

## ON THE INDEX OF CACTUSES WITH $n$ VERTICES

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ABSTRACT. Among all connected cactuses with  $n$  vertices we find a unique graph whose largest eigenvalue (index, for short) is maximal.

### 1. Introduction

We consider only simple graphs in this paper. Let  $G$  be a graph with  $n$  vertices, and let  $A(G)$  be the  $(0, 1)$ -adjacency matrix of  $G$ . Since  $A(G)$  is symmetric, its eigenvalues are real. Without loss of generality we can write them in non-increasing order as  $\lambda_1(G) \geq \lambda_2(G) \geq \dots \geq \lambda_n(G)$  and call them the eigenvalues of  $G$ . The characteristic polynomial of  $G$  is just  $\det(\lambda I - A(G))$ , and denoted by  $P(G, \lambda)$ . The largest eigenvalue  $\lambda_1(G)$  is called the index of  $G$  (or the spectral radius of  $G$ ). If  $G$  is connected, then  $A(G)$  is irreducible and it is well-known that  $\lambda_1(G)$  has multiplicity one and there exists a unique positive unit eigenvector corresponding to  $\lambda_1(G)$ , by the Perron–Frobenius theory of non-negative matrices. We shall refer to such an eigenvector as the Perron vector of  $G$ .

The investigation of the index of graphs is an important topic in the theory of graph spectra. The reference [9] is an excellent survey which includes a large number of references on this topic.

Let  $\mathcal{H}(n, n + t)$  be the set of all connected graphs with  $n$  vertices and  $n + t$  edges ( $t \geq -1$ ). The corresponding extremal index problems have been solved for certain values of  $t$  [2, 3, 8, 13, 16, 17].

The recent developments on this topic [1, 4, 5, 6, 11, 14, 18] also involve the problem concerning graphs with maximal or minimal index of a given class of graphs.

In this paper we study the index of cactuses with  $n$  vertices. We say that the graph  $G$  is a *cactus* if any two of its cycles have at most one common vertex (about

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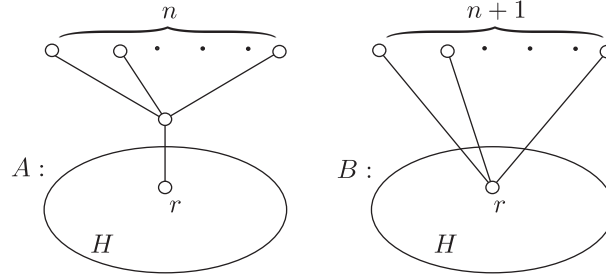


FIGURE 1

cactuses see e.g., [15]). If all cycles of the cactus  $G$  have exactly one common vertex we say that they form a *bundle*.

Denote by  $C(n)$  the set of all connected cactuses with  $n$  vertices. In this paper we determine the graphs with the largest index in the class  $C(n)$ .

## 2. Preliminaries

Denote by  $C_n$  the cycle on  $n$  vertices. Let  $G - x$  or  $G - xy$  denote the graph that arises from  $G$  by deleting the vertex  $x \in V(G)$  or the edge  $xy \in E(G)$ . Similarly,  $G + xy$  is a graph that arises from  $G$  by adding an edge  $xy \notin E(G)$ , where  $x, y \in V(G)$ .

For  $v \in V(G)$ ,  $d(v)$  denotes the degree of vertex  $v$  and  $N(v)$  denotes the set of all neighbors of vertex  $v \in G$ . Also, by  $d(v, w)$  we will denote the distance between vertices  $v$  and  $w$  in  $G$ .

In order to complete the proof of our main result we need the following lemmas.

LEMMA 2.1. [18] *Let  $G$  be a connected graph and let  $\lambda_1(G)$  be the index of  $A(G)$ . Let  $u, v$  be two vertices of  $G$  and let  $d(v)$  be the degree of the vertex  $v$ . Suppose that  $v_1, v_2, \dots, v_s \in N(v) \setminus N(u)$  ( $1 \leq s \leq d(v)$ ) and  $x = (x_1, x_2, \dots, x_n)$  is the Perron vector of  $A(G)$ , where  $x_i$  corresponds to the vertex  $v_i$  ( $1 \leq i \leq n$ ). Let  $G^*$  be the graph obtained from  $G$  by deleting the edges  $vv_i$  and adding the edges  $uv_i$  ( $1 \leq i \leq s$ ). If  $x_u \geq x_v$ , then  $\lambda_1(G) < \lambda_1(G^*)$ .*

Lemma 1 was first given by Wu, Xiao and Hong and it is a stronger version of a similar lemma in [16].

LEMMA 2.2. [16] *Let  $H$  be any connected graph with at least two vertices. If  $A$  and  $B$  are the graphs as in Figure 1, then  $P(A, \lambda) > P(B, \lambda)$  for  $\lambda \geq \lambda_1(A)$ . In particular,  $\lambda_1(A) < \lambda_1(B)$ .*

If  $H$  is a spanning subgraph of  $G$  we shall write  $H \leq G$ ; in particular if it is a proper spanning subgraph, we then write  $H < G$ .

LEMMA 2.3. [12], [10, p. 50] *Let  $G$  be a connected graph. If  $H$  is connected and  $H < G$ , then  $\lambda_1(H) < \lambda_1(G)$ .*

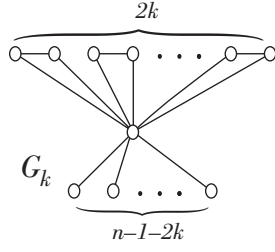


FIGURE 2.

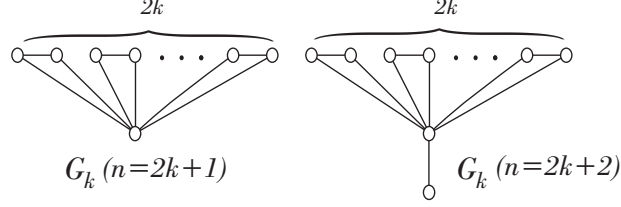


FIGURE 3.

### 3. Main result

Let  $G_k$  be the bundle with  $n$  vertices and  $k$  cycles of length 3 depicted in Figure 2. In particular, if  $k = \lfloor \frac{n-1}{2} \rfloor$  we have a bundle with  $n$  vertices and  $k$  cycles of length 3 depicted in Figure 3.

**THEOREM 3.1.** *Let  $G$  be a graph in  $C(n)$ . Then*

$$\lambda_1(G) \leq \lambda_1(G_k),$$

where  $G_k$  is the graph depicted in Figure 3 ( $k = \lfloor \frac{n-1}{2} \rfloor$ ), and equality holds if and only if  $G \cong G_k$ .

**PROOF.** Choose  $G \in C(n)$  such that the index of  $G$  is as large as possible. Denote the vertex set of  $G$  by  $V(G) = \{v_1, v_2, \dots, v_n\}$  and the Perron vector of  $G$  by  $x = (x_1, x_2, \dots, x_n)$ , where  $x_i$  corresponds to the vertex  $v_i$  ( $1 \leq i \leq n$ ).

We first prove that the graph  $G$  is a bundle. In order to do that we will prove the following two claims.

**CLAIM 1.** *Any two cycles of the graph  $G$  have one common vertex.*

**PROOF.** Assume, on the contrary, that there are two disjoint cycles  $C_p$  and  $C_q$ . Then, there exists a path  $v_1 v_2 \dots v_k$  joining the cycles  $C_p$  and  $C_q$  of length  $k - 1 \geq 1$ , where the vertex  $v_1$  belongs to the cycle  $C_p$  and the vertex  $v_k$  belongs to the cycle  $C_q$ . Note, any path joining the cycles  $C_p$  and  $C_q$  starts from  $v_1$  and ends to  $v_k$  (in the opposite case  $G$  is not a cactus). Without loss of generality we may assume that  $x_1 \geq x_k$ . Denote by  $v_{k+1}$  and  $v_{k+2}$  neighbors of  $v_k$  which belong to  $C_q$ .

Let

$$G^* = G - \{v_k v_{k+1}, v_k v_{k+2}\} + \{v_1 v_{k+1}, v_1 v_{k+2}\}.$$

Then  $G^* \in C(n)$  and by Lemma 1 we have  $\lambda_1(G^*) > \lambda_1(G)$ , a contradiction.  $\square$

Hence, any two cycles have one common vertex.

**CLAIM 2.** *Any three cycles have exactly one common vertex.*

PROOF. In the opposite case the graph  $G$  is not a cactus, because there exist cycles which have more than one common vertex.  $\square$

By Claims 1 and 2, all cycles of the graph  $G$  have exactly one common vertex, i.e., they form a bundle. Let us denote by  $v_1$  the common vertex of all cycles in this bundle.

Secondly we prove that if  $G$  contains a tree  $T$  attached to a cycle at some vertex  $v$  (called the *root* of  $T$ ) then  $T$  consists only of edges containing  $v$ . That is:

CLAIM 3. *Any tree  $T$  attached to a vertex  $v$  of one of the cycles in the graph  $G$  contains only vertices at distance one from its root  $v$ .*

PROOF. In the opposite case, there exists a tree  $T$  (with root  $v_i \in C_p$ ) and a vertex of  $T$  whose distance from  $v_i$  is greater than one. Let  $v_j \in T$  be a vertex furthest from the root  $v_i$ . Then,  $d(v_i, v_j) \geq 2$  and there exists a path  $v_i \dots v_{j-2}v_{j-1}v_j$  joining  $v_i$  and  $v_j$  of length  $\geq 2$ . Now, if we take the vertex  $v_{j-2}$  as the root  $r$  of the graph  $A$  from Figure 1 and apply Lemma 2 we will get a graph  $G^* \in C(n)$ , such that  $\lambda_1(G^*) > \lambda_1(G)$ , a contradiction.  $\square$

Hence, any tree  $T$  attached to a vertex of some cycle of  $G$  consists only of edges with exactly one common vertex.

We further prove the following claim.

CLAIM 4. *Any tree  $T$  of the graph  $G$  is attached to the common vertex  $v_1$  of all cycles of the bundle.*

PROOF. In the opposite case, there exists a tree  $T$  attached to a vertex  $v_i$  ( $v_i \neq v_1$ ), and let  $v_i \in C_p$ . Let this tree  $T$  consist of vertices  $y_1, y_2, \dots, y_k$  (at distance one from the root  $v_i$ ) and  $w_1, w_2, \dots, w_l \in N(v_1) \setminus V(C_p)$ . If  $x_1 \geq x_i$ , let

$$G^* = G - \{v_i y_1, v_i y_2, \dots, v_i y_k\} + \{v_1 y_1, v_1 y_2, \dots, v_1 y_k\}.$$

If  $x_1 < x_i$ , let

$$G^* = G - \{v_1 w_1, v_1 w_2, \dots, v_1 w_l\} + \{v_i w_1, v_i w_2, \dots, v_i w_l\}.$$

Then, in either case,  $G^* \in C(n)$ , and by Lemma 1, we have  $\lambda_1(G^*) > \lambda_1(G)$ , a contradiction.  $\square$

Hence  $G$  is a bundle with a unique tree attached to the common vertex of all cycles of  $G$ , and this tree contains only vertices at distance one from the root.

Finally, we prove:

CLAIM 5. *All cycles of  $G$  have length three.*

PROOF. Suppose on the contrary that there exists a cycle  $C_p$  of length  $p \geq 4$ . Let  $C_p = v_1 v_2 \dots v_p v_1$  and let  $w_1, w_2, \dots, w_l \in N(v_1) \setminus V(C_p)$ . If  $x_1 \geq x_2$ , let

$$G^* = G - \{v_2 v_3\} + \{v_1 v_3\}.$$

If  $x_1 < x_2$ , let

$$G^* = G - \{v_1 v_p, v_1 w_1, \dots, v_1 w_l\} + \{v_2 v_p, v_2 w_1, \dots, v_2 w_l\}.$$

Then, in either case,  $G^* \in C(n)$ , and by Lemma 1, we have  $\lambda_1(G^*) > \lambda_1(G)$ , a contradiction.  $\square$

Combining arguments from Claims 1–5, we have that  $G = G_k$ , where  $G_k$  is a bundle with  $n$  vertices and  $k$  cycles of length 3 (Figure 2).

We notice that  $G_0 = K_{1,n-1}$  and

$$G_0 < G_1 < \cdots < G_{\lfloor \frac{n-1}{2} \rfloor}.$$

By Lemma 3 we have

$$\lambda_1(G_0) < \lambda_1(G_1) < \cdots < \lambda_1(G_{\lfloor \frac{n-1}{2} \rfloor}).$$

So, we obtain that the graph  $G_k$  ( $k = \lfloor \frac{n-1}{2} \rfloor$ ) depicted in Figure 3, is the graph with maximal index in the set  $C(n)$  of all connected cactuses with  $n$  vertices. This completes the proof.  $\square$

**COROLLARY 3.1.** *The graph  $G_k$  depicted in Figure 2 is the graph with maximal index in the set of all connected cactuses with  $n$  vertices and  $k$  cycles ( $1 \leq k \leq \lfloor \frac{n-1}{2} \rfloor$ ).*

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