, P_{i_{r-1}}, P_{i_{r-1}} \mathcal{L} P_{i_r}, P_{i_r}, (P_{i_{r+1}} \mathcal{L} P_{i_r}, P_{i_{r+1}})$ such that successive edges are distinct. The parentheses indicate that the first and last edges of a chain may have either orientation. Note that going from one vertex to the next in a chain, the orientation of successive edges alternates.

A chain C_2 extends a chain C_1 at the right end (or C_1 extends C_2 at the left end) if the first vertex of C_2

is the same as the last vertex of C_1 and identifying these vertices makes C_1 followed by C_2 a chain. The chain C_1 is said to be a cycle if it extends itself.

A chain branches at one end if it can be extended by at least two distinct edges at that end.

Let $A = A' \dot{+} N$ be a decomposition of A into the vector space direct sum of its radical and a semisimple subalgebra A'. Let $A' = \sum_{i=1}^{m} A_i$ be the ring direct sum of simple two-sided ideals A_i, each with a unity element ϵ_i. Let $P_1,\ldots,P_n$ be vertices and let $P_i \mathcal{L} P_j$ if $\epsilon_i N \epsilon_j \neq 0$. N^2 is assumed to be zero here.

Let M be the relation matrix of $\mathcal{L}$. It should be noted here that if I is the identity matrix of degree n then $I + M$ is a matrix with a non zero entry only in positions where C, the Cartan Matrix, of A has a non zero entry. For the definition and properties of the Cartan Matrix of an algebra see [1].

In the case that the radical squared is zero, sufficient conditions that an algebra be of strongly unbounded type can now be described in terms of the graph defined above.

If there exists a vertex of order 4 or more then the algebra satisfies the hypothesis of Theorem 1.2.A. If the graph has a cycle then the algebra satisfies the hypothesis of Theorem 1.2.B. The fourth sufficient condition that an algebra be strongly unbounded type is given by Thrall in [7] in terms of the graph. That condition appears in the following theorem.

Theorem 1.2.D: If the graph G as defined above has a chain which branches at each end then A is of strongly unbounded type.

In Chapter V, the definition and use of the graph is extended to the case where N^2 is not necessarily zero. Theorem 5.3.A then implies the proof of Theorem 1.2.D. Hence, the proof of Theorem 1.2.D is omitted here.

3. Necessary and Sufficient Conditions for Strongly Unbounded Type

If certain other conditions are imposed on an algebra, necessary and sufficient conditions that it be of strongly unbounded type can be given. If the underlying field is infinite in each such case, the algebra is either of strongly unbounded type or finite type.

D. G. Higman [4] has shown, using only group representation theory, that the group algebra over a field of characteristic p is of unbounded type if and only if it has a noncyclic Sylow p-subgroup.

R. M. Thrall [7] has shown that the four conditions of Nakayama, Brauer and Thrall are necessary and sufficient conditions that an algebra be of strongly unbounded type if the underlying field is algebraically closed and the square of the radical is zero.

In Theorem 3.3.B the author shows that if the algebra is commutative, the necessary and sufficient condition that it be of strongly unbounded type is that its two-sided ideal lattice be infinite. If a commutative algebra is not of strongly unbounded type it is of finite type.

The underlying field here is assumed to be algebraically closed.

CHAPTER II

It is necessary, before attempting the proofs of the main results, to consider certain preliminary concepts. Use is made here of the assumption that the underlying field is algebraically closed to investigate the properties of basic algebras and their representations. Theorem 2.4.B exhibits the relationship between the two-sided ideal structures of an algebra and its basic algebra.

In this chapter a fundamental tool, Lemma 2.5.A, is provided which is used for showing the existence of indecomposable representations of large degree. In Lemma 2.5.C a method for constructing representations of large degree is given.

1. Structure Theory

Considering the Wedderburn Structure Theory for algebras as given in Artin, Nesbitt and Thrall [1] an algebra A with unity element, over k, an algebraically closed field, can be decomposed into the vector space direct sum,

(2.1) $$A = A' \dot{+} N,$$

where A' is a semisimple subalgebra and N is the radical of A. A' can be further decomposed into the ring direct sum,

(2.2) $$A' = \dot{+} \sum_{i=1}^{n} A_i,$$

where each A_i is a simple ideal of A'. Since the underlying field k is algebraically closed, each A_i is a total matrix algebra over k. A basis of matrix units $C_{i\mu\nu}$ $\mu,\nu = 1 \ldots f(i)$ can be chosen for each A_i. The set $C_{i\mu\nu}$ $i = 1 \ldots n$ $\mu,\nu = 1 \ldots f(i)$, where f(i) is the degree of the matrix set for A_i, forms a basis for A'. Since (2.2) is a direct sum in the ring sense and the $C_{i\mu\nu}$ multiply like matrix entries, they have the following multiplication formula.

(2.3) $$C_{i\mu\nu}C_{j\rho\tau} = \delta_{ij}\,\delta_{\nu\rho}\,C_{i\mu\tau}$$ (δ_{ij} the Kronecker delta symbol)

Also, the unity element of A is written

(2.4) $$1 = \sum_{i=1}^{n} \sum_{\mu=1}^{f(i)} C_{i\mu\mu}$$

Now let $\hat{1}$ be defined as

(2.5) $$\hat{1} = \sum_{i=1}^{n} C_{i\,11}.$$

2. Basic Algebras

Definition 1.2.A: The set $\hat{1}\,A\,\hat{1} = \hat{A}$ is called the basic algebra of A.

Clearly, $\hat{A}$ is a subalgebra of A. Basic algebras were first introduced by Nesbitt and Scott [6] and were also treated by Wall [9]. It is shown by Wall that, although the basic algebra $\hat{A}$ depends on the choice of a semisimple subalgebra A' and on the choice of matrix units, a different choice yields an isomorphic basic algebra.

The unity element of $\hat{A}$ is obviously $\hat{1}$. The radical of $\hat{A}$ is $\hat{1}\,N\,\hat{1} = \hat{N}$. Thus, the decomposition (2.1) implies a decomposition of $\hat{A}$.

$$\hat{A} = \sum_{i=1}^{n} k\, C_{i11} \dot{+} \hat{N}. \tag{2.6}$$

The C_{i11} in $\hat{A}$ will be labeled e_i. They are orthogonal idempotents by (2.3). The semisimple subalgebra $\hat{A}'$ of $\hat{A}$ corresponding to A' in (2.1) is the ring direct sum of the one dimensional two-sided ideals $k\, e_i$ of $\hat{A}'$. Since irreducible representations of $\hat{A}$ have maximal kernels in $\hat{A}/\hat{N}$, it is clear that irreducible representations of $\hat{A}$ are one dimensional over k.

Using the unity element $\hat{1}$ of $\hat{A}$ a decomposition of the radical $\hat{N}$ is possible.

$$\hat{N} = \dot{+} \sum_{i,j=1}^{n} e_i \hat{N} e_j. \tag{2.7}$$

Since the idempotents are orthogonal, (2.7) is a vectorspace direct sum. (2.6) and (2.7) imply that any element α in $\hat{A}$ can be written,

$$\alpha = \sum_{i=1}^{n} x_i(\alpha) e_i + \sum_{i,j=1}^{n} e_i\, \nu_\alpha\, e_j, \tag{2.8}$$

where $x_i(\alpha)$ is in k and $e_i\, \nu_\alpha\, e_j$ is in $e_i \hat{N} e_j$.

Now let $\hat{V}$ be a representation space for $\hat{A}$ and choose in $\hat{V}$ a composition series $\hat{V} = \hat{V}_1 \supset \hat{V}_2 \supset \ldots \supset \hat{V}_{t+1} = 0$ of $\hat{A}$ subspaces of $\hat{V}$. Each $\hat{V}_i/\hat{V}_{i+1}$ is irreducible and hence one dimensional. Pick v_i in $\hat{V}_i$ but not in $\hat{V}_{i+1}$. Let $\hat{1}$ act like the identity on $\hat{V}$, then $\hat{1}v_i = v_i$. There must exist $e_{j(i)}$ such that $e_{j(i)}v_i$ is not in $\hat{V}_{i+1}$, because if all $e_j v_i$ were in $\hat{V}_{i+1}$, $\sum_{j=1}^{n} e_j v_i = v_i$ would be there too. Let $v_i' = e_{j(i)} v_i$. Clearly $\{v_i' \mid i=1 \ldots t\}$ is a basis for $\hat{V}$.

Note that orthogonality of the idempotents e_i implies $e_p v_i' = 0$ unless $p = j(i)$, and then $e_{j(i)} v_i' = v_i'$. Let α be an element of $\hat{A}$. Using expression (2.8),

$$(2.9) \qquad \alpha v_i' = x_{j(i)}(\alpha) v_i' + \sum_{p=1}^{n} e_p \nu_\alpha e_{j(i)} v_i',$$

where each $e_p \nu_\alpha e_{j(i)} v_i'$ is in $\hat{V}_{i+1}$. With respect to this basis $\{v_i'\}$, the representation of $\hat{A}$ has the matrix form

$$(2.10) \quad R(\alpha) = \begin{pmatrix} x_{j(1)}(\alpha) & & & & 0 \\ & x_{j(2)}(\alpha) & & & \\ & & \cdot & & \\ & * & & \cdot & \\ & & & & x_{j(t)}(\alpha) \end{pmatrix}.$$

It is evident that if ν is an element in the radical each $x_i(\nu) = 0$, so $R(\nu)$ has zeros on and above the diagonal. Also by the choice of the basis, if e_p is one of the previously defined idempotents, $R(e_p)$ has zeros off the diagonal and $\delta_{pj(i)}$ in the i^{th} diagonal position. Let ν be an element of $e_p \hat{N} e_r$, then

$$(2.11) \qquad R(e_p) R(\nu) R(e_r) = R(\nu).$$

By the description of $R(e_p)$ and $R(e_r)$ given above, it follows that $R(\nu)$ can be non-zero only in entries directly below $x_{j(i)}$ where $j(i) = r$ and directly to the left of $x_{j(i)}$ where $j(i) = p$. When considering the matrix form of a representation of $\hat{A}$, it will always be assumed to be in the form (2.10) and the above mentioned facts will hold for it.

3. Representations of Algebras and Basic Algebras

The reason for considering representations of basic algebras is seen in the following development. Let V be a representation space for the algebra A.

Define

(2.12) $$\hat{1}\,V = \hat{V}.$$

Since elements of $\hat{A}$ are of the form $\hat{1}\alpha\hat{1}$, it is clear that $\hat{V}$ is an $\hat{A}$ space. If $\hat{1}\,V = 0$, then $0 = C_{i11}\,V = C_{i1\mu}\,V$. It follows that $C_{i\mu 1}C_{i1\mu}V = C_{i\mu\mu}\,V = 0$ implying $1V = 0$. Hence if 1 acts like the identity on V, $\hat{V}$ is not zero and $\hat{1}$ acts like the identity on $\hat{V}$. Also if $V = V_1 \dotplus V_2$ (A direct) then $\hat{V} = \hat{1}(V_1 \dotplus V_2) = \hat{V}_1 \dotplus \hat{V}_2$ ($\hat{A}$ direct). Thus, if V is decomposable so is $\hat{V}$.

An important fact concerning this process of going from A spaces to $\hat{A}$ spaces is that it has an inverse. Let $\hat{V}$ be an $\hat{A}$ space, let $\{v_i'\}$ be a basis chosen so that the matrix form of the representation of $\hat{A}$ is as (2.10). Recall that $e_{j(i)}v_i' = v_i' = C_{j(i)11}v_i'$, where $C_{j(i)11}$ is equal to $e_{j(i)}$. Now adjoin to $\hat{V}$ the additional basis vectors $C_{j(i)\mu 1}v_i'$; $\mu = 2, \ldots f(j(i))$. Call the new space V and define a representation of A on V in the following manner for the given basis of V. Let α be in A

(2.13) $$\alpha(C_{j\mu 1}v_i') = \sum_{t=1}^{n}\sum_{\rho=1}^{f(t)} C_{t\rho\rho}\,\alpha\, C_{j\mu 1}v_i'$$

$$= \sum_{t=1}^{n}\sum_{\rho=1}^{f(t)} C_{t\rho 1}(C_{t1\rho}\alpha C_{j\mu 1}v_i').$$

The element $C_{t1\rho}\alpha\, C_{j\mu 1}$ is in $\hat{A}$, so the expression in parentheses is a linear combination of basis vectors in $\hat{V}$. When this is multiplied on the left by $C_{t\rho 1}$, the resulting element is well defined in V. If $\hat{V}$ is an $\hat{A}$ direct sum, V is an A direct sum. Clearly, these two processes are the inverses of each other. Wall [9] has proved the following theorem highlighting the value of the preceding development.

<u>Theorem 2.3.A:</u> <u>If V and V' are A-spaces, $\hat{V}$ and $\hat{V}'$ their corresponding $\hat{A}$-spaces, then V is A-isomorphic to V' if and only if $\hat{V}$ is $\hat{A}$-isomorphic to $\hat{V}$</u>.

With the exception of Theorem 2.3.A all the above results are in Nesbitt and Scott [6]; they are presented here in condensed form. In addition, Nesbitt and Scott showed that the composition length of V as an A space is the same as the composition length of $\hat{V}$ as an $\hat{A}$ space.

Thus, in studying algebras over an algebraically closed field with respect to their representation theory, it is sufficient merely to study representations of basic algebras. Every indecomposable representation of A leads to a corresponding indecomposable representation of $\hat{A}$ and conversely. Since the factor space of two successive steps in a composition series is irreducible, its dimension is equal to the degree of that irreducible representation. There are only a finite number of such irreducible representations for any given algebra. Hence, there exist an infinite number of inequivalent indecomposable representations of a certain degree of A if and only if there exist an infinite number of inequivalent indecomposable representations of $\hat{A}$ of some smaller degree. The following theorem sums up the above results.

Theorem 2.3.B: If A is an algebra over k, an algebraically closed field, and $\hat{A}$ is its basic algebra, then A is of strongly unbounded, bounded or finite type if and only if $\hat{A}$ is of the same type.

4. Two-Sided Ideals in A and $\hat{A}$

In addition to using the correspondence between the representation theory for A and for $\hat{A}$, use will be made of the structures of their two-sided ideals. Let L_A be the lattice of two-sided ideals in A and let $L_{\hat{A}}$ be the lattice of two-sided ideals in $\hat{A}$. (For a discussion of the lattice concepts used here see Birkhoff [2].) Let A_0 be a two-sided ideal in A. Define the function $\emptyset$ from L_A to $L_{\hat{A}}$,

(2.14) $$\emptyset\colon\ L_A \longrightarrow L_{\hat{A}} \quad \text{by} \quad \emptyset(A_0) = \hat{1}A_0\hat{1}.$$

Lemma 2.4.A: $\emptyset$ is a lattice homomorphism of L_A into $L_{\hat{A}}$.

Proof: Since A_0 is a two-sided ideal $\hat{1}A_0\hat{1} \subseteq A_0 \cap \hat{A}$. Also $\hat{1}$ is the unity element in $\hat{A}$, so $A_0 \cap \hat{A} \subseteq \hat{1}A_0\hat{1}$. Therefore $\hat{1}A_0\hat{1} = A_0 \cap \hat{A}$.

Then form $\emptyset(A_1) + \emptyset(A_2) = \hat{1}A_1\hat{1} + \hat{1}A_2\hat{1}$. Distributivity of multiplication implies this equals $\hat{1}(A_1 + A_2)\hat{1} = \emptyset(A_1 + A_2)$.

Let $\emptyset(A_1) \cap \emptyset(A_2) = (A_1 \cap \hat{A}) \cap (A_2 \cap \hat{A})$. But this is $A_1 \cap A_2 \cap \hat{A}$ because set intersection is associative, commutative, and $\hat{A} \cap \hat{A} = \hat{A}$. Therefore $\emptyset(A_1) \cap \emptyset(A_2) = \emptyset(A_1 \cap A_2)$. This completes the proof of the lemma.

Let $\hat{A}_0$ be a two-sided ideal in $\hat{A}$ and define the function ψ from $L_{\hat{A}}$ to L_A,

(2.15) $$\psi\colon\ L_{\hat{A}} \longrightarrow L_A \quad \text{by} \quad \psi(\hat{A}_0) = \left\{\hat{A}_0\right\}_A,$$

where $\{\hat{A}_0\}_A$ is the two-sided ideal in A generated by elements in $\hat{A}_0$. The following theorem gives the desired relation between L_A and $L_{\hat{A}}$.

Theorem 2.4.B: Both $\phi\psi$ and $\psi\phi$ are identity functions and L_A and $L_{\hat{A}}$ are lattice isomorphic under ϕ and ψ.

Proof: Since $\phi(A_0) \subseteq A_0$, $\psi\phi(A_0) \subseteq A_0$ by the definition of ψ. Let α be in A_0. A_0 is a two-sided ideal so that $C_{i\mu\nu}\alpha C_{j\rho\tau}$ is also in A_0 and $C_{i1\nu}\alpha C_{j\rho 1}$ is in $\phi(A_0)$ because $\hat{1}$ leaves it invariant on both sides. It follows that $C_{i\nu 1}C_{i1\nu}\alpha C_{j\rho 1}C_{j1\rho}$ is in $\psi\phi(A_0)$. But then α is in $\psi\phi(A_0)$ because $\alpha = 1\alpha 1 = \sum_{i,\nu,j,\rho} C_{i\nu\nu}\alpha C_{j\rho\rho}$. Hence $\psi\phi$ is identity.

Certainly, $\phi\psi(\hat{A}_0) \supseteq \hat{A}_0$ because $\hat{A}_0 \subseteq (\hat{A}_0)_A$ and $\hat{1}\hat{A}_0\hat{1} = \hat{A}_0$. Let β be in $\phi\psi(\hat{A}_0)$, then $\beta = \hat{1}\beta_0\hat{1}$ where $\beta_0 = \sum_{\tau\rho} \beta_\tau \hat{\beta}_{\tau\rho} \beta_\rho$ and the $\hat{\beta}_{\tau\rho} = \hat{1}\hat{\beta}_{\tau\rho}\hat{1}$ are in $\hat{A}_0$. Then $\beta = \sum_{\tau\rho} (\hat{1}\beta_\tau\hat{1})\hat{\beta}_{\tau\rho}(\hat{1}\beta_\rho\hat{1})$ is in $\hat{A}_0$ because $\hat{A}_0$ is a two-sided ideal in $\hat{A}$. Therefore $\phi\psi$ = identity. ϕ is one to one, onto, and a lattice homomorphism, hence ϕ is a lattice isomorphism and ψ is its inverse.

Corollary 2.4.C: If a two-sided ideal in $\hat{A}$ is principal, then its image under ψ is principal in A.

Proof: If $\hat{A}_0 = \{\alpha_0\}$ then, by the definition of ψ, $\psi(A_0) = \{\alpha_0\}_A$ is the two-sided ideal generated by α_0 in A and is therefore principal.

5. Representation of Large Degree

In the proofs of the main theorems in Chapters III, IV, and V, certain algebras are shown to have indecomposable representations of arbitrarily high degree. Those proofs depend heavily on the following lemma, which can be used to show the existence of indecomposable representations of a certain degree without actually exhibiting the representations themselves.

Lemma 2.5.A: If A is an algebra over an algebraically closed field, V is a representation space for A, L is the commutator algebra of the representation (the set of all A-homomorphisms of V into itself), and if every B in L has more than d equal eigenvalues, then V has an indecomposable direct summand V_0 of dimension greater than d.

Proof: Suppose the contrary. Let V be decomposed into $V = V_1 \dotplus \ldots \dotplus V_t$ (A direct sum) where each V_i has dimension $d_i \leq d$. Let $y_1 \ldots y_t$ be distinct elements in the field k. Let B be the homomorphism of V into V which maps vectors v_i in V_i onto $y_i v_i$. On each direct summand V_i, B acts like y_i times the identity matrix. This commutes with every homomorphism of V_i into V_i, including those caused by elements in A. Thus B is in L. But this contradicts the hypotheses of the lemma, because the eigenvalues of B are the y_i, each appearing as often as the dimension d_i of V_i and $d_i \leq d$. This contradiction establishes the proof of Lemma 2.5.A.

A result on commuting matrices which is used extensively in Chapters III, IV, and V is given in the following development. Let

$$(2.16) \qquad P_c = \begin{pmatrix} c & & & & & & \\ 1 & c & & & & & \\ & 1 & \cdot & & & & \\ & & \cdot & \cdot & & & \\ & & & \cdot & \cdot & & \\ & & & & \cdot & \cdot & \\ & & & & & \cdot & \cdot \\ & & & & & 1 & c \\ & & & & & & 1 \quad c \end{pmatrix}$$

be a square matrix with a single eigenvalue c and 1's just below the diagonal. P_c is called a primary matrix. Let V be the space on which P_c acts. Since V is generated by powers of P_c acting on a single vector, V is indecomposable. In [1] it is shown that V is indecomposable if and only if the commutator algebra of V is completely primary. If the underlying field is algebraically closed, the following lemma is a corollary to that result.

Lemma 2.5.B: If B is a matrix which commutes with P_c, $P_cB = BP_c$, then B has exactly one eigenvalue.

In the proofs of the main results, it is necessary to construct representations of large degree. One method is to form direct sums of representations. Another method used by Thrall in [7] is given in the following.

Let $R(\alpha)$ be a representation of A by matrices. Let

$$(2.17) \qquad R(\alpha) = (C_{ij}(\alpha))$$

be the matrix cut into submatrices $C_{ij}(\alpha)$, so that the $C_{ii}(\alpha)$ are all square and $C_{ij}(\alpha)$ has the same number of rows as $C_{ii}(\alpha)$ and the same number of columns as $C_{jj}(\alpha)$. Then

$$(2.18) \qquad R(\alpha)R(\beta) = R(\alpha\beta) \text{ implies } \sum_{\nu=1}^{t} C_{i\nu}(\alpha)C_{\nu j}(\beta) = C_{ij}(\alpha\beta).$$

Let $D = (D_{ij})$ also be a matrix whose coefficients are matrices such that (D_{ij}) has the same number of rows and columns as $(C_{ij}(\alpha))$ and the diagonal blocks D_{ii} are square. Let $D_{ij} \times C_{ij}(\alpha)$ be the Kronecker product of the two matrices D_{ij} and $C_{ij}(\alpha)$. (For the definition and properties of the Kronecker product of two matrices see van der Waerden [8].)

Lemma 2.5.C: If $D_{ij}D_{jk} = D_{ik}$ holds for the positions where $C_{ij}(\alpha)$, $C_{jk}(\alpha)$, $C_{ik}(\alpha)$ are not identically zero then $Q(\alpha) = (D_{ij} \times C_{ij}(\alpha))$ is a representation of A. Further D_{ii} = identity implies $Q(1)$ = identity.

Proof: The properties of Kronecker products that are used here are $A \times (B + C) = (A \times B) + (A \times C)$ and $(A \times B)\cdot(C \times D) = (A \cdot C) \times (B \cdot D)$ if all products are defined.

$Q(\alpha) + Q(\beta) = (D_{ij} \times C_{ij}(\alpha)) + (D_{ij} \times C_{ij}(\beta)) =$
$(D_{ij} \times C_{ij}(\alpha) + D_{ij} \times C_{ij}(\beta)) = (D_{ij} \times [C_{ij}(\alpha) + C_{ij}(\beta)])$
$= (D_{ij} \times C_{ij}(\alpha + \beta)) = Q(\alpha + \beta)$.

Also $Q(\alpha)Q(\beta) = \left(\sum_{\nu=1}^{t} (D_{i\nu} \times C_{i\nu}(\alpha)) \cdot (D_{\nu j} \times C_{\nu j}(\beta)\right)$

$$= \left(\sum_{\nu=1}^{t}(D_{i\nu} \cdot D_{\nu j}) \times (C_{i\nu}(\alpha) \cdot C_{\nu j}(\beta))\right)$$

$$= \left(\sum_{\nu=1}^{t} D_{ij} \times (C_{i\nu}(\alpha) \cdot C_{\nu j}(\beta))\right) = (D_{ij} \times C_{ij}(\alpha\beta)) = Q(\alpha\beta).$$

Then $Q(\alpha)$ is clearly a representation of A.

CHAPTER III

In this chapter it is shown that every algebra over an algebraically closed field with an infinite two-sided ideal lattice is of strongly unbounded type. Finiteness of the two-sided ideal lattice is equivalent to distributivity of that lattice. This is proved by Corollary 3.1.G. If the algebra is commutative and has only a finite number of two-sided ideals it is shown to be of finite representation type. The proof of this fact also establishes Corollary 3.3.C which states that every commutative algebra with a finite two-sided ideal lattice is the direct sum of polynomial algebras over the field k. Before proving these results it is necessary to establish certain results about lattices and in particular about the two-sided ideal lattice of a basic algebra.

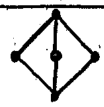
1. Two-Sided Ideal Lattices

The following facts concerning lattices will be used in this and later chapters. A modular lattice is one which has the property that $D \supseteq B$ implies $D \cap (B + C) = (D \cap B) + (D \cap C)$. Lattices generated by subspaces of a vector space, where $D \cap B$ means set intersection and $D + B$ means the subspace generated by D and B, are modular lattices. Since left, right, or two-sided ideals are subspaces of an algebra and $D \cap B$, $D + B$ are again left, right or two-sided ideals if both D and B are, the lattice of left ideals, the lattice of right ideals and the lattice of two-sided ideals are all modular lattices. The dimension of a modular lattice is the number of elements in a chain from the least element to the greatest: Modularity of the lattice is needed to show that the dimension is independent of the choice of the chain.

A lattice is *distributive* if $D \cap (B + C) = (D \cap B) + (D \cap C)$ always holds. Clearly, a distributive lattice is modular. Proofs of the following two lemmas can be found in [2].

Lemma 3.1.A: *A finite dimensional distributive lattice is necessarily finite.*

Lemma 3.1.B: *A modular lattice is distributive if and only if it fails to contain a sublattice of the form*



The sublattice appearing in the previous lemma is called a *projective root*. A lattice homomorphism of a projective root either maps it onto a single element or maps it isomorphically.

If U is the greatest element in a lattice and 0 the smallest, an element D is said to have a *complement* D' if $D + D' = U$ and $D \cap D' = 0$. If every element is complemented, the lattice is called a *complemented lattice*. A complemented distributive lattice is called a *Boolean algebra*. An element D *covers* B if $D \supset B$ and $D \supset C \supseteq B$ implies $C = B$. The following lemma gives a fact that will be used in Chapter V.

Lemma 3.1.C: In a distributive lattice, the covers of a single element generate a Boolean algebra.

Now let A be an algebra over an algebraically closed field k and consider L_A, its lattice of two-sided ideals. By the lattice isomorphism of Theorem 2.4.B it is sufficient to consider only basic algebras. Throughout the remainder of this section A will represent a basic algebra.

Using the theory developed in Chapter II for basic algebras, the algebra A can be written

$$A = \sum_{i,j=1}^{n} e_i A e_j,$$

where $e_i A e_j$ is an additive subspace of A. The e_i are the n orthogonal idempotents, forming a basis for the semisimple subalgebra A' of A.

Definition 3.1.D: Let ϕ_{ij} be a function from L_A to subspaces of $e_i A e_j$ defined by $\phi_{ij}(A_0) = A_0 \cap e_i A e_j = e_i A_0 e_j$.

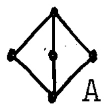
Lemma 3.1.E: For each pair i,j, ϕ_{ij} is a lattice homomorphism.

Proof: $\phi_{ij}(A_1 + A_2) = e_i(A_1 + A_2)e_j = e_i A_1 e_j + e_i A_2 e_j$ $= \phi_{ij}(A_1) + \phi_{ij}(A_2)$ because of the distributive law for multiplication in A. Secondly, $\phi_{ij}(A_1 \cap A_2) = A_1 \cap A_2 \cap e_i A e_j =$ $(A_1 \cap e_i A e_j) \cap (A_2 \cap e_i A e_j) = \phi_{ij}(A_1) \cap \phi_{ij}(A_2)$. Thus, ϕ_{ij} is a lattice homomorphism.

Distributivity of the two-sided ideal lattice L_A can now be described in terms of these n^2 lattice homomorphisms ϕ_{ij}.

Lemma 3.1.F: $\phi_{ij}(L_A)$ is a chain for each pair i,j if and only if L_A is distributive.

Proof: Suppose L_A is not distributive, then, since L_A is a modular lattice L_A contains a projective root R which is either collapsed entirely or mapped isomorphically by a lattice homomorphism. Let

(3.1) R =  A_1. A_2

But there exists a pair i,j such that $\phi_{ij}(A_1) \neq \phi_{ij}(A_2)$. For if $\phi_{ij}(A_1) = \phi_{ij}(A_2)$ for all i,j, then summing over all i,j, $A_1 = A_2$. Hence for that pair i,j $\phi_{ij}(L_A)$ contains an isomorphic image of the projective root and is therefore not a chain.

To prove the "if" part, let $\phi_{ij}(L_A)$ not be a chain. Starting up from 0 in e_iAe_j, let $\phi_{ij}(A_0)$ be the smallest element in $\phi_{ij}(L_A)$ which has at least two covers in $\phi_{ij}(L_A)$. Let A_δ be the sum of all two-sided ideals A_α of L_A for which $\phi_{ij}(A_\alpha) \subseteq \phi_{ij}(A_0)$. Since ϕ_{ij} is a lattice homomorphism $\phi_{ij}(A_\delta) = \phi_{ij}(A_0)$. Let $\phi_{ij}(A_1)$ and $\phi_{ij}(A_2)$ be two distinct covers of $\phi_{ij}(A_0)$.

Choose an element α_1 in $\phi_{ij}(A_1)$ but not in $\phi_{ij}(A_\delta)$ or in $\phi_{ij}(A_2)$ and form its principal ideal $(\alpha_1) = A\alpha_1A$. Since α_1 is chosen in e_iAe_j, $e_t\alpha_1 = 0$ unless $t = i$. For the same reason, $\alpha_1 e_r = 0$ unless $r = j$. Using expression (2.6) for A it is clear that (α_1) can be written as

$$(3.2) \qquad (\alpha_1) = k\alpha_1 \dot{+} \alpha_1 N + N\alpha_1 + N\alpha_1 N.$$

The first $\dot{+}$ in the above expression is direct because the summands after that are in a power of the radical one higher than the power of the radical that α_1 is in. Clearly, the expression $\alpha_1 N + N\alpha_1 + N\alpha_1 N$ is a two-sided ideal because N is a two-sided ideal. By the choice of α_1 in $e_i A e_j$ it is clear that

$$(3.3) \qquad \phi_{ij}(\alpha_1 N + N\alpha_1 + N\alpha_1 N) \subset \phi_{ij}((\alpha_1)) \subseteq \phi_{ij}(A_1),$$

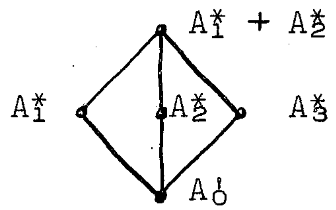
where the first is properly contained in the last. By the choice of A_0^*, $\alpha_1 N + N\alpha_1 + N\alpha_1 N$ is contained in A_0^*. Therefore $A_1^* = k\alpha_1 \dotplus A_0^*$ is a two-sided ideal and the sum is direct because α_1 was chosen not in A_0^*. Since α_1 was chosen not in $\phi_{ij}(A_2)$, $\phi_{ij}(A_1^*)$ and $\phi_{ij}(A_2)$ are two distinct covers of $\phi_{ij}(A_0^*)$ in $e_i A e_j$.

Running through an identical argument with the subscript 2 replacing 1, $A_2^* = k\alpha_2 \dotplus A_0^*$ is a two-sided ideal, and the sum is direct. A_1^* and A_2^* both cover A_0^*, for their quotients are one dimensional. The sum $A_1^* + A_2^*$ is

$$k\alpha_1 + k\alpha_2 + A_0^*.$$

Let $\alpha_3 = \alpha_1 + \alpha_2$, then $A_3^* = k\alpha_3 + A_0^*$ is a third cover of A_0^* and

(3.4)



is clearly a projective root. Hence L_A is not distributive. This completes the proof of the lemma.

Note that the previous proof implies that $k(k_1\alpha_1 + k_2\alpha_2) + A_0^*$ is a different two-sided ideal for every distinct ratio k_1/k_2. This leads directly to the following corollary.

Corollary 3.1.G: L_A is finite if and only if L_A is distributive.

Proof: If L_A is not distributive the previous lemma implies $\emptyset_{ij}(L_A)$ is not a chain for some pair i,j and there exist two-sided ideal $A\delta$ and the elements α_1, α_2 of the previous lemma. By the comment above, $k(k_1\alpha_1 + k_2\alpha_2) + A\delta$ is a distinct two-sided ideal for each ratio k_1/k_2. Since the field is infinite, the lattice of two-sided ideals is necessarily infinite.

Conversely, a finite dimensional distributive lattice is necessarily finite by Lemma 3.1.A.

2. The Two-Sided Ideal Lattice and Strongly Unbounded Type

It can now be shown that infiniteness of the two-sided ideal lattice implies that the algebra is of strongly unbounded type.

There is a pattern that is repeated in the proofs of each of the theorems in which an algebra is shown to be of strongly unbounded type. This pattern is outlined in the following.

In the first part of the proof, the hypotheses of the theorem are used to show the existence of certain special elements $\alpha_1, \ldots, \alpha_r$ in the algebra and to construct a representation R_{cs}, s an integer, c a parameter in k. R_{cs} has a degree which is a fixed integral multiple of s.

An element B in the commutator algebra of R_{cs} must satisfy the commutator equation $R_{cs}(\alpha)B = BR_{cs}(\alpha)$ for all α in A. This equation is examined for α equal to each of the special elements. Among the conditions this imposes on

B is that B have at least s equal eigenvalues. Lemma 2.5.A then implies that R_{cs} has an indecomposable direct summand T_{cs} of degree at least s. At this point the conclusion can be drawn that A is of unbounded type.

In the second part of the proof, it is shown that for distinct values of the parameter, $c \neq d$, T_{cs} is not similar to T_{ds}. Since the field is infinite there are an infinite number of such inequivalent indecomposable representations with degrees between the degree of R_{cs} and s. So for some integer d_s between, there must be an infinite number of inequivalent indecomposable representations with degree d_s. Clearly the algebra A is of strongly unbounded type.

The method for showing that T_{cs} and T_{ds} are not similar is to assume they are and show this produces a contradiction. If T_{cs} and T_{ds} were similar there would exist an intertwining matrix P satisfying $PR_{cs}(\alpha) = R_{ds}(\alpha)P$ and P, when cut down to the space V_T of T_{cs}, would represent an isomorphism. By letting α equal each of the special elements in the intertwining equation, certain conditions are imposed in P. Among these conditions, is that $P_{11}P_{ds} = P_{cs}P_{11}$ where P_{11} is a certain block cut out of P and P_{cs}, P_{ds} are primary matrices with eigenvalues c and d respectively. It is then seen that P_{11} represents P cut down to a certain subspace V_0 contained in V_T. Then P_{11} represents an isomorphism and has an inverse P_{11}^{-1}. The above equation is then impossible. This contradiction establishes that T_{cs} and T_{ds} cannot be equivalent.

The main theorem of this chapter is now proved according to the above scheme.

<u>Theorem 3.2.A: If A is an algebra over an algebraically closed field k, and if L_A, its two-sided ideal lattice, is infinite, then A is of strongly unbounded representation type.</u>

Proof: It is sufficient to consider only basic algebras, for an algebra has the above conditions if and only if its basic algebra has them. Hence let A be a basic algebra.

By 3.1.F and 3.1.G, infiniteness of the two-sided ideal lattice is equivalent to the existance of a pair i,j such that $\emptyset_{ij}(L_A)$ is not a chain.

If $i \neq j$ then $e_iNe_j = e_iAe_j$. If $i = j$, then e_iAe_j is a subalgebra of A with a single idempotent e_i. If A_o is a two-sided ideal of A, $\emptyset_{ii}(A_o)$ is a two-sided ideal of e_iAe_i. If $\emptyset_{ii}(A_o)$ contains e_i, it is all of e_iAe_i; if not, it is nilpotent and is contained in e_iNe_i the radical of e_iAe_i. Hence in the case $i = j$ the part of $\emptyset_{ii}(L_A)$ which is not a chain must be contained in e_iNe_i. Thus, in any case, there exist two two-sided ideals A_1, A_2 such that $\emptyset_{ij}(A_1)$ and $\emptyset_{ij}(A_2)$ are incomparable and both are contained in e_iNe_j. Hence there exist α_1, α_2 in e_iNe_j such that α_1 is in A_1 and not in A_2, α_2 is in A_2 and not in A_1. These two ideals A_1, A_2 and these two special elements are used in proving A is of strongly unbounded type.

Let R_1 and R_2 be representations of A with kernels A_1 and A_2 respectively. (There always exist such representations, for instance, let R_t be the regular representation of A/A_t.) By the choice of the elements α_1 and α_2 relative to the kernels A_1 and A_2 of R_1 and R_2,

$$(3.5) \qquad R_t(\alpha_t) = 0 \qquad R_t(\alpha_r) \neq 0 \quad t,r = 1,2.$$

Assume R_1 and R_2 are in the diagonal form (2.10). α_2 is in N so $R_1(\alpha_2)$ is non zero only below the diagonal. Since $R_1(\alpha_2) \neq 0$, there exists a top non zero row of $R_1(\alpha_2)$ the r^{th}, $r \geq 2$, and in that r^{th} row, there exists a non zero entry b_{rt}, $t < r$, farthest to the right. Let R_1' be the representation induced by R_1 by taking the square diagonal block of R_1 having b_{rt} in the lower left hand corner. Since R_1 is in the diagonal form (2.10), R_1' is the representation induced by V_t/V_{r+1} in the composition series for R_1. Note that $R_1'(\alpha_1)$ is still zero. Induce a representation R_2' from R_2 in an analogous manner.

$R_2'(\alpha_1)$ and $R_1'(\alpha_2)$ are non zero only in the lower left corner. Now by multiplying α_1 and α_2 by appropriate scalars, the non zero entries may be assumed to be 1 in k. Both α_1 and α_2 were chosen in e_iNe_j, so that by the development of Chapter II, R_1' and R_2' are assumed to be in the diagonal form (2.10).

$$(3.6)\quad R_1'(\alpha) = \begin{pmatrix} x_j(\alpha) & & \\ P_1(\alpha) & Q_1(\alpha) & \\ y_1(\alpha) & S_1(\alpha) & x_i(\alpha) \end{pmatrix}, \quad R_2'(\alpha) = \begin{pmatrix} x_j(\alpha) & & \\ P_2(\alpha) & Q_2(\alpha) & \\ y_2(\alpha) & S_2(\alpha) & x_i(\alpha) \end{pmatrix}$$

By the choice of the representations R_1' and R_2' and the choice of the special elements α_1, α_2 the following relations hold.

(3.7) $x_i(\alpha_t)$, $x_j(\alpha_t)$, $P_r(\alpha_t)$, $Q_r(\alpha_t)$, $S_r(\alpha_t)$ are all zero, for $r,t = 1$ or 2; $y_r(\alpha_t) = 0$ when $t = r$, 1 when $t \neq r$.

Using the two representations R_1' and R_2', form their direct sum $R_1' \dot{+} R_2' = R'$. Clearly,

$$(3.8)\qquad R' \cong \begin{pmatrix} x_j & & & & & \\ 0 & x_j & & & & \\ P_1 & 0 & Q_1 & & & \\ 0 & P_2 & 0 & Q_2 & & \\ y_1 & 0 & S_1 & 0 & x_i & \\ 0 & y_2 & 0 & S_2 & 0 & x_i \end{pmatrix} \cong \left(\begin{array}{c|ccccc} x_j & & & & & \\ \hline 0 & x_j & & & & \\ P_1 & P_1 & Q_1 & & & \\ 0 & P_2 & 0 & Q_2 & & \\ y_1 & y_1 & S_1 & 0 & x_i & \\ 0 & y_2 & 0 & S_2 & 0 & x_i \end{array}\right)$$

where $\cong$ indicates similarity. Let R be the representation induced below and to the right of the dotted lines.

Let s be an arbitrary positive integer, I the unit matrix of degree s. Let P_{cs} be a primary matrix of degree s with the single eigenvalue c. Then

$$(3.9)\qquad D_{cs} = \begin{pmatrix} I & & & & \\ I & I & & & \\ I & 0 & I & & \\ I & I & 0 & I & \\ P_{cs} & 0 & P_{cs} & 0 & I \end{pmatrix}$$

is seen to satisfy the hypotheses of Lemma 2.5.C, hence

$$(3.10)\qquad R'_{cs} = \begin{pmatrix} I \times x_j & & & & \\ I \times P_1 & I \times Q_1 & & & \\ I \times P_2 & 0 & I \times Q_2 & & \\ I \times y_1 & I \times S_1 & 0 & I \times x_i & \\ P_{cs} \times y_2 & 0 & P_{cs} \times S_2 & 0 & I \times x_i \end{pmatrix}$$

is a representation. R'_{cs} is seen to be similar to

$$(3.11)\qquad R^{*}_{cs} = \left(\begin{array}{c|cc|c:c}
I \times x_j & & & & \\
\hline
I \times P_1 & I \times Q_1 & & & \\
I \times P_2 & 0 & I \times Q_2 & & \\
\hline
I \times y_1 + P_{cs} \times y_2 & I \times S_1 & P_{cs} \times S_2 & I \times x_i & \\
\hdashline
P_{cs} \times y_2 & 0 & P_{cs} \times S_2 & 0 & I \times x_i
\end{array}\right)$$

Let R_{cs} be the representation induced above and to the left of the dotted lines. R_{cs} is now shown to contain an indecomposable direct summand of degree at least 2s. R_{cs} is first evaluated at the special elements α_1 and α_2 by equations (3.7).

$$(3.12)\qquad R_{cs}(\alpha_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ P_{cs} & 0 & 0 \end{pmatrix}, \; R_{cs}(\alpha_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ I & 0 & 0 \end{pmatrix}$$

Let B be in the commuting algebra of R_{cs}, B is broken up into submatrices to correspond to the divisions of R_{cs} given by the solid lines in (3.11).

$$(3.13)\qquad B = \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{pmatrix}$$

B must satisfy the commutator equation,

$$(3.14)\qquad R_{cs}(\alpha)B = BR_{cs}(\alpha),$$

for all α in A. In particular it must satisfy (3.14) for $\alpha = \alpha_2$ and $\alpha = \alpha_1$. Combining (3.12) and (3.14), it follows that

$$(3.15)\qquad B_{11} = B_{33};\ B_{12},\ B_{13},\ B_{23} \text{ are all zero; } B_{11}P_{cs} = P_{cs}B_{33}.$$

By Lemma 2.5.B, matrices commuting with a primary matrix P_{cs} have only one eigenvalue. Since $B_{11} = B_{33}$, $B_{11}P_{cs} = P_{cs}B_{33}$ implies that B_{11} has only one eigenvalue. Considering (3.15) and the form (3.13) of B, it is clear that B must have 2s equal eigenvalues. Lemma 2.5.A implies R_{cs} must have an indecomposable direct summand T_{cs} of degree at least 2s. Since s is an arbitrary integer, A is clearly of unbounded type. This completes the first part of the proof.

It is now shown that A is of strongly unbounded type. For $c \neq d$ let the two representations R_{cs} and R_{ds} be as in the first part of the proof. Let T_{cs} and T_{ds} be the two indecomposable direct summands of degree at least 2s shown above to be in R_{cs} and R_{ds} respectively. It is shown that T_{cs} and T_{ds} cannot be similar.

Suppose they were similar. Let V be the space of R_{cs} and let V_T be the space for the summand T_{cs}. If T_{cs} and T_{ds} were similar, there would exist a matrix P intertwining R_{cs} and R_{ds}, which, when restricted to the space V_T, represents an isomorphism. It is also an isomorphism when restricted to any space contained in V_T.

A particular subspace V_0 is shown to be contained in V_T. Let $R_{cs}(\alpha_2)V = V_0$. Equations (3.12) imply that V_0 has for a basis the last s basis vectors of V, when R_{cs} is in the form indicated by (3.11). Let B_T be the commuting matrix of R_{cs} which is identity on V_T and 0 on its complement in V.

(3.16) $$B_T = \begin{pmatrix} B_{11} & & \\ & B_{22} & \\ * & & B_{11} \end{pmatrix}$$

because B_T is a commuting matrix and must therefore satisfy equations (3.15). By the choice of B_T, B_{11} has unit eigenvalues and is non singular. Hence $B_T V_0 = V_0$. Now apply the commutator equation $R_{cs}(\alpha_2)B_T = B_T R_{cs}(\alpha_2)$ to the whole space V. Clearly,

$$V_T \supseteq R_{cs}(\alpha_2)V_T = R_{cs}(\alpha_2)B_T V = B_T R_{cs}(\alpha_2)V = B_T V_0 = V_0$$

so that V_0 is contained in V_T.

Let P be the intertwining matrix mentioned above, P satisfies the equation

$$(3.17) \qquad PR_{cs}(\alpha) = R_{ds}(\alpha)P.$$

Note that $R_{cs}(\alpha_2) = R_{ds}(\alpha_2)$, so that for $\alpha = \alpha_2$ (3.17) looks like the commutator equation (3.14). Let P be divided up as B is. Then $P_{11} = P_{33}$, and P_{13}, P_{12}, P_{32} are all zero. Now let $\alpha = \alpha_1$ in (3.17), then using equations (3.11), $P_{33}P_{cs} = P_{ds}P_{11}$ or $P_{11}P_{c_s} = P_{ds}P_{11}$ because $P_{11} = P_{33}$. But P cut down to V_0 is represented by P_{11} which must be an isomorphism. So P_{11} is nonsingular. But then $P_{11}P_{cs} = P_{ds}P_{11}$ is impossible because P_{cs} and P_{ds} have unequal eigenvalues c and d respectively. This shows that for $c \neq d$, T_{cs} and T_{ds} cannot be similar. For every c in the infinite field k there is such an indecomposable representation T_{cs} with degree between 2s and the degree of R_{cs}. Thus, there exists an integer d_s between 2s and the degree of R_{cs} such that A has an infinite number of inequivalent indecomposable representations of degree d_s. Hence A is of strongly unbounded type and, by Theorem 2.3.B, so is any algebra with A for a basic algebra. This completes the proof of Theorem 3.2.A.

An interesting consequence of finiteness of the two-sided ideal lattice is given in the following theorem.

<u>Theorem 3.2.B: If A is an algebra over an algebraically closed field, and if L_A its two-sided ideal lattice is finite, then every two-sided ideal is principal.</u>

Proof: If every two-sided ideal in the basic algebra is principal, then every two-sided ideal in the algebra is also principal. This follows from 2.4.C. It therefore is sufficient to consider only basic algebras.

If L_A is finite, 3.1.F and 3.1.G imply that $\phi_{ij}(L_A)$ is a chain for each of the n^2 lattice homomorphisms ϕ_{ij}.

Let A_0 be a two-sided ideal in A. Pick α^1_{ij} in $\phi_{ij}(A_0)$. Let (α^1_{ij}) be the principal two-sided ideal generated by α^1_{ij}. Either $\phi_{ij}((\alpha^1_{ij})) = \phi_{ij}(A_0)$ or $\phi_{ij}((\alpha^1_{ij}))$ is properly contained in $\phi_{ij}(A_0)$. If the latter, pick α^2_{ij} in $\phi_{ij}(A_0)$ but not in $\phi_{ij}((\alpha^2_{ij}))$. Since $\phi_{ij}(L_A)$ is a chain $\phi_{ij}(A_0) \supseteq \phi_{ij}((\alpha^2_{ij})) \supset \phi_{ij}((\alpha^1_{ij}))$. Since e_iAe_j is finite dimensional there exists an element α^0_{ij} such that

$$\phi_{ij}((\alpha^0_{ij})) = \phi_{ij}(A_0). \tag{3.18}$$

For each pair i,j obtain such an element and let α_0 be the sum of all of these. Since each α^0_{ij} is in $\phi_{ij}(A_0) \subseteq A_0$, the principal ideal (α_0) is contained in A_0. The n idempotents e_i are orthogonal and the elements α^0_{ij} were picked in e_iAe_j, so $e_i\alpha_0 e_j = \alpha^0_{ij}$. Hence the principal ideal (α_0) contains each of the principal ideals (α^0_{ij}) and

$$\phi_{ij}((\alpha_0)) \supseteq \phi_{ij}((\alpha^0_{ij})) = \phi_{ij}(A_0) \tag{3.19}$$

by equation (3.18). Summing equation (3.19) for all i,j implies $(\alpha_0) \supseteq A_0$. Hence A_0 is the principal ideal generated by α_0.

The results of this section are referred to throughout the rest of this paper. The purpose of this paper is to classify algebras with respect to the number of inequivalent indecomposable representations; a part of that task is accomplished.

3. Commutative Algebras

The following lemma provides a tool used in the proof of Theorem 3.3.B, the main result of this section.

Lemma 3.3.A: If A is a basic algebra with a finite two-sided ideal lattice L_A, then the subalgebra e_iAe_i is the homomorphic image of a polynomial algebra over k.

Proof: e_iAe_i has e_i for its unity and only idempotent so it can be written

$$e_iAe_i = ke_i + e_iNe_i. \tag{3.20}$$

By 3.1.F and 3.1.G finiteness of the two-sided ideal lattice is equivalent to $\emptyset_{ij}(L_A)$ being a chain for every pair i,j. By the proof of Theorem 3.2.B this implies that there exists an element α_i in e_iNe_i such that the principal ideal $(\alpha_i) = A\alpha_iA$ has the same image under $\emptyset_{ii}$ as does N,

$$e_iNe_i = e_iAe_i\alpha_ie_iAe_i. \tag{3.21}$$

Use equation (3.20) in (3.21) to obtain (3.21').

$$\begin{aligned} e_iNe_i &= k\alpha_i + e_iNe_i\alpha_i + \alpha_ie_iNe_i + e_iNe_i\alpha_ie_iNe_i \\ &= k\alpha_i + (e_iNe_i)^2 \end{aligned} \tag{3.21'}$$

Now use (3.21') recursively,

$$e_iNe_i = k\alpha_i + k\alpha_i^2 + (e_iNe_i)^3. \tag{3.21''}$$

Continue in this manner until $(e_iNe_i)^{t_i} = 0$. Since e_iNe_i

is the radical of e_iAe_i such a t_i exists. Then e_iNe_i can be written

(3.21'') $$e_iNe_i = k\alpha_i + k\alpha_i^2 + \cdots + k\alpha_i^{t_i-1} .$$

Clearly e_iNe_i has a basis of powers of a single element α_i. Equation (3.20) then becomes

(3.20') $$e_iAe_i = ke_i + k\alpha_i + k\alpha_i^2 + \cdots + k\alpha_i^{t_i-1} .$$

e_i is the unity element of e_iAe_i and powers of α_i commute. Let $k[x]$ be the polynomial algebra over k in an indeterminant x. Let $\theta(x^t) = \alpha_i^t$, $\theta(1) = e_i$ for 1 in k and extend linearly. θ is a ring homomorphism of $k[x]$ onto e_iAe_i with (x^{t_i}) for a kernel. This completes the proof of the lemma.

The main theorem of this section can now be proved.

<u>Theorem 3.3.B</u>: <u>If A is a commutative algebra, A is of strongly unbounded type if and only if L_A is infinite. If L_A is finite A is of finite representation type</u>.

Proof: Theorem 3.2.A establishes that if L_A is infinite then A is of strongly unbounded type.

If A is commutative, it is its own basic algebra, for any subalgebra of A isomorphic to a total matrix algebra must already be of degree one. Let e_i and e_j be distinct orthogonal idempotents, then $e_iAe_j = e_ie_jA = 0$. Hence A can be written

(3.22) $$A = \dot{+}\sum_{i=1}^{n} e_iAe_i ,$$

where the sum is direct in the ring sense, and the e_iAe_i are two-sided ideals here. If R is an indecomposable representation of A, then the kernel of R contains all the e_iAe_i but one. For if not, R could be decomposed into a direct sum.

Assume L_A is finite then by Lemma 3.3.A each e_iAe_i is the homomorphic image of a polynomial algebra and the indecomposable representation R can be considered as a representation of $k[x]$. Such representations are studied in elementary matrix theory. R(x) has a characteristic function f(x) which must equal m(x) the minimum function of the representation of x if R is to be indecomposable. But $R(x^{t_i}) = 0$ so m(x) must divide x^{t_i}, and the degree of f(x) equals the degree of R, hence the degree of $R \leq t_i$. A is therefore bounded type.

Two such indecomposable representations of $k[x]$ are known to be similar if and only if they have the same minimum function. But x^{t_i} has only t_i distinct nontrivial divisors so e_iAe_i has only t_i distinct indecomposable representations. Hence A had only $\sum_{i=1}^{n} t_i$ non equivalent indecomposable representations. A is therefore of finite representation type.

Contained in the above proof is the following structure theorem for commutative algebras.

<u>Corollary 3.3.C</u>: <u>A commutative algebra over an algebraically closed field k is the direct sum of polynomial algebras over k if and only if it has a finite two-sided ideal lattice</u>.

Proof: If A has an infinite two-sided ideal lattice, it is of strongly unbounded type. But polynomial algebras and direct sums of them are of finite type.

4. Other Consequences of a Finite Two-Sided Ideal Lattice

A fruitful approach to the representation theory for algebras is to consider subalgebras generated by one or a few elements. The commutator algebra of a representation of the subalgebra is given by the set of matrices which commute with each of the generators of the subalgebra. This is because a matrix commuting with two matrices also commutes with all linear combinations of products of the two, that is, with the subalgebra they generate. In the previous section, it was the fact that e_iNe_i was a subalgebra generated by a single element that lead to the proof of Theorem 3.3.B.

The following two lemmas show that finiteness of the two-sided ideal lattice allows the choice of a certain few elements of the basic algebra A which generate A in the subalgebra sense. Then to find the commutator algebra of a representation of A, it is necessary only to look at the commutator algebra of the images of these few elements.

Lemma 3.4.A: If A is a basic algebra and if L_A is finite then there exists α_0 in N the radical of A such that the subalgebra generated by the idempotents $e_1, \ldots e_n$ and α_0 is all of A.

Proof: Let α_i be chosen as in Lemma 3.3.A such that powers of α_i generate e_iNe_i. Since N is a two-sided ideal in A, there exists α_{ij} in e_iNe_j for which $\emptyset_{ij}((\alpha_{ij}))$ equals e_iNe_j. α_{ij} is chosen as in Theorem 3.2.B. Then

$$(3.23) \qquad e_iNe_j = e_iAe_i(\alpha_{ij})e_jAe_j.$$

But by the choice of the α_i, e_iAe_i has a basis consisting of e_i and powers of α_i, and similarly for e_jAe_j. Then

(3.23) can be written

(3.23') $$e_iNe_j = \sum_{t,r} k\alpha_i{}^t \alpha_{ij} \alpha_j{}^r, \quad \alpha_i{}^o = e_i$$

where the sum on the right is not necessarily direct.

Let $\alpha_o = \sum_{i \neq j}^{n} \alpha_{ij} + \sum_{i=1}^{n} \alpha_i$ and let S be the subalgebra generated by α_o and the n orthogonal idempotents e_i. Orthogonality of the idempotents implies $e_i\alpha_o e_j = \alpha_{ij}$ if $i \neq j$ or $e_i\alpha_o e_i = \alpha_i$. Hence α_{ij} and α_i are in S. By the choice of α_i, e_iAe_i is in S and by (3.23') e_iNe_j which equals e_iAe_j for $i \neq j$ is also in S. But the sum of all of these is A so S = A.

A further refinement of this is given in the following lemma.

<u>Lemma 3.4.B: If L_A is finite then there exist two elements α_o, α_1 such that the subalgebra generated by them is all of A</u>.

Proof: Let α_o be chosen as in the previous lemma. Let $c_1, \ldots, c_n$ be distinct elements of the field k, and let $\alpha_1 = \sum_{i=1}^{n} c_ie_i$. Let f(x) be a polynomial in an indeterminant x with coefficients in the field k. Since the idempotents are orthogonal

$$f(\alpha_1) = \sum_{i=1}^{n} f(c_i)e_i.$$

Then let $$f_i(x) = \prod_{\mu \neq i}^{n} \frac{(x - c_\mu)}{(c_i - c_\mu)}.$$

It follows that $f_i(\alpha_1) = e_i$. So the subalgebra generated by α_1 contains each of the idempotents e_i. Lemma 3.4.A then gives the result.

This chapter ends with a lemma which indicates that infiniteness of the two-sided ideal lattice is a generalization, in the case the field is algebraically closed, of the hypothesis of Theorem 1.2.C.

Lemma 3.4.C: If A has a representation R

$$R = \begin{pmatrix} F_j & & & \\ P & Q & & \\ Y_1 & S_1 & F_i & \\ Y_2 & S_2 & 0 & F_i \end{pmatrix}$$

a top constituent of a representation induced by a left ideal Ae where e is a primitive idempotent and Y_1 has power s_1, Y_2 has power $s_2, s_2 \geq s_1$ and Q has no F_j appearing in the top $s_2 - s_1$ Loewy constituents of the upper Loewy series for Q, then A has an infinite two-sided ideal lattice.

Proof: Starting with the primitive idempotent e it is possible to pick a set of matrix units for a semi-simple subalgebra A' of A such that $C_{j11} = e$. Going over to the basic algebra $\hat{A}$ of A, $e = e_j$. The representation of A induced by Ae goes over into a representation R' of $\hat{A}$ induced by $\hat{1}Ae_j$. The top constituent R' corresponding to R can be taken to be

$$(3.24) \qquad R' = \begin{pmatrix} x_j & & & \\ P' & Q' & & \\ Y_1' & S_1' & x_j & \\ Y_2' & S_2' & 0 & x_j \end{pmatrix}$$

R has composition factors that induce irreducible representations F_t, and R' has corresponding composition factors x_t. That no F_j appear in the top s_2-s_1 constituents of the upper Loewy series for Q means that no x_j appears in the top s_2-s_1 constituents of the corresponding series for Q'. Then $e_j\hat{N}e_j = e_j\hat{N}^{s_2-s_1+1}e_j$, for if

$$e_j\hat{N}e_j/e_j\hat{N}^{s_2-s_1+1}e_j$$

is not zero there would exist an x_j in the top s_2-s_1 Loewy blocks of Q'. By the definition of "power" Y_1' having power s_1 means $Y_1'(\alpha) = 0$ for all α in $\hat{N}^{s_1}$ but there exists α_1 in $e_1\hat{N}^{s_1-1}e_j$ such that $Y_1'(\alpha_1) \neq 0$. Since R' was induced by $\hat{A}e_j$, the first column of R' is completely independent. That is, α_1 can be chosen so that $Y_2'(\alpha_1)$, $P'(\alpha_1)$, $x_j(\alpha_1)$ are all zero. Similarly Y_2' has power s_2 so α_2 can be chosen in $e_1\hat{N}^{s_2-1}e_j$ such that $Y_2'(\alpha_2) \neq 0$ but $x_j(\alpha_2)$, $P'(\alpha_2)$, $Y_1'(\alpha_2)$ are all zero.

Define $\hat{A}_1 = \hat{A}\alpha_1\hat{A}$, $\hat{A}_2 = k\alpha_2 + \hat{N}^{s_2}$. By the choice of α_2, $\hat{A}_2$ is a two-sided ideal. $\hat{A}_1$ obviously is too. $Y_1'(\alpha) = 0$ for all α in $\hat{A}_2$ because $Y_1'(\alpha_2) = 0$ and $Y_1'(\alpha) = 0$ for all α in $\hat{N}^{s_1} \supseteq N^{s_2}$. But $Y_1'(\alpha_1) \neq 0$. Hence

$$(3.25) \qquad \phi_{1j}(\hat{A}_2) \not\supseteq \phi_{1j}(\hat{A}_1),$$

for if $\phi_{1j}(\hat{A}_2)$ contained $\phi_{1j}(\hat{A}_1)$, $Y_1'(\alpha_1)$ would have to be zero.

Form $\emptyset_{ij}(A_1) = e_i\hat{A}e_i\alpha_1 e_j\hat{A}e_j$. By the remark above, $e_j\hat{A}e_j = ke_j + e_j\hat{N}^{s_2-s_1+1}e_j$. Use this in the above equation

(3.26) $\emptyset_{ij}(\hat{A}_1) = e_i\hat{A}\alpha_1 + e_i\hat{A}e_i\alpha_1 e_j\hat{N}^{s_2-s_1+1}e_j$.

The right hand summand is in $\hat{N}^{s_2}$ because α_1 is in $\hat{N}^{s_1-1}$, so $Y\tfrac{1}{2}(\alpha) = 0$ for α in the right hand summand.

$$(3.27) \qquad R'(\alpha_1) = \begin{pmatrix} 0 & & \\ \cdot & & \\ \cdot & & \\ \cdot & * & \\ 0 & & \\ 1 & & \\ 0 & & \end{pmatrix}$$

By looking at R' in (3.24) it is clear that

$$(3.28) \qquad R'(\alpha)R'(\alpha_1) = \begin{pmatrix} * & & & \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ * & & & \\ 0 & * & \ldots & * \end{pmatrix}$$

for all α in $\hat{A}$. Therefore $Y\tfrac{1}{2}(\alpha) = 0$ for all elements α in $e_i\hat{A}\alpha_1$ the left summand of (3.26). But then $Y\tfrac{1}{2}(\alpha) = 0$ for all α in $\emptyset_{ij}(\hat{A}_1)$ and $Y\tfrac{1}{2}(\alpha_2) \neq 0$. Hence

$$(3.29) \qquad \emptyset_{ij}(\hat{A}_1) \not\supseteq \emptyset_{ij}(\hat{A}_2),$$

for if $\emptyset_{ij}(\hat{A}_1)$ contained $\emptyset_{ij}(\hat{A}_2)$, $Y\tfrac{1}{2}(\alpha_2)$ would also be zero. Expressions (3.25) and (3.29) imply that the image of L_A under $\emptyset_{ij}$ is not a chain. Then 3.1.F and 3.1.G imply that the basic algebra $\hat{A}$ has an infinite two-sided ideal lattice $L_{\hat{A}}$. Theorem 2.4.B implies that A also has an infinite two-sided ideal lattice.

CHAPTER IV

1. Left and Right Ideals

Although infiniteness of the two-sided ideal lattice is a sufficient condition for an algebra to be of strongly unbounded type, it is not, in general, a necessary one. In this chapter another condition is given which implies that an algebra is of strongly unbounded type. The condition given in this chapter is a generalization of Brauer's second condition stated in Chapter I. As before, the underlying field k of the algebra A is assumed to be algebraically closed. It is then possible to center attention on basic algebras, and the results in this chapter are stated in terms of basic algebras.

In this and the following chapter, consideration is restricted to ideals in the radical N of A. Designate by L_N the lattice of two-sided ideals of A that are contained in N.

Definition 4.1.A: For a two-sided ideal A_o in the radical N of a basic algebra A, let $\emptyset_i(A_o) = A_o e_i = A_o \cap Ne_i$. (Let ${}_i\emptyset(A_o) = e_i A_o = A_o \cap e_i N$.)

Lemma 4.1.B: $\emptyset_i$ (${}_i\emptyset$) is a lattice homomorphism of L_N into the lattice of left (right) ideals in Ne_i ($e_i N$).

Proof: $(A_1 + A_2)e_i = A_1 e_i + A_2 e_i$ and $A_1 \cap A_2 \cap Ne_i = (A_1 \cap Ne_i) \cap (A_2 \cap Ne_i)$, so $\emptyset_i$ preserves both $+$ and $\cap$. (Similarly for ${}_i\emptyset$.)

Theorem 4.1.C: If, for any i and any two-sided ideal A_o in N, $\emptyset_i(A_o)$ $[{}_i\emptyset(A_o)]$ has more than three covers in $\emptyset_i(L_N)$ $[{}_i\emptyset(L_N)]$, then A is of strongly unbounded type.

Proof: The proof given here is in terms of the lattice homomorphism $\emptyset_i$. A similar proof holds for ${}_i\emptyset$.

Let $\emptyset_i(A_t)$ cover $\emptyset_i(A_o)$ in Ne_i for t = 1,2,3,4. Assume L_N is distributive, for if not, A is already of strongly unbounded type. The sublattice L_o of $\emptyset_i(L_N)$ generated by these elements is complemented (every element in it is a unique sum of covers of $\emptyset_i(A_o)$), so L_o is a Boolean algebra.

Let $A_t' = \sum_{t \neq m} A_m$; $\emptyset_i(A_t') = \sum_{t \neq m} \emptyset_i(A_m)$ is the complement of $\emptyset_i(A_t)$ in L_o. Let α_t' be picked in $\emptyset_i(A_t)$ but not in $\emptyset_i(A_o)$. If $e_j\alpha_t'$ is in $\emptyset_i(A_o)$ for all j then $\alpha_t' = \sum_{j=1}^{n} e_j\alpha_t'$ is there also. So there exists $e_{i(t)}$ such that $e_{i(t)}\alpha_t' = \alpha_t$ is in $\emptyset_i(A_t)$ but not in $\emptyset_i(A_o)$. Also $e_{i(t)}\alpha_t = \alpha_t$ because $e_{i(t)}$ is an idempotent. By the choice of α_t and the definition of A_t', α_t is not in A_t', but α_t is in A_m' for $m \neq t$.

Let R_t be a representation with kernel A_t'. By the choice of the special elements α_m with respect to the kernels A_t' it is clear that $R_m(\alpha_t) = 0$ for $t \neq m$. Moreover, for each $t, R_t(\alpha_t) \neq 0$.

From R_t induce a representation $\bar{R}_t$, where $\bar{R}_t(\alpha_t)$ is non-zero only in the lower left hand corner. Since each α_t is in the radical this can always be done. Multiply α_t by a suitable scalar in k to obtain α_t', so that the non-zero entry in $\bar{R}_t(\alpha_t')$ is 1 in k.

By the choice of α_t' in $e_{i(t)}Ne_i$, each $\overline{R}_t$ is in the form

$$(4.1) \qquad \overline{R}_t(\alpha) = \begin{pmatrix} x_i(\alpha) & & \\ P_t(\alpha) & Q_t(\alpha) & \\ Y_t(\alpha) & S_t(\alpha) & x_{i(t)}(\alpha) \end{pmatrix} .$$

The special elements α_m' were selected in such a way that

$$(4.2) \qquad \begin{array}{l} P_t(\alpha_m'),\ Q_t(\alpha_m'),\ S_t(\alpha_m'),\ x_i(\alpha_m'),\ x_{it}(\alpha_m') \text{ are all zero} \\ \text{for all } m,t, \text{ but } Y_t(\alpha_m') = \delta_{tm}. \end{array}$$

Now, form the representation R which is seen to be similar to the direct sum $\dot{+}\sum_{m=1}^{4} \overline{R}_m$.

$$R = \left(\begin{array}{ccc|ccccccccc}
x_i & & & & & & & & & & & \\
 & x_i & & & & & & & & & & \\
 & & x_i & & & & & & & & & \\
\hline
 & & & x_i & & & & & & & & \\
P_1 & & & P_1 & Q_1 & & & & & & & \\
 & P_2 & & P_2 & & Q_2 & & & & & & \\
 & & P_3 & P_3 & & & Q_3 & & & & & \\
 & & & P_4 & & & & Q_4 & & & & \\
Y_1 & & & Y_1 & S_1 & & & & x_{i_1} & & & \\
 & Y_2 & & Y_2 & & S_2 & & & & x_{i_2} & & \\
 & & Y_3 & Y_3 & & & S_3 & & & & x_{i_3} & \\
 & & & Y_4 & & & & S_4 & & & & x_{i_4}
\end{array}\right)$$

Induce the representation R' below and to the right of the dotted lines. Let s be an integer, c be an element of k. Let I_{2s} be the unit matrix of degree 2s, I be the unit matrix of degree s. Denote by P_{cs} a primary matrix of degree s with the single repeated eigenvalue c. Form from these matrices the matrix D_{cs}.

$$D_{cs} = \begin{pmatrix} I_{2s} \\ (I,0) & I \\ (0,I) & & I \\ (I,I) & & & I \\ (I,P_{cs}) & & & & I \\ (I,0) & I & & & & I \\ (0,I) & & I & & & & I \\ (I,I) & & & I & & & & I \\ (I,P_{cs}) & & & & I & & & & I \end{pmatrix}$$

This is seen to satisfy the hypothesis of Lemma 2.5.C.

Then, using the construction given in that lemma, form the representation R_{cs} from the representation R' and the matrix D_{cs}

$$(4.3)\quad R_{cs} = \left(\begin{array}{c|cccccccc} I_{2s} \times x_i \\ \hline (I,0)\times P_1 & I\times Q_1 \\ (0,I)\times P_2 & & I\times Q_2 \\ (I,I)\times P_3 & & & I\times Q_3 \\ (I,P_{cs})\times P_4 & & & & I\times Q_4 \\ (I,0)\times Y_1 & I\times S_1 & & & & I\times x_{i_1} \\ (0,I)\times Y_2 & & I\times S_2 & & & & I\times x_{i_2} \\ (I,I)\times Y_3 & & & I\times S_3 & & & & I\times x_{i_3} \\ (I,P_{cs})\times Y_4 & & & & I\times S_4 & & & & I\times x_{i_4} \end{array}\right)$$

It is now shown that every commuting matrix B of R_{cs} has at least 6s equal eigenvalues. Let B be divided up as follows.

$$(4.4) \qquad B = \left(\begin{array}{cc|cccc} B_{11} & B_{12} & B_{13} & \cdots & & B_{110} \\ B_{21} & B_{22} & B_{23} & \cdots & & B_{210} \\ \hline B_{31} & B_{32} & B_{33} & \cdots & & B_{310} \\ \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & \cdot & & & \cdot \\ B_{101} & B_{102} & B_{103} & \cdots & & B_{1010} \end{array}\right)$$

where $\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$ corresponds to $I_{2s} \times x_i$, $(B_{31}\ B_{32})$ corresponds to $(I,0) \times P_1$ etc., but below and to the left of the dotted lines the $B_{\mu\nu}$ correspond to the similarly placed entries below and to the left of the dotted lines in R_{cs}. If B is to be a commuting matrix B must satisfy

$$(4.5) \qquad R_{cs}(\alpha)B = BR_{cs}(\alpha)$$

for all α in A. Consider equation (4.5) when $\alpha = \alpha_1^{i}$. Using equations (4.1) and (4.2) to evaluate (4.3) for $\alpha = \alpha_1^{i}$ and substituting in (4.5), it is shown that

(4.6) $\quad B_{11} = B_{77}$ and $B_{1\nu}$, $B_{\mu 7}$ are zero for $\nu \neq 1$, $\mu \neq 7$.

Now let $\alpha = \alpha_2^{i}$ and again evaluate $R_{cs}(\alpha_2^{i})$ by means of relations (4.2). Equation (4.5) then implies

(4.7) $\quad B_{22} = B_{88}$ and $B_{2\nu}$, $B_{\mu 8}$ are zero for $\nu \neq 2$, $\mu \neq 8$.

Let $\alpha = \alpha_3^{i}$, evaluate (4.3) and substitute in (4.5). By equations (4.6) and (4.7) the only non-zero entries in the first two rows of B are B_{11} and B_{22}. It then follows that

(4.8) $\quad B_{11} = B_{99}$, $B_{22} = B_{99}$ and $B_{\mu 9}$ are zero for $\mu \neq 9$.

Finally let $\alpha = \alpha_4$ and consider (4.5) again to obtain

(4.9) $B_{11} = B_{1010}$, $P_{cs}B_{22} = B_{1010}P_{cs}$ and $B_{\mu 10}$ are zero for $\mu \neq 10$.

Consolidating equations (4.6) through (4.9), B must have the form

$$(4.10)\quad B = \begin{pmatrix} B_{11} & 0 & \cdots & & & & & 0 \\ 0 & B_{11} & 0 & \cdots & \cdot & & & 0 \\ * & & & * & \vdots & & & \\ & & & & 0 & \ddots & & \\ & & & & B_{11} & 0 & \ddots & \\ & & & & 0 & B_{11} & 0 & \vdots \\ * & & & * & 0 & 0 & B_{11} & 0 \\ & & & & 0 & 0 & 0 & B_{11} \end{pmatrix}$$

and equation (4.9) also implies B_{11} has a single eigenvalue. Thus B must have 6s equal eigenvalues and by Lemma 2.5.A, R_{cs} must have a direct summand T_{cs} of degree at least 6s. A is of unbounded type and so is any algebra that has A for a basic algebra.

Continuing with the proof according to the scheme given in Chapter III, it is now shown that two such indecomposable direct summands T_{cs} and T_{ds} cannot be equivalent for $d \neq c$.

Let V be the space of R_{cs} and let V_T be the direct summand corresponding to T_{cs}. Let $R_{cs}(\alpha_4)V$ be defined to be V_o. From the form of $R_{cs}(\alpha_4)$, V_o is an A subspace of V, and V_o has for a basis the last s basis vectors of V when R_{cs} is in the form (4.3). It must be shown that V_o is a subspace of V_T.

Let B_T be the matrix of the linear transformation of V which is identity on V_T and zero on its complement in V, $B_T V = V_T$. B_T commutes with $R_{cs}(\alpha)$ for all α. Then B_T has the form (4.10) and the part of B_T corresponding to B_{11} in (4.10) has unit eigenvalues. This means B_T cut down to V_o is an isomorphism of V_o onto itself, $B_T V_o = V_o$. Now apply the commutator equation $B_T R_{cs}(\alpha_4) = R_{cs}(\alpha_4) B_T$ to V itself,

$$V_T \geq R_{cs}(\alpha_4) V_T = R_{cs}(\alpha_4) B_T V = B_T R_{cs}(\alpha_4) V = B_T V_o = V_o,$$

hence $V_T \geq V_o$.

Using this subspace V_o, it is shown that T_{cs} and T_{ds} cannot be equivalent. Suppose T_{cs} and T_{ds} were similar. Then there would exist a matrix P intertwining R_{cs} and R_{ds} which, when cut down to V_T, is an isomorphism. P is also an isomorphism when cut down to the subspace V_o in V_T. P satisfies the intertwining equation

(4.11) $$PR_{cs}(\alpha) = R_{ds}(\alpha)P$$

for all α in A. Using equations (4.1) and (4.2) in (4.3), it is clear that $R_{cs}(\alpha_t) = R_{ds}(\alpha_t)$ for t = 1,2,3. Hence for $\alpha = \alpha_t$, t = 1,2,3, equation (4.11) is identical with equation (4.5) where P replaces B. Let P be divided up according to the same scheme as B. Then equations (4.6), (4.7), (4.8) hold for $P_{\mu\nu}$ replacing $B_{\mu\nu}$. From these it follows that

(4.12) $$P_{11} = P_{22} = P_{77} = P_{88} = P_{99}.$$

Finally, let $\alpha = \alpha_4$ in (4.11) and using (4.1) and (4.2) it is clear that $P_{\mu 10}$ are zero for $\mu \neq 10$ and

(4.13) $$P_{11} = P_{1010}, \quad P_{11} P_{cs} = P_{ds} P_{11}.$$

It follows that P has the form of B in (4.10). But P cut down to V_0 is

$$\begin{pmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ P_{11} \end{pmatrix}$$ which must be an isomorphism so P_{11} is non-

singular. However, the equation $P_{11} P_{cs} = P_{ds} P_{11}$ is seen to be impossible, for P_{cs} and P_{ds} have distinct eigenvalues c and d respectively. Hence for $c \neq d$, T_{cs} cannot be similar to T_{ds}.

Since the field is infinite there exist an infinite number of inequivalent indecomposable representations with degrees between 6s and degree R_{cs}. It follows that A is of strongly unbounded type. By the development in Chapter II, any algebra which has A for a basic algebra is also of strongly unbounded type.

The following is a corollary to Theorem 4.1.C.

Corollary 4.1.D: If $\phi_1(L_N)$ $_1\phi(L_N)$ contains a sublattice which is a Boolean algebra with more than 2^3 elements, then A is of strongly unbounded type.

Proof: The Stone Representation Theorem as given in Birkhoff [2] implies that Boolean algebras are fields of subsets of a certain set. Thus, the orders of finite Boolean algebras are 2^n, n an integer. If the hypothesis of this corollary holds, then $\phi_1(L_N)$ $_1\phi(L_N)$ has a sublattice which is a Boolean algebra with 2^4 elements. The smallest element in that sublattice has four covers in $\emptyset_1(L_N)$, so by Theorem 4.1.C, A is of strongly unbounded bype and so is any algebra which has A for a basic algebra.

Still another statement of this type of condition is given in terms of the graphs developed in Chapter V.

CHAPTER V

1. Graphs

In this chapter two additional sufficient conditions are given which imply that an algebra A is of strongly unbounded type. These conditions are given in terms of a graph which is defined in the following development. Throughout this chapter only basic algebras are considered, although Lemma 5.1.A is true in general. Let A be a basic algebra and let A_1 and A_2 be two two-sided ideals of A. Let $\emptyset_{ij}$ and L_A be defined as in Chapter III.

Lemma 5.1.A: If A_1 covers A_2, then $NA_1 \subseteq A_2$, $A_1N \subseteq A_2$.

Proof: NA_1 is a two-sided ideal contained in A_1. Then $NA_1 + A_2 \subseteq A_1$. Suppose equality holds in the previous expression. $N(NA_1 + A_2) = NA_1$ that is, $N^2A_1 + NA_2 + A_2 = A_1$ so $N^2A_1 + A_2 = A_1$. Continue in this manner, $N^rA_1 + A_2 = A_1$ implies $N^{2r}A_1 + A_2 = A_1$. But for some integer r_o, $N^{r_o} = 0$. This would imply $A_2 = A_1$ contrary to the initial assumption. Hence $NA_1 + A_2 \subset A_1$, proper inclusion. Therefore, $NA_1 \subseteq A_2$ because A_1 covers A_2. The same argument with right multiplication by N replacing left shows $A_1N \subseteq A_2$.

Lemma 5.1.B: If A_1 covers A_2 then there exists exactly one pair i,j such that $\emptyset_{ij}(A_1)$ covers $\emptyset_{ij}(A_2)$.

Proof: There exists at least one pair i,j for which $\emptyset_{ij}(A_1)$ properly contains $\emptyset_{ij}(A_2)$. For if $\emptyset_{ij}(A_1)$ equals

$\phi_{ij}(A_2)$ for all i,j, the sums over all i,j of these are also equal, so $A_1 = A_2$.

Let α_{ij} be chosen in $\phi_{ij}(A_1)$ not in $\phi_{ij}(A_2)$. Let A_3 be $k\alpha_{ij} + A_2$. Clearly $e_tA_3 \subseteq A_3$ and $A_3e_t \subseteq A_3$ for all idempotents e_t in A. By Lemma 5.1.A $N\alpha_{ij} \subseteq A_2$ and $\alpha_{ij}N \subseteq A_2$ because α_{ij} was chosen in A_1, a cover of A_2. Hence A_3 is a two-sided ideal covering A_2 and contained in A_1. The hypothesis implies that $A_3 = A_1$. Clearly $\phi_{ij}(A_1)$ covers $\phi_{ij}(A_2)$, since their quotient is one dimensional. Also from the form of $A_3 = A_1$, it is clear that $\phi_{pr}(A_1) = \phi_{pr}(A_2)$ for all pairs p,r not equal to i,j. This completes the proof of the lemma.

Throughout the remainder of this chapter, only basic algebras with a finite two-sided ideal lattice L_A will be considered. According to 3.1.F and 3.1.G, L_A is distributive and $\phi_{ij}(L_A)$ is a chain for every pair i,j. Also by 3.2.B, every two-sided ideal in A is principal.

Of primary importance in this chapter is the sublattice L_N of L_A consisting of two-sided ideals of A which are contained in the radical N. $\phi_{ij}(L_N)$ is also a chain for each pair i,j.

Let A_o be a two-sided ideal in L_N and let $A_1, \ldots, A_q$ be all the covers of A_o in L_N. Corresponding to the two-sided ideal A_o, construct the oriented graph $G(A_o)$ as follows. Let $P_1, \ldots, P_n$ be n vertices and let $P_i \mathcal{L} P_j$, a binary relation, hold if for some cover A_p of A_o, $\phi_{ij}(A_p)$ covers $\phi_{ij}(A_o)$. Recall the definitions of the terms used in describing graphs in Chapter I. Lemma 5.1.B insures that there exists exactly one edge for each cover of A_o.

For each edge in the graph $G(A_o)$ a special element in the algebra can be selected and a special representation constructed, the properties of which are described in the following lemma.

Lemma 5.1.C: For the graph $G(A_o)$, there exist special elements $\alpha_1, \ldots, \alpha_q$, in A, and representations $R_1, \ldots, R_q$, of A, one special element and one representation for each edge in the graph such that:

a. If $P_i \mathcal{L} P_j$ represents the pth edge in $G(A_o)$ then

$$R_p(\alpha) = \begin{pmatrix} x_j & & \\ P_p & Q_p & \\ y_p & S_p & x_i \end{pmatrix}$$ is the representation R_p.

b. The special element α_p for that edge is chosen in $e_i N e_j$.

c. $R_p(\alpha_r) = 0$ if $r \neq p$, $R_p(\alpha_p)$ has only a 1 in the lower left corner, the rest is zero.

Proof: L_N, the sublattice of a distributive lattice, is distributive so that the covers of A_o generate a sublattice L that is a Boolean algebra. The complement A_t' of A_t in L is given by $\sum_{r \neq t} A_r$.

Let α_t' be picked in $\phi_{ij}(A_t)$ not in $\phi_{ij}(A_o)$. By the proof of Lemma 5.1.B, $A_t = k\alpha_t' + A_o$. Clearly α_t' is not in A_t' the complement of A_t in L, for if it were then A_t would be in A_t'.

Let R_t' be a representation with kernel A_t'. $R_t'(\alpha_t') \neq 0$ because α_t' is not in A_t', but α_r', for $r \neq t$ is in A_r hence is in A_t' so $R_t(\alpha_r') = 0$.

As in Chapters III and IV induce a representation R_t out of R_t' such that $R_t(\alpha_t')$ is zero except in the lower left corner, the entry there being b_t in k. Note that $R_t(\alpha_r)$ for $r \neq t$, being part of $R_t'(\alpha_r)$ is still zero for $r \neq t$. Now let α_t be $1/b_t \alpha_t'$. Clearly, the R_t and α_t thus defined satisfy the conclusions of the lemma.

A fact concerning cycles which is used later in this chapter is given in the following lemma.

Lemma 5.1.D: If the graph $G(A_0)$ has a chain C with a repeated edge then $G(A_0)$ contains a cycle.

Proof: If $P_i \mathcal{L} P_j$ appears in a chain, P_i must appear on one side of it and P_j on the other. There are two cases to consider depending on whether P_i is on the same side of $P_i \mathcal{L} P_j$ each time $P_i \mathcal{L} P_j$ appears or not.

If the first case the chain C is

$$\ldots, P_i, P_i \mathcal{L} P_j, P_j, P_k \mathcal{L} P_j, \ldots, P_i, P_i \mathcal{L} P_j, P_j, \ldots$$

Let C_0 be the chain obtained from C by cutting off C before the first P_j and after the second P_j.

Examine C_0 followed by C_0 with the P_j's identified. It is clear that the orientation of successive edges alternates. That the edges $P_i \mathcal{L} P_j$ and $P_k \mathcal{L} P_j$ are distinct is true because they appear in succession in the chain C. Hence C_0 is a cycle.

In the second case the chain C is

$$\ldots, P_i, P_i \mathcal{L} P_j, P_j, P_k \mathcal{L} P_j, P_k, \ldots, P_r, P_r \mathcal{L} P_j, P_j, P_i \mathcal{L} P_j, P_i, \ldots$$

Let $P_i \mathcal{L} P_j$ be the repeated edges closest together, then $P_k \mathcal{L} P_j$ can be assumed not equal to $P_r \mathcal{L} P_j$. Let C_0 be the chain C cut off before the first P_j and after the second P_j.

Examine C_o followed by C_o with the P_j's identified. The orientation of successive edges alternates and, by the remark above, the successive edges $P_k \mathcal{L} P_j$ and $P_r \mathcal{L} P_j$ are distinct. C_o is therefore a cycle. This completes the proof of 5.1.D.

2. Graph with Cycle

The following theorem gives a third sufficient condition that a basic algebra be of strongly unbounded type. This condition is described in terms of the graphs investigated in the previous section.

Theorem 5.2.A: If the graph $G(A_o)$ associated with any two-sided ideal $A_o \subset N$ has a cycle then A is of strongly unbounded type.

Proof: Let C be a cycle. C equals

$$P_{i_1},\ P_{i_1} \mathcal{L} P_{j_1},\ P_{j_1},\ P_{i_2} \mathcal{L} P_{j_1}, \ldots, P_{j_r},\ P_{i_1} \mathcal{L} P_{j_r},\ P_{i_1}.$$

The proof of Lemma 5.1.D insures that all the edges may be taken to be distinct. Let $R_{11}, R_{21}, \ldots, R_{rr}, R_{1r}$ be the representations associated with the 2r distinct edges of C by Lemma 5.1.C.

$$(5.1) \qquad R_{\mu\nu}(\alpha) = \begin{pmatrix} x_{j_\nu} & & \\ P_{\mu\nu} & Q_{\mu\nu} & \\ y_{\mu\nu} & S_{\mu\nu} & x_{i_\mu} \end{pmatrix} \qquad (\mu,\nu) = (1,1),\ (2,1), \ldots (1,r)$$

From the submatrices of these $R_{\mu\nu}$ construct a matrix function R_{cs} of A,

$$(5.2) \qquad R_{cs} = \begin{pmatrix} X_T & & \\ P & Q & \\ Y & S & X_B \end{pmatrix}$$

as follows.

Let $X_T(\alpha)$ be the direct sum of $I_s \times x_{j_\nu}(\alpha)$ for $\nu = 1,\ldots,r$ and let $X_B(\alpha)$ be the direct sum of $I_s \times x_{i_\mu}(\alpha)$ for $\mu = 1,\ldots r$, where $\times$ means Kronecker product and I_s is an identity matrix of degree s. Let $Q(\alpha)$ be the direct sum of $I_s \times Q_{\mu\nu}(\alpha)$ for the 2r Q's. By Lemma 2.5.C, $X_T(\alpha)$, $X_B(\alpha)$ and $Q(\alpha)$ are all representations of A.

Let $P(\alpha)$ have $I_s \times P_{\mu\nu}(\alpha)$ directly below $I_s \times x_{j_\nu}(\alpha)$ in X_T and directly to the left of $I_s \times Q_{\mu\nu}(\alpha)$ in Q for (μ,ν) = (1,1), ... , (r,r). Directly below $I_s \times x_{j_r}$ and directly to the left of $I_s \times Q_{1r}$ put $P_{cs} \times P_{1r}(\alpha)$ where P_{cs} is the primary matrix with eigenvalue c. Fill out the rest of P with zeros.

Let $S(\alpha)$ have $I_s \times S_{\mu\nu}(\alpha)$ directly below $I_s \times Q_{\mu\nu}$ in Q and directly to the left of $I_s \times x_{i_\mu}$ in X_B for (μ,ν) = (1,1), (2,1), ... , (r,r), (1,r). Fill out the rest of $S(\alpha)$ with zeros.

Let $Y(\alpha)$ have $I_s \times y_{\mu\nu}(\alpha)$ directly below $I_s \times x_{j_\nu}$ in X_T and directly to the left of $I_s \times x_{i_\mu}$ in X_B for (μ,ν) = (1,1), (2,1), ... , (r,r). Directly below $I_s \times x_{j_r}$ and directly to the left of $I_s \times x_{i_1}$ put $P_{cs} \times y_{1r}(\alpha)$. Fill out the rest of $Y(\alpha)$ with zeros.

The form of the block $Y(\alpha)$ will play an important part in the proof of the theorem.

$$(5.3) \qquad Y(\alpha) = \begin{pmatrix} I_s \times y_{11} & & & & P_{cs} \times y_{1r} \\ I_s \times y_{21} & I_s \times y_{22} & & & \\ & \cdot & \cdot & & \\ & & \cdot & I_s \times y_{r-1\,r-1} & \\ & & & I_s \times y_{r\;r-1} & I_s \times y_{rr} \end{pmatrix}$$

It is now shown that R_{cs} is a representation of A with an indecomposable direct summand of degree at least 2rs. It must first be shown that R_{cs} is indeed a representation of A. It certainly is an additive matrix function of A because all of the blocks that went into its construction were. Examine the expression

$$(5.4) \qquad R_{cs}(\alpha\beta) - R_{cs}(\alpha)\, R_{cs}(\beta)$$

block by block. As was noted before, the diagonal blocks of of R_{cs} are themselves representations, so (5.4) can be non zero only in the positions P, S, and Y.

The expression

$$(5.5) \qquad P(\alpha)X_T(\beta) + Q(\alpha)P(\beta)$$

appears in the position P of $R_{cs}(\alpha)\, R_{cs}(\beta)$. It has

$$[I_s \times P_{\mu\nu}(\alpha)] \cdot [I_s \times x_{j_\nu}(\beta)] + [I_s \times Q_{\mu\nu}(\alpha)] \cdot [I_s \times P_{\mu\nu}(\beta)]$$

which equals $I_s \times [P_{\mu\nu}(\alpha) \cdot x_{j_\nu}(\beta) + Q_{\mu\nu}(\alpha) \cdot P_{\mu\nu}(\beta)]$ directly below $I_s \times x_{j_\nu}$ and directly to the left of $I_s \times Q_{\mu\nu}$ in Q for $(\mu,\nu) = (1,1), \ldots, (r,r)$. But this expression is $I_s \times P_{\mu\nu}(\alpha\beta)$ from the rule for $P_{\mu\nu}(\alpha\beta)$ given by the form of $R_{\mu\nu}$ in (5.1).

Below $I_s \times x_{j_r}$ and to the left of $I_s \times Q_{1r}$ in $R_{cs}(\alpha)\, R_{cs}(\beta)$ is

$$[P_{cs} \times P_{1r}(\alpha)] \cdot [I_s \times x_{j_r}(\beta)] + [I_s \times Q_{1r}(\alpha)] \cdot [P_{cs} \times P_{1r}(\beta)]$$

which equals $P_{cs} \times [P_{1r}(\alpha) \cdot x_{j_r}(\beta) + Q_{1r}(\alpha) \cdot P_{1r}(\beta)]$ by the multiplication rule for Kronecker products. But this last expression is seen to be the corresponding entry in $P(\alpha\beta)$. Thus $R_{cs}(\alpha\beta) - R_{cs}(\alpha)\, R_{cs}(\beta)$ is zero in all the positions corresponding to $P(\alpha)$.

By a repetition of the same methods, it is shown that $S(\alpha)Q(\beta) + X_B(\alpha)S(\beta) - S(\alpha\beta) = 0$ and that

$$Y(\alpha)X_T(\beta) + S(\alpha)P(\beta) + X_B(\alpha)Y(\beta) - Y(\alpha\beta) = 0.$$

Then the expression (5.4) is zero and $R_{cs}(\alpha)$ is a representation of A.

Recall that R_{cs} was constructed out of the representations $R_{\mu\nu}$ $(\mu,\nu) = (1,1), \ldots, (1,r)$ associated with the 2r edges of the cycle C by Lemma 5.1.C. This same lemma also showed the existence of 2r special elements in N associated with those edges.

Evaluate R_{cs} at the special element associated with $P_{i_\mu} \mathcal{L} P_{j_\nu}$ in C. $R_{cs}(\alpha_{\mu\nu})$ is zero everywhere except in Y directly below $I_s \times x_{j_\nu}$ in X_T and directly to the left of $I_s \times x_{i_\mu}$ in X_B. In that non zero position is I_s if $(\mu,\nu) = (1,1), \ldots, (r,r)$ or P_{cs} if $(\mu,\nu) = (1,r)$. All of this follows from the properties of the $R_{\mu\nu}$ of Lemma 5.1.C and the construction of R_{cs}.

Now let B be in the commutator algebra of R_{cs}.

$$B = \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{pmatrix}, \tag{5.6}$$

where B is divided up to correspond to R_{cs} in (5.2). B satisfies the commutator equation $BR_{cs}(\alpha) = R_{cs}(\alpha)B$ for all α in A. Evaluate $R_{cs}(\alpha)$ at the 2r special elements. Since $Y(\alpha)$ is the only non zero part of $R_{cs}(\alpha)$, the commutator equation implies the following equations:

$$\begin{aligned} &Y(\alpha)B_{11} = B_{33}Y(\alpha), \\ &Y(\alpha)B_{12} = 0, \quad Y(\alpha)B_{13} = 0, \quad B_{23}Y(\alpha) = 0, \end{aligned} \tag{5.7}$$

where α is one of the special elements. According to the form of Y in (5.3) and the description of Y evaluated at the special elements, (5.7) implies that B_{12}, B_{13}, B_{23} are all zero and that B_{11} and B_{33} are direct sums of an $s \times s$ block B_0, repeated r times in each. For $\alpha = \alpha_{1r}$, the commutator equation implies $B_0P_{cs} = P_{cs}B_0$ so that by Lemma 2.5.B, B_0 has s equal eigenvalues. Then B has 2rs equal eigenvalues and, by Lemma 2.5.A, R_{cs} has an indecomposable direct summand T_{cs} of degree at least 2rs. Clearly, A is of unbounded type.

It is now shown that A is of strongly unbounded type. Let V be the space of R_{cs} and let V_T be the space of T_{cs}. Let $R_{cs}(\alpha_{rr})V = V_0$. It is clear that V_0 has for a basis the last s basis vectors of V when R_{cs} is in the form (5.2). Let B_T be a commuting matrix of R_{cs} that is identity on V_T and zero on its complement in V. B_T must have the previously described form for commuting matrices, B_{33T} is the direct sum of an $s \times s$ matrix B_0 which must have unit eigenvalues and is therefore non singular. Then it follows that $B_TV_0 = V_0$. Now apply the commutator equation $R_{cs}(\alpha_{rr})B_T = B_TR_{cs}(\alpha_{rr})$ to all of V.

$$V_T \supseteq R_{cs}(\alpha_{rr})V_T = R_{cs}(\alpha_{rr})B_TV = B_TR_{cs}(\alpha_{rr})V = V_0$$

hence $V_T \supseteq V_0$.

The space V_0 is used in proving that T_{cs} and T_{ds} cannot be equivalent when $d \neq c$. Suppose that they were similar, then there would exist a matrix P intertwining R_{cs} and R_{ds}, which when cut down to V_T or any subspace of it, represents an isomorphism. P satisfies the intertwining equation,

(5.8) $$R_{ds}(\alpha)P = PR_{cs}(\alpha),$$

for all α in A. But for $\alpha = \alpha_{11}, \ldots, \alpha_{rr}$, $R_{cs}(\alpha)$ equals $R_{ds}(\alpha)$. Therefore, for those α, (5.8) looks like the commutator equation. Let P be divided into submatrices,

$$P = \begin{pmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{pmatrix},$$

corresponding to the divisions of B. It is clear from the previous argument concerning the commutator equation that P_{12}, P_{13}, P_{23} are all zero and P_{11} and P_{33} are direct sums of an $s \times s$ matrix P_o. Finally let $\alpha = \alpha_{1r}$ in the intertwining equation, this implies

$$P_{ds}P_o = P_oP_{cs}. \tag{5.9}$$

But P cut down to V_o is P_o so that P_o is non singular. Then, the above equation (5.9) is impossible, because P_{cs} and P_{ds} are primary matrices with distinct eigenvalues $d \neq c$. Therefore no such P can exist and T_{cs} and T_{ds} must be inequivalent. Since the field k is infinite there exists an infinite number of such inequivalent indecomposable representations T_{cs} with degrees between 2rs and the degree of R_{cs}. A is of strongly unbounded type and, by Theorem 2.3.B, so is any algebra with A for a basic algebra. This completes the proof of Theorem 5.2.A.

3. Graph with a Chain that Branches at Each End

The fourth sufficient condition that an algebra be of strongly unbounded type is described in the following theorem.

Theorem 5.3.A: If the graph $G(A_o)$ associated with any two-sided ideal $A_o \subset N$ contains a chain which branches at each end then A is of strongly unbounded type.

Proof: Let C and its branches be as follows:

$$(5.10)\qquad \begin{matrix} P_{k_1}, P_{k_1}\mathcal{L}P_{j_1}, & & P_{k_3}\mathcal{L}P_{j_r}, P_{k_3} \\ & P_{j_1}, P_{i_1}\mathcal{L}P_{j_1}, P_{i_1}, \ldots, P_{j_r}, & \\ P_{k_2}, P_{k_2}\mathcal{L}P_{j_1}, & & P_{k_4}\mathcal{L}P_{j_r}, P_{k_4} \end{matrix}$$

Note that the vertices P_{j_1} and P_{j_r} at the ends of C each have order three, so that there are four cases to consider depending on whether these two vertices have left or right order three. It is assumed here that both have left order three. The other three cases are proved by analogous methods.

It is also assumed that all of the edges appearing in (5.10) are distinct. For if not, Lemma 5.1.D implies that the graph has a cycle and the algebra A, by Theorem 5.2.A, is already of strongly unbounded type.

Let $R_{\mu\nu}$ be the representations associated with the edges in the chain C by Lemma 5.1.C,

$$(5.11)\quad R_{\mu\nu} = \begin{pmatrix} x_{j_\nu} & & \\ P_{\mu\nu} & Q_{\mu\nu} & \\ y_{\mu\nu} & S_{\mu\nu} & x_{i_\mu} \end{pmatrix} \quad (\mu,\nu) = (1,1),(1,2),\ldots,(r-1,r).$$

Let R^ρ_ν be the representations associated with the branches in (5.10) by Lemma 5.1.C,

$$(5.12)\quad R^\rho_\nu = \begin{pmatrix} x_{j_\nu} & & \\ P^\rho_\nu & Q^\rho_\nu & \\ y^\rho_\nu & S^\rho_\nu & x_{k_\rho} \end{pmatrix} \quad (\rho,\nu) = (1,1),(2,1),(3,r),(4,r).$$

From the submatrices of these representations form the matrix function R_{cs},

$$(5.13) \qquad R_{cs} = \begin{pmatrix} X_T & & \\ P & Q & \\ Y & S & X_B \end{pmatrix},$$

as follows.

Let $X_T(\alpha)$ be the direct sum of $I_{2s} \times x_{j_\nu}(\alpha)$ for $\nu = 1,\ldots,r$ and let $Q(\alpha)$ be the direct sum of $I_{2s} \times Q_{\mu\nu}(\alpha)$, $I_{2s} \times Q^{\rho}_{\nu}(\alpha)$ for (μ,ν) equal to $(1,1),\ldots,(r-1,r)$ and for (ρ,ν) equal to $(1,1)$, $(2,1)$, $(3,r)$, $(4,r)$. Let $X_B(\alpha)$ have $I_s \times x_{k_1}(\alpha) \dot{+} I_s \times x_{k_2}(\alpha)$ in the top diagonal block, $I_s \times x_{k_3}(\alpha) \dot{+} I_s \times x_{k_4}(\alpha)$ in the bottom diagonal block and the direct sum of $I_{2s} \times x_{i_\mu}(\alpha)$, $\mu = 1, \ldots, r-1$ in the middle diagonal block. $X_T(\alpha)$, $Q(\alpha)$ and $X_B(\alpha)$ are all representations of A, by Lemma 2.5.C.

Let $P(\alpha)$ have $I_{2s} \times P_{\mu\nu}(\alpha)$ directly below $I_{2s} \times x_{j_\nu}$ and directly to the left of $I_{2s} \times Q_{\mu\nu}$ for $(\mu,\nu) = (1,1),\ldots,(r-1,r)$ and $I_{2s} \times P^{\rho}_{\nu}(\alpha)$ directly below $I_{2s} \times x_{j_\nu}$ and directly to the left of $I_{2s} \times Q^{\rho}_{\nu}$ for $(\rho,\nu) = (1,1)$, $(2,1)$, $(3,r)$, $(4,r)$. Fill out the rest of $P(\alpha)$ with zeros.

Let $S(\alpha)$ have $I_{2s} \times S_{\mu\nu}(\alpha)$ directly below $I_{2s} \times Q_{\mu\nu}$ in Q and directly to the left of $I_{2s} \times x_{i_\mu}$ in X_B for $(\mu,\nu) = (1,1),\ldots,(r-1,r)$. Put $(I_s,0) \times S^1_1(\alpha)$ below $I_{2s} \times Q^1_1$ to the left of $I_s \times x_{k_1}$, $(0,I_s) \times S^2_1(\alpha)$ below $I_{2s} \times Q^2_1$ to the left of $I_s \times x_{k_2}$, $(I_s,I_s) \times S^3_r(\alpha)$ below $I_{2s} \times Q^3_r$ to the left of $I_s \times x_{k_3}$, $(I_s,P_{cs}) \times S^4_r(\alpha)$ below $I_{2s} \times Q^4_r$ to the left of $I_s \times x_{k_4}$, where P_{cs} is the primary matrix of degree s with eigenvalue c. Fill out the rest of $S(\alpha)$ with zeros.

Define $Y(\alpha)$ similarly. Let $I_{2s} \times y_{\mu\nu}$ appear directly below $I_{2s} \times x_{j_\nu}$ directly to the left of $I_{2s} \times x_{i_\mu}$ for $(\mu,\nu) = (1,1), \ldots, (r-1,r)$. Put $(I_s,0) \times y^1_1$ below $I_{2s} \times x_{j_1}$ to the left of $I_s \times x_{k_1}$, $(0, I_s) \times y_1^2$ below $I_{2s} \times x_{j_1}$ and to the left of $I_s \times x_{k_2}$, $(I_s, I_s) \times y_r^3$ below $I_{2s} \times x_{j_r}$ and to the left of $I_s \times x_{k_3}$, $(I_s, P_{cs}) \times y_r^4$ below $I_{2x} \times x_{j_r}$ and to the left of $I_s \times x_{k_4}$. Fill out the rest of Y with zeros.

Of particular importance in the proof of this theorem is the form of $Y(\alpha)$.

(5.14)

$$Y(\alpha) = \begin{pmatrix} (I_s,0)\times y_1^1 & & & & & \\ (0,I_s)\times y_1^2 & & & & & \\ I_{2s}\times y_{11} & I_{2s}\times y_{12} & & & & \\ & I_{2s}\times y_{22} & \cdot & & & \\ & & \cdot & \cdot & & \\ & & & & I_{2s}\times y_{r-1r-1} & I_{2s}\times y_{r-1r} \\ & & & & & (I_s,I_s)\times y_r^3 \\ & & & & & (I_s,P_{cs})\times y_r^4 \end{pmatrix}$$

It is now shown that R_{cs} is a representation of A with an indecomposable direct summand which has a degree an integral multiple of s. It is necessary to first show that R_{cs} is a representation of A. It is an additive function of A because all of the blocks that went into its construction were. Examine the expression,

$$R_{cs}(\alpha)R_{cs}(\alpha) - R_{cs}(\alpha\beta), \tag{5.15}$$

block by block. The diagonal blocks of R_{cs} are representations themselves so (5.15) can be non zero only in the positions corresponding to P, S, and Y.

Examine the position corresponding to $(I_s,0)\times y_1^1$ in $R_{cs}(\alpha)R_{cs}(\beta)$. Appearing in that position is

$[(I_s,0)\times y_1^1(\alpha)] \cdot [I_{2s}\times x_{j_1}(\beta)] + [(I_s,0)\times S_1^1(\alpha)]\cdot[I_{2s}\times P_1^1(\beta)] + [I_s\times x_{k_1}(\alpha)] \cdot [(I_s,0)\times y_1^1(\beta)]$ which equals

$(I_s,0)\times(y_1^1(\alpha)x_{j_1}(\beta) + S_1^1(\alpha)P_1^1(\beta) + x_{k_1}(\alpha)y_1^1(\beta)\ )$ by the rules for operation with Kronecker products. But this last expression is $(I_s,0)\times y_1^1(\alpha\beta)$ by the rule for $y_1^1(\alpha\beta)$ given by the form of the representation R_1^1 in (5.12). Hence (5.15) is zero in the position corresponding to $(I_s,0)\times y_1^1$. A repetition of these same methods implies that all of the positions in (5.15) are zero and that R_{cs} is a representation of A.

Recall that R_{cs} was constructed out of the representations associated with edges of the figure (5.10) by Lemma 5.1.C. The same lemma associated a special element in A with each of these representations.

Consider R_{cs} evaluated at the special element $\alpha_{\mu\nu}$ associated with $R_{\mu\nu}$ by Lemma 5.1.C. $R_{cs}(\alpha_{\mu\nu})$ is zero everywhere except in Y directly below $I_{2s}\times x_{j_\nu}$ and directly to the left of $I_{2s}\times x_{i_\mu}$. I_{2s} appears in that position. The above fact is true for $(\mu,\nu) = (1,1), \ldots, (r-1,r)$. All of this follows from the properties of the $R_{\mu\nu}$ and $\alpha_{\mu\nu}$ of Lemma 5.1.C and the construction of R_{cs}.

Now consider R_{cs} evaluated at the special elements $\alpha_1^1, \alpha_1^2, \alpha_r^3, \alpha_r^4$ associated with $P_{k_1}\mathcal{L}P_{j_1}$, $P_{k_2}\mathcal{L}P_{j_1}$, $P_{k_3}\mathcal{L}P_{j_r}$, $P_{k_4}\mathcal{L}P_{j_r}$ respectively by Lemma 5.1.C. $R_{cs}(\alpha_\nu^\rho)$ is zero except in the positions corresponding to Y. The only non zero entries of $R_{cs}(\alpha_\nu^\rho)$ are the following: $R_{cs}(\alpha_1^1)$ has $(I_s,0)$ below $I_{2s}\times x_{j_1}$ to the left of $I_s\times x_{k_1}$, $R_{cs}(\alpha_1^2)$ has $(0,I_s)$ below $I_{2s}\times x_{j_1}$ to the left of $I_s\times x_{k_2}$, $R_{cs}(\alpha_r^3)$ has (I_s,I_s) below $I_{2s}\times x_{j_r}$ to the left of $I_s\times x_{k_3}$, $R_{cs}(\alpha_r^4)$ has (I_s,P_{cs}) below $I_{2s}\times x_{j_r}$ to the left of $I_s\times x_{k_4}$.

Now let B be in the commutator algebra of R_{cs}. Let B be divided into submatrices to correspond to the division

(5.13) of R_{cs},

$$B = \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{pmatrix} . \tag{5.16}$$

B must satisfy the commutator equation,

$$R_{cs}(\alpha)B = BR_{cs}(\alpha),$$

for all α in A. Consider this equation for α equal to the special elements $\alpha_{\mu\nu}$ and α_{ν}^{ρ} . Since $Y(\alpha)$ is the only non zero part of R_{cs}, the commutator equation implies the following equations,

$$\begin{aligned} &Y(\alpha)B_{11} = B_{33}Y(\alpha) \\ &Y(\alpha)B_{12} = 0, \quad Y(\alpha)B_{13} = 0, \quad B_{23}Y(\alpha) = 0, \end{aligned} \tag{5.17}$$

where α is one of the special elements. From the form of Y in (5.14) and the description of Y evaluated at the special elements, (5.17) implies that B_{12}, B_{13}, and B_{23} are all zero.

By letting α equal $\alpha_1^1, \alpha_1^2, \alpha_{11}, \ldots, \alpha_{r-1r}$ in (5.17), it follows that B_{11} is a direct sum of a $2s \times 2s$ block B_0,

$$B_0 = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} . \tag{5.18}$$

B_{33} is the direct sum of the block B_0 down to the last $2s \times 2s$ diagonal block B',

$$B' = \begin{pmatrix} B_3 & 0 \\ 0 & B_4 \end{pmatrix} . \tag{5.19}$$

Letting $\alpha = \alpha_r^3$ in (5.17), and using (5.18) and (5.19) in (5.17) implies $B_1 = B_3 = B_2$. Finally, letting $\alpha = \alpha_r^4$ in

(5.17) implies that $B_1 = B_4$ and $B_1 P_{cs} = P_{cs} B_1$. Since P_{cs} is a primary matrix, Lemma 2.5.B implies that B_1 has a single eigenvalue. Then from the above description of B, B has at least $(4r + 2)s$ equal eigenvalues. By Lemma 2.5.A, R_{cs} has an indecomposable direct summand T_{cs} of degree at least $(4r + 2)s$ and A is clearly of unbounded type.

To show A is of strongly unbounded type proceed as in the previous theorems. Let V be the space of R_{cs}, let V_T be the space of T_{cs} and let V_o be the space $R_{cs}(\alpha_r^4)V$. V_o has for a basis the last s basis vectors of V when R_{cs} is in the form (5.13). Let B_T be the commuting matrix which is identity on V_T and zero on its complement in V. The bottom $s \times s$ diagonal block B_1 in B_T has unit eigenvalues and is non singular. By the form of such commuting matrices B_T has only zeros above B_1. Then $B_T V_o = V_o$. Now apply $R_{cs}(\alpha_r^4)B_T = B_T R_{cs}(\alpha_r^4)$ to all of V. $V_T \supseteq R_{cs}(\alpha_r^4)V_T = R_{cs}(\alpha_r^4)B_T V = B_T R_{cs}(\alpha_r^4)V = B_T V_o = V_o$, hence $V_T \supseteq V_o$.

Now it can be shown that T_{cs} and T_{ds} are not equivalent for $c \neq d$. Suppose they were equivalent, then there would exist P, intertwining R_{cs} and R_{ds} such that P, when cut down to V_T or V_o, represents an isomorphism. P satisfies the intertwining equation,

$$PR_{cs}(\alpha) = R_{ds}(\alpha)P, \tag{5.20}$$

for all α in A. Note that for all the special elements except α_r^4, $R_{cs}(\alpha) = R_{ds}(\alpha)$. For these α the intertwining equation looks exactly like the commutator equation. Let P be divided into submatrices corresponding to those of B,

$$P = \begin{pmatrix} P_{11} & 0 & 0 \\ P_{21} & P_{22} & 0 \\ P_{31} & P_{32} & P_{33} \end{pmatrix} .$$

The blocks in P above the diagonal may be taken to be 0. Also P_{11} is the direct sum of an $s \times s$ matrix P_0 and P_{33} is the direct sum of the same P_0 down to the last $s \times s$ diagonal block which is P_1. All of this follows from the previous arguments concerning commuting matrices and the above comment about (5.20).

Now let $\alpha = \alpha_r^4$ in (5.20). This implies that $P_1 = P_0$ and $P_0 P_{cs} = P_{ds} P_0$. Note that P cut down to V_0 is P_0, which is therefore non singular. Under these circumstances the above equation is impossible because P_{cs} and P_{ds} have distinct eigenvalues $d \neq c$.

The conclusion can then be drawn that T_{cs} and T_{ds} are inequivalent for $c \neq d$. Since the field k is infinite, A has an infinite number of inequivalent indecomposable representations T_{cs} with degrees between 4rs and the degree of R_{cs}. A is therefore of strongly unbounded representation type.

4. Graph with a Vertex of Order Four

A restatement of the condition in Theorem 4.1.C of Chapter IV can now be made in terms of the graph.

Theorem 5.4.A: If the graph G_0 associated with any two-sided ideal $A_0 \subset N$ has a vertex of order four or more then A is of strongly unbounded type.

Proof: If G_0 has a vertex of order four, there exist four covers A_1, A_2, A_3, A_4 of A_0 such that $\phi_{i_r j}(A_r)$ covers

$\phi_{i_r j}(A_o)$ (or $\phi_{i j_r}(A_r)$ covers $\phi_{i j_r}(A_o)$) for r = 1,2,3,4. The first case occurs when the vertex of order four has left order four, the other case occurs when it has right order four. ϕ_{ij} is the lattice homomorphism of L_A into subspaces of $e_i A e_j$. By Lemma 5.1.B, A_r can be written $A_r = k\alpha_r + A_o$ where α_r is chosen in $e_{i_r} N e_j$ (or in $e_i N e_{j_r}$). Recalling the lattice homomorphism ϕ_j (and $_i\phi$) of L_A into the lattice of left ideals (or right ideals), $\phi_j(A_r) = k\alpha_r + A_o e_j$ (or $_i\phi(A_r) = k\alpha_r + e_i A_o$). But then $\phi_j(A_o)$ has four covers in $\phi_j(L_A)$ (or $_i\phi(A_o)$ has four covers in $_i\phi(L_A)$) and, by Theorem 4.1.C, A is of strongly unbounded type.

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