and take the values 1 (with probability 2/3) and -2 (with probability 1/3). $X_1 + X_2 + X_3$ takes the values 3 (with probability 8/27), 0 (with probability 12/27), -3 (with probability 6/27) and -6 (with probability 1/27). Hence $E(|X_i|) = 4/5$, and $E(|X_1 + X_2 + X_3|) = 48/27 = 16/9 = 4/3E(|X_i|)$; which is not $\geq 3/2E(|X_i|)$.

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ON THE INDEPENDENCE OF THE EXTREMES IN A SAMPLE[1]

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In a previous article [1] the assumption was used that the mth observation in ascending order (from the bottom) and the mth observation in descending order (from the top) are independent variates, provided that the rank m is small compared to the sample size n. In the following it will be shown that this assumption holds for the usual distributions.

Let x be a continuous, unlimited variate, let $\Phi(x)$ be the probability of a value equal to, or less than, x; let $\varphi(x)$ be the density of probability, henceforth called the initial distribution. The mth observation from the bottom is written ${}_mx$ and the kth observation from the top is written x_k. Thus, the bivariate distribution $\mathfrak{w}_n({}_mx, x_k)$ of ${}_mx$ and x_k, is such that there are $m - 1$ observations less than ${}_mx$; $k - 1$ observations greater than x_k and $n - m - k$ observations between ${}_mx$ and x_k.

For simplicity's sake write

$$\Phi({}_mx) = {}_m\Phi; \qquad \Phi(x_k) = \Phi_k.$$
$$\varphi({}_mx) = {}_m\varphi; \qquad \varphi(x_k) = \varphi_k.$$

Then

$$\mathfrak{w}_n({}_mx, x_k) = C\,{}_m\Phi^{m-1}\,{}_m\varphi(\Phi_k - {}_m\Phi)^{n-m-k}\varphi_k(1 - \Phi_k)^{k-1}, \tag{1}$$

where

$$C = \frac{n!}{(m-1)!(k-1)!(n-m-k)!}. \tag{1'}$$

In the expression (1) no assumption about dependence or independence of ${}_mx$ and x_k is implied except that these values are taken from the same population.

The distribution (1) is now modified by introducing three conditions. First,

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that the two variates are extreme, namely that the ranks m and k are of the same order of magnitude and small compared to the sample size n.

$$n >> m \simeq k = 0(1). \tag{2}$$

Furthermore it is assumed that the initial distribution $\varphi(x)$ is, for small and for large values of the variate, subject to L'Hospital's rules

$$\lim_{x \to -\infty} \frac{\varphi'(x)}{\varphi(x)} = \lim_{x \to -\infty} \frac{\varphi(x)}{\Phi(x)}; \qquad \lim_{x \to \infty} \frac{\varphi'(x)}{\varphi(x)} = -\lim_{x \to \infty} \frac{\varphi(x)}{1 - \Phi(x)}. \tag{3}$$

Finally it is assumed that n is so large that the equality of the limits may be replaced by the equality of the quotients. Then it is legitimate to write

$$\frac{{}_m\varphi'}{{}_m\varphi} = \frac{{}_m\varphi}{{}_m\Phi}; \qquad \frac{\varphi_k'}{\varphi_k} = -\frac{\varphi_k}{1 - \Phi_k}. \tag{3'}$$

Clearly, the three conditions do not imply any assumption about dependence or independence of the two extremes.

From (1) the most probable mth value from the bottom, ${}_mu$, and the most probable kth value from the top, u_k, are the solutions of

$$\frac{m-1}{{}_m\Phi}\,{}_m\varphi + \frac{{}_m\varphi'}{{}_m\varphi} = \frac{n-m-k}{\Phi_k - {}_m\Phi}\,{}_m\varphi.$$

$$\frac{n-m-k}{\Phi_k - {}_m\Phi}\,\varphi_k + \frac{\varphi_k'}{\varphi_k} = \frac{k-1}{1-\Phi_k}\,\varphi_k.$$

These two equations may be written by virtue of (3′)

$$\frac{m}{{}_m\Phi} = \frac{n-m-k}{\Phi_k - {}_m\Phi} = \frac{k}{1-\Phi_k}.$$

Consequently the probabilities of the most probable mth and kth values ${}_mu$ and u_k are

$$\Phi({}_mu) = \frac{m}{n}; \qquad \Phi(u_k) = 1 - \frac{k}{n}. \tag{4}$$

The expansion of the probabilities ${}_m\Phi$ and Φ_k around the modes ${}_mu$ and u_k leads [2, 3] by virtue of (2), (3), (4), to

$${}_m\Phi = \frac{m}{n}\, e^{{}_my}; \qquad \Phi_k = 1 - \frac{k}{n}\, e^{-y_k}. \tag{5}$$

where

$${}_my = \frac{n}{m}\,\varphi({}_mu)({}_mx - {}_mu); \qquad y_k = \frac{n}{k}\,\varphi(u_k)(x_k - u_k). \tag{6}$$

Therefore, distributions, subject to L'Hospital's rules (3), may be said to be of the exponential type. Since the derivatives ${}_m\varphi$ and φ_k are

$$(7)\qquad {}_m\varphi = {}_m\alpha_m\Phi; \qquad \varphi_k = \alpha_k(1 - \Phi_k),$$

where

$$(7')\qquad {}_m\alpha = \frac{m}{n}\varphi({}_mu); \qquad \alpha_k = \frac{n}{k}\varphi(u_k),$$

the product of the first two and the last two functions in formula (1) may be written as a product of two functions

$$(8)\qquad {}_m\Phi^{m-1}{}_m\varphi\,\varphi_k(1 - \Phi_k)^{k-1} = \left({}_m\alpha\frac{m^m}{n^m}e^{m_my}\right)\left(\alpha_k\frac{k^k}{n^k}e^{-ky_k}\right)$$

Clearly, each factor in (8) depends only on one variable.

In the same way the function of ${}_mx$ and x_k in the middle of (1) can be split up into a product of two independent functions, each depending only on one variate. By virtue of (5)

$$\Phi_k - {}_m\Phi = 1 - \frac{1}{n}(me^{my} + ke^{-y_k})$$

and by virtue of (2)

$$(9)\qquad (\Phi_k - {}_m\Phi)^{n-m-k} = \exp(-me^{my})\exp(-ke^{-y_k}),$$

where

$$\exp(x) = e^x.$$

From (2) the constant factor (1′) may also be split into a product

$$(10)\qquad \frac{n!}{(m-1)!(k-1)!(n-m-k)!} = \frac{n^m}{(m-1)!}\cdot\frac{n^k}{(k-1)!}.$$

Introducing (10), (9) and (8) into (1), the bivariate distribution of the mth extreme value from the bottom and the kth extreme value from the top is obtained as a product of two independent distributions

$$(11)\qquad \mathfrak{w}_n({}_mx, x_k) = {}_mf({}_mx)\cdot f_k(x_k)$$

where

$$(12)\qquad {}_mf({}_mx) = \frac{{}_m\alpha m^m}{(m-1)!}\exp(m\,{}_my - me^{my})$$

and

$$(12')\qquad f_k(x_k) = \frac{\alpha_k k^k}{(k-1)!}\exp(-ky_k - ke^{-y_k})$$

are the distributions of the mth extreme values from the bottom, alone, and of the kth extreme values from the top, alone.

In the special case $m = k$ and for a symmetrical initial distribution with mean zero, the following equations hold

(13) $$_m\alpha = \alpha_k = \alpha_m; \qquad _mu = -u_k = -u_m.$$

(13′) $$_m\Phi = 1 - \Phi_k = 1 - \Phi_m; \qquad _m\varphi = \varphi_k = \varphi_m.$$

and the bivariate distribution of the mth values from the bottom $_mx$, and from the top x_m, is

(14) $$\mathfrak{w}_n(_mx, x_m) = {_mf}(_mx)\cdot f_m(x_m),$$

where

(14′) $$_mf(_mx) = f_m(-x_m)$$

is the expression used in the beginning of article [1]

It follows from (11) that the mth observation in ascending order, and the kth observation in descending order, may be dealt with as independent variates provided that n is large, the ranks m and k are small, and that the initial continuous unlimited distribution is of the exponential type as defined by equations (3).

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A NOTE ON SAMPLING INSPECTION

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In designing an industrial sampling plan conformable to the Pearson-Neyman approach, the operating characteristic is made to pass as nearly as possible through two predetermined points. Wald [1] has used this method for setting up sequential sampling plans.

A similar type of single sampling plan can be designed by using tables of the incomplete Beta function. Unfortunately, tables of this function are not generally available, and the existing tables do not cover the range for large sample sizes.

An approximate solution of the problem for single sampling can be based on the widely available tables of percentage points of the chi-square distribution. This is equivalent to assuming a Poisson distribution of defectives in the sample, utilizing the well known fact that for even degrees of freedom the chi-square distribution gives the summation of a Poisson series.

We use the following well established notation: