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On the inductive definition with quantifiers of second order

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§ 1. Introduction.

In a former paper [4], the author defined the system of ordinal diagrams and proved that the system is well-ordered. By using ordinal diagrams he proved in [5] the consistency of a fairly impredicative theory. The theory developed in [4] was generalized in [6]. In this paper we shall generalize the result of [5] by using [6] and show the consistency of a theory which has inductive definitions with quantifiers of second order.

Let $I(a)$ and $a <^* b$ be two primitive recursive predicates. Let us assume that the following condition is satisfied:

$<^*$ is a well-ordering of I, where I is $\{a \mid I(a)\}$.

Now we shall consider the formal system obtained as follows from G^1LC. (G^1LC is a simple type theory of second order as defined in [2], [3].)

1. Every beginning sequence is of the form $D \rightarrow D$ or of the form $a=b$, $A(a) \rightarrow A(b)$ or the 'mathematische Grundsequenz' in Gentzen [1] or the following form

$$I(a),\ A_i(a,b) \rightarrow G_i(a, b\ \{x,y\}(A_i(x,y) \wedge x <^* a))$$

$$I(a),\ G_i(a,b,\{x,y\}(A_i(x,y) \wedge x <^* a)) \rightarrow A_i(a,b) \qquad i=0,1,2,\cdots.$$

Here $\{x,y\}$ is used instead of usual notations $\hat{x}\hat{y}$, λxy and $A_0, A_1, A_2, \cdots$ are new symbols for predicates. Moreover, G_i $(i=0,1,2,\cdots)$ are arbitrary formulas satisfying the following conditions:

a) $G_i(a,b,\alpha)$ does not contain $A_i, A_{i+1}, A_{i+2}, \cdots$.

b) If $G_i(a,b,\alpha)$ contains the figures of the form $\forall\varphi A(\varphi)$ or $\exists\varphi A(\varphi)$, then $A(\beta)$ does not contain any bound f-variable. (The bound f-variable means the quantifier of second order.)

2. The following inference-schema called 'induction' is added.

$$\frac{A(a),\ \Gamma \rightarrow \Delta,\ A(a')}{A(0),\ \Gamma \rightarrow \Delta,\ A(t)}$$

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where t is an arbitary term and a is contained in none of $A(0)$, Γ, Δ.

3. The inference $\forall$ left and $\exists$ right on an f-variable of the form

$$\frac{F(V),\ \Gamma\rightarrow\Delta}{\forall\varphi F(\varphi),\ \Gamma\rightarrow\Delta} \quad\text{and}\quad \frac{\Gamma\rightarrow\Delta,\ F(V)}{\Gamma\rightarrow\Delta,\ \exists\varphi F(\varphi)}$$

are restricted by the condition that $F(\alpha)$ does not contain any bound f-variable. It should be remarked that $F(\alpha)$ may contain $A_0, A_1, A_2, \cdots$ and V may contain bound f-variables and $A_0, A_1, A_2, \cdots$.

Then the following theorem holds.

THEOREM. *The consistency of our system can be proved by using the well-ordering of the system of the ordinal diagrams of*

$$((2\cdot I)\cdot\omega+1)^{\omega}\times N^2+2$$

and

$$((2\cdot I)\cdot\omega+1)^{\omega}\times N^2.$$

Here N denotes the set of integers. The symbols ω, $\times$ etc. have the ordinary meanings, the exact definitions of which will be given in 2.

REMARK. The transfinite induction over I is provable in our system. Let $J(a)$ and $D(a,\alpha)$ be the abbreviations of $\forall\varphi(\forall x(I(x)\wedge\forall y(y<^*x\vdash\varphi[y])\vdash\varphi[x])\vdash\varphi[a])$ and $\forall x(x<^*a\vdash\alpha[x])\vdash J(a)$ respectively. The following sequences are beginning sequence of our system:

$$I(i),\ C(i)\rightarrow D(i,\ \{x\}(C(x)\wedge x<^*i));$$

$$I(i),\ D(i,\ \{x\}(C(x)\wedge x<^*i))\rightarrow C(i).$$

We see easily that the following sequences are provable in our system:

$$\forall x(x<^*i\vdash C(x)),\ j<^*i\rightarrow C(j)\wedge\forall y(y<^*j\vdash C(y));$$

$$j<^*i,\ C(j)\wedge\forall y(y<^*j\vdash C(y))\rightarrow J(j).$$

From above two sequences we have

$$\forall x(x<^*i\vdash C(x)),\ j<^*i\rightarrow J(j).$$

On the other hand, we see easily that the following sequence is provable in our system:

$$I(i),\ \forall x(x<^*i\vdash J(x))\rightarrow J(i).$$

Thus we have

$$I(i),\ \forall x(x<^*i\vdash C(x))\rightarrow J(j).$$

From this and our beginning sequence we have

$$I(i)\rightarrow C(i),$$

and then

$$I(i)\rightarrow J(i)$$

which states the transfinite induction over I.

§ 2. Consistency proof of our system.

1. First we define a system of ordinal diagrams on which our proof is based.
1.1. $\tilde{I}$ is defined to be $\{j \mid j \in I$ or j is of the form $\tilde{i}$ where $i \in I\}$. $<_*$ is a well ordering of $\tilde{I}$ which is defined as follows:
1.1.1. If $i \in I$, then $i <_* \tilde{i}$.
1.1.2. If $i <^* j$, then $\tilde{i} <_* \tilde{j}$.
1.1.3. If $i <^* j$, then $i <_* \tilde{j}$.
1.2. Let n be an integer. I_n is defined to be $\{(i, n) \mid i \in \tilde{I}\}$. $<_n$ is a well-ordering of I_n which is defined as follows:
1.2.1. If $i <_* j$, then $(i, n) <_n (j, n)$.
1.3. I_∞ is defined to be $\{\infty\} \cup I_0 \cup I_1 \cdots$.
$<_\infty$ is a well-ordering of I_∞ defined as follows:
1.3.1. If $i \in I_n$, then $i <_\infty \infty$ $(n = 0, 1, 2, \cdots)$,
1.3.2. If $i \in I_n$, $j \in I_m$ and $n < m$, then $i <_\infty j$.
1.3.3. If $i <_n j$, then $i <_\infty j$.
1.4. $\hat{I}$ is defined to be a set consisting of elements of the form

$$[i_0, i_1, \cdots, i_n;\ k_1, k_2],$$

where $i_0, i_1, \cdots, i_n$ are elements of I_∞, $i_0 \geqq_\infty i_1 \geqq_\infty \cdots \geqq_\infty i_n$, and n, k_1, k_2 are integers. $\tilde{<}$ is a lexicographical well-ording of $\hat{I}$.
1.5. To prove the theorem, we assume that there are proof-figures to the sequence $\rightarrow$ in our system, and we assign an o. d. (ordinal diagram) of $O(\{\infty_1, \infty_2\} \cup \hat{I}, \hat{I})$, where ∞_1 and ∞_2 are the maximal elements of $\{\infty_1\} \cup \hat{I}$ and $\{\infty_1, \infty_2\} \cup \hat{I}$ respectively, to each of these proof-figures and define the reduction similarly as in [**1**] and in [**5**].

2. Let $\mathfrak{P}$ be a proof-figure in our system.
2.1. The degree of A_n contained in $\mathfrak{P}$ is defined as follows:
2.1.1. If A_n is contained in $\mathfrak{P}$ and is of the form $A_n(j, b)$, where $j \in I$, then the degree of A_n is $(\tilde{j}, n)$.
2.1.2. If A_n is contained in $\mathfrak{P}$ and is of the form $A_n(x, b) \wedge x <^* i$, where x is a variable or else '$\neg I(x)$ or $i \leqq^* x$' is probable, then the degree is A_n is (i, n).
2.1.3. If A_n is contained in $\mathfrak{P}$ and is of the form $A_n(x, b)$ with $x \notin I$ and not of the form $A_n(x, b) \wedge x <^* i$, then the degree of A_n is ∞.
2.2. We define the degree of a formula F in $\mathfrak{P}$ to be

$$[i_0, i_1, \cdots, i_n;\ k_1, k_2],$$

where $i_0, i_1, \cdots, i_n$ are the non-increasing series consisting of all the degrees of A_m contained in F $(m = 0, 1, 2, \cdots)$, k_1 is the number of $\forall$'s on f-variable in F and k_2 is the number of logical symbols except $\forall$ on an f-variable in F.
2.3. We add the inference 'substitution' with the following restriction in our

system (cf. [**5**]).

2.3.1. To every substitution is attached an element of $\hat{I}$, which is called the degree of the substitution, satisfying the following condition: The degree of every implicit formula in the upper sequence of a substitution is less than that of the substitution.

2.3.3. The eigenvariable of a substitution is not tied by any $\forall$ on an f-variable in the upper sequence of the substitution.

2.3.4. If an implicit formula in the upper sequence of a substitution contains $\forall$ on an f-variable which ties a free f-variable, then it is $\forall$ right in the concerned proof-figure and the degree of the substitution is ∞_1.

2.4. Let $i \in \hat{I}$, $\mathfrak{P}$ be a proof-figure and $\mathfrak{S}$ be a sequence in $\mathfrak{P}$. The i-loader of $\mathfrak{S}$ is the upper sequence of the uppermost substitution under $\mathfrak{S}$, whose degree is not greater than i in $\{\infty_1, \infty_2\} \cup \hat{I}$, if such exsists; otherwise the i-loader of $\mathfrak{S}$ is the end-sequence.

2.5. Now we assign an o. d. of $O(\{\infty_1, \infty_2\} \cup \hat{I}, \hat{I})$ to every sequence of a proof-figure recursively as follows:

2.5.1. The o. d. of a beginning sequence of the form $D \to D$ is the degree of D.

2.5.2. The o. d. of a beginning sequence of the form $a=b,\ A(a) \to A(b)$ is the degree of $A(a)$.

2.5.3. The o. d. of a beginnnig sequence of the form

$$I(i),\ A_n(i, a) \to G_n(i, a, \{x, y\}(A_n(x, y) \wedge x <^* i))$$

or $$I(i),\ G_n(i, a, \{x, y\}(A_n(x, y) \wedge x <^* i)) \to A_n(i, a)$$

is the degree of $A_n(i, a)$.

2.5.4. If $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are the upper sequence and the lower sequence of an inference on structure, then the o. d. of $\mathfrak{S}_2$ is equal to that of $\mathfrak{S}_1$.

2.5.5. If $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are the upper sequence and the lower sequence of an inference $\neg$, $\wedge$ left, $\forall$ on a t-variable or $\forall$ right on an f-variable respectively, then the o. d. of $\mathfrak{S}_2$ is $(\infty_2, 0, \sigma)$ where 0 denotes the first element of $\hat{I}$ and σ is the o. d. of $\mathfrak{S}_1$.

2.5.6. If $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are the upper sequences and $\mathfrak{S}$ is the lower sequence of an inference $\wedge$ right, then the o. d. of $\mathfrak{S}$ is $(\infty_2, 0, \sigma_1 \# \sigma_2)$, where σ_1 and σ_2 are the o. d.'s of $\mathfrak{S}_1$ and $\mathfrak{S}_2$ respectively.

2.5.7. If $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are the upper sequence and the lower sequence of an $\forall$ left $\mathfrak{J}$ on an f-variable respectively, then the o. d. of $\mathfrak{S}_2$ is

$$(\infty_2, [i_0, \cdots, i_m;\ k_1, k_2+2], \sigma),$$

where $[i_0, \cdots, i_m;\ k_1, k_2]$ is the degree of the subformula of $\mathfrak{J}$ and σ is the o. d. of $\mathfrak{S}_1$.

2.5.8. If $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are the upper sequences and $\mathfrak{S}$ is the lower sequence of a cut $\mathfrak{J}$, then the o. d. of $\mathfrak{S}$ is $(\infty_2, [i_0, \cdots, i_m;\ k_1, k_2+1], \sigma_1 \# \sigma_2)$, where σ_1 and σ_2 are the o. d.'s of $\mathfrak{S}_1$ and $\mathfrak{S}_2$ respectively and $[i_0, \cdots, i_m;\ k_1, k_2]$ is the degree of the cut-formula of $\mathfrak{J}$.

2.5.9. If $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are the upper sequence and the lower sequence of a substitution with the degree i respectively, then the o. d. of $\mathfrak{S}_2$ is $(i, 0, \sigma)$ where σ is the o. d. of $\mathfrak{S}_1$,

2.5.10. If $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are the upper sequence and the lower sequence of an induction, then the o. d. of $\mathfrak{S}_2$ is $(\infty_2, [i_0, \cdots, i_m; k_1, k_2+2], \sigma)$, where σ is the o. d. of $\mathfrak{S}_1$ and $[i_0, \cdots, i_m; k_1, k_2]$ is the degree of $A(a)$ in the schema.

The ordinal diagram of the end-sequence of a proof-figure is called the ordinal diagram of the proof-figure.

3. Suppose that the sequence $\rightarrow$ is provable in our system. In the following, we shall reduce a proof-figure $\mathfrak{P}$ to $\rightarrow$ to a proof-figure with the o. d. less than that of $\mathfrak{P}$. Without loss of generality, we may assume that every free variable used as an eigenvariable in a proof-figure is different from each other. Let $\mathfrak{P}$ be a proof-figure to $\rightarrow$.

3.1. First we substitute 0 for every free variable in $\mathfrak{P}$ except in case it is used as an eigenvariable. In this alteration the proof-figure is still correct and the end-sequence of $\mathfrak{P}$ and the o. d. of $\mathfrak{P}$ are invariable.

3.2. We may assume that $\mathfrak{P}$ contains no free variable other than those used as an eigenvariable in $\mathfrak{P}$. If the end-place of $\mathfrak{P}$ contains an induction, apply the '*VJ*-Reduktion' in Gentzen [**1**], where every substitution in the reduced proof-figure has the same degree as the corresponding one in $\mathfrak{P}$.

3.3. In the following we may assume besides the condition assumed in 3.2, that the end-place of $\mathfrak{P}$ contains no induction. Let the end-place of $\mathfrak{P}$ contain a beginning sequence of the form $m=n,\ A(m)\rightarrow A(n)$, where m and n are of the form $0'^{\cdots\prime}$. Then either $m=n\rightarrow$ or $\rightarrow m=n$ is a 'mathematische Grundsequenz.'

3.3.1. If $m=n\rightarrow$ is a 'mathematische Grundsequenz,' replace the beginning sequence to the proof-figure

$$\cfrac{\cfrac{m=n\rightarrow}{\text{Weakenings and an exchange}}}{m=n,\ A(m)\rightarrow A(n)\,.}$$

3.3.2. If $\rightarrow m=n$ is a 'mathematische Grundsequenz,' then m is n. Replace the beginning sequence by the proof-figure

$$\frac{A(m)\rightarrow A(n)}{m=n,\ A(m)\rightarrow A(n)\,.}$$

3.4. We may assume besides the conditions assumed in 3.3, that the end-place of $\mathfrak{P}$ contains no beginning sequence of the form $m=n,\ A(m)\rightarrow A(n)$. We can reduce $\mathfrak{P}$ to a proof-figure which contains no beginning sequence of the form $D\rightarrow D$ in the end-place in the same way as in [**5**].

3.5. Then we can remove a weakening cut-formula in the end-place in the same way as in [3, 6.4].

3.6. We may assume that the end-place of $\mathfrak{P}$ contains no free variable, induction, beginning sequence of the form $m=n$, $A(m)\rightarrow A(n)$, or $D\rightarrow D$, or weakening cut-formula. Suppose that $\mathfrak{P}$ contains a beginning sequence of the form

(*) $\quad I(i),\ A_n(i, a)\rightarrow G_n(i, a, \{x, y\}(A_n(x, y)\wedge x<^* i))$

or $\quad I(i),\ G_n(i, a, \{x, y\}(A_n(x, y)\wedge x<^* i))\rightarrow A_n(i, a)$,

where i and a are of the form $0'^{\cdots}{}'$, and n is an integer. Since each case is treated similarly, we treat here only the case that $\mathfrak{P}$ contains a beginnig sequence (*). By our assumption, either $I(i)\rightarrow$ or $\rightarrow I(i)$ is provable without an induction, or a substition or an $\forall$ on an f-variable.

3.6.1. In case that $I(i)\rightarrow$ is provable, replace the beginning sequence by the following proof-figure:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ I(i)\rightarrow\end{array}}{\text{Weakenings and an exchange}}}{I(i),\ A_n(i, a)\rightarrow G_n(i, a, \{x, y\}(A_n(x, y)\wedge x<^* i))}\,.$$

The ordinal diagram of the above proof-figure is less than that of (*). In the same way as in [**5**], we see that the ordinal diagram of the proof-figure to $\rightarrow$ decreases by this alteration.

3.6.2. The case that $\rightarrow I(i)$ is provable: Since every formula in $\mathfrak{P}$ is implicit, there exists a cut $\mathfrak{J}$ where one of the cut-formulas of $\mathfrak{J}$ is a descendant of $A_n(i, a)$ in (*). Let $\mathfrak{P}$ be of the following form:

$$\begin{array}{c}\dfrac{\begin{array}{c}A_n(i, a)\rightarrow A_n(i, a)\\ \vdots\\ \Gamma\overset{\sigma_1}{\rightarrow}\Delta,\ A_n(i, a)\end{array}\qquad\begin{array}{c}I(i),\ A_n(i, a)\overset{(*)}{\rightarrow}G_n(i, a, A_n{}^i)\\ \vdots\\ A_n(i, a),\ \Pi\overset{\sigma_2}{\rightarrow}\Lambda\end{array}}{\Gamma, \Pi\rightarrow\Delta, \Lambda}\,\mathfrak{J}\\ \vdots\\ \rightarrow\end{array}$$

where $A_n{}^i$ is the abbreviation of $\{x, y\}(A_n(x, y)\wedge x<^* i)$, and $A_n(i, a)\rightarrow A_n(i, a)$ in the figure may not appear.

Consider the following proof-figure $\mathfrak{P}'$:

$$\dfrac{\begin{array}{c}\vdots\\ \xrightarrow{\tau} I(i)\end{array} \qquad \dfrac{\dfrac{\dfrac{\begin{array}{c}\dfrac{I(i),\ A_n(i, a)\to G_n(i, a, A_n^{i})}{A_n(i, a),\ I(i)\to G_n(i, a, A_n^{i})}\\ \vdots\\ \Gamma,\ I(i)\xrightarrow{\sigma_1}\Delta,\ G_n(i, a, A_n^{i})\end{array} \qquad \begin{array}{c} G_n(i, a, A_n^{i})\xrightarrow{(*')}G_n(i, a, A_n^{i})\\ \vdots \\ G_n(i, a, A_n^{i}),\ \Pi\xrightarrow{\sigma_2'}\Lambda\end{array}}{\Gamma,\ I(i),\ \Pi\to\Delta,\ \Lambda}\,\mathfrak{J}'}{\text{Some exchanges}}}{I(i),\ \Gamma,\ \Pi\to\Delta,\ \Lambda}}{\begin{array}{c}\Gamma,\ \Pi\xrightarrow{\sigma'}\Delta,\ \Lambda\\ \vdots\\ \to .\end{array}}$$

Every substitution in $\mathfrak{P}'$ has the same degree as the corresponding one in $\mathfrak{P}$. $\sigma=(\infty_2,[(\tilde{i}, n); 0, 0], \sigma_1 \# \sigma_2)$ and $\sigma'=(\infty_2,[0; 0, k], \tau\#(\infty_2, [(i, n), \cdots; k_1, k_2], \sigma_1\#\sigma_2'))$ where $[0; 0, k]$ and $[(i, n), \cdots; k_1, k_2]$ denote the degrees of $I(i)$ and $G_n(i, a, A_n^{i})$ respectively. By an analogous method as in [**5**], we see that $\sigma' <_0 \sigma$. Then the ordinal diagram of $\mathfrak{P}'$ is less than that of $\mathfrak{P}$.

4. Now we may assume that the end-place of $\mathfrak{P}$ contains no free variable, induction, weakening cut-formula or beginning sequence except a 'mathematische Grundsequenz.' If all the sequences in $\mathfrak{P}$ are in the end-place, we treat in the same way as in [**1**]. Then we may assume that $\mathfrak{P}$ contains an inference on logical symbols and that the end-place contains a suitable cut in the same way as in [**3**, § 6]. To define the essencial reduction, we treat separately several cases according to the form of the outermost logical symbol of the cut-formulas of the suitable cut of $\mathfrak{P}$. Since other cases are treated in the same way as in [**5**], we treat here only the case that the outermost logical symbol of the suitable cut $\mathfrak{J}$ of $\mathfrak{P}$ is $\forall$ on an f-variable.

Thus let $\mathfrak{P}$ be of the following form:

$$\dfrac{\begin{array}{c}\vdots\\ \dfrac{\Gamma_1\to\Delta_1,\ F(\alpha)}{\Gamma_1\to\Delta_1,\ \forall\varphi F(\varphi)}\\ \vdots\\ \Gamma_2\to\Delta_2,\ \forall\varphi\tilde{F}(\varphi)\end{array} \qquad \begin{array}{c}\vdots\\ \dfrac{F_1(V),\ \Pi_1\to\Lambda_1}{\forall\varphi F_1(\varphi),\ \Pi_1\to\Lambda_1}\\ \vdots\\ \forall\varphi\tilde{F}(\varphi),\ \Pi_2\to\Lambda_2\end{array}}{\begin{array}{c}\Gamma_2,\ \Pi_2\to\Delta_2,\ \Lambda_2\\ \vdots\\ \Gamma_3\to\Delta_3\\ \vdots\\ \to .\end{array}}\,\mathfrak{J}$$

Let $[i_0, \cdots, i_n; k_1, k_2]$ be the degree of $\tilde{F}(\alpha)$. Let i mean ∞_1 or $[i_0, \cdots, i_n; k_1, k_2+1]$

according as $\forall\varphi\tilde{F}(\varphi)$ contains a free f-variable or not. Let $\Gamma_3 \rightarrow \Delta_3$ be the i-loader of $\Gamma_2, \Pi_2 \rightarrow \Delta_2, \Lambda_2$. We can prove easily that $\forall\varphi F_1(\varphi)$ is $\forall\varphi\tilde{F}(\varphi)$.

A reduction $\mathfrak{P}'$ of $\mathfrak{P}$ is given in the following form:

$$\vdots$$

$$\dfrac{\Gamma_1 \rightarrow \Delta_1, F(\alpha)}{\Gamma_1 \rightarrow F(\alpha), \Delta_1, \forall\varphi F(\varphi)} \text{ Some exchanges and a weakening}$$

$$\vdots$$

$$\dfrac{\dfrac{\Gamma_3 \rightarrow \Delta_3, \tilde{F}(\alpha)}{\Gamma_3 \rightarrow \Delta_3, \tilde{F}(V)}\,\mathfrak{I}_1 \qquad \overset{\vdots}{\tilde{F}(V), \Pi_1 \rightarrow \Lambda_1}}{\dfrac{\Gamma_3, \Pi_1 \rightarrow \Delta_3, \Lambda_1}{\forall\varphi\tilde{F}(\varphi), \Pi_1, \Gamma_3 \rightarrow \Delta_3, \Lambda_1} \text{ Some exchanges and a weakening}}$$

$$\dfrac{\overset{\vdots}{\Gamma_2 \rightarrow \Delta_2, \forall\varphi\tilde{F}(\varphi)} \qquad \overset{\vdots}{\forall\varphi\tilde{F}(\varphi), \Pi_2, \Gamma_3 \rightarrow \Delta_3, \Lambda_2}}{\dfrac{\Gamma_2, \Pi_2, \Gamma_3 \rightarrow \Delta_2, \Delta_3, \Lambda_2}{\Gamma_2, \Pi_2, \Gamma_3 \rightarrow \Delta_3, \Delta_2, \Lambda_2} \text{ Some exchanges}}$$

$$\vdots$$

$$\dfrac{\Gamma_3, \Gamma_3 \rightarrow \Delta_3, \Delta_3}{\Gamma_3 \rightarrow \Delta_3} \text{ Some exchanges and contractions}$$

$$\vdots$$

$$\rightarrow$$

where $\mathfrak{I}_1$ is a substitution whose eigenvariable is α and whose degree is defined to be i. Every substitution in $\mathfrak{P}'$ except $\mathfrak{I}_1$ has the same degree as the corresponding one in $\mathfrak{P}$. Following 4.2 in [**5**], we can show that substitutions in $\mathfrak{P}'$ satisfy 1.3.1-2.3.4. and that the ordinal diagram of $\mathfrak{P}'$ is less than that of $\mathfrak{P}$.

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