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On the infinite divisibility of inverse Beta distributions

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We show that all negative powers $\beta_{a,b}^{-s}$ of the Beta distribution are infinitely divisible. The case $b \leq 1$ follows by complete monotonicity, the case $b > 1, s \geq 1$ by hyperbolically complete monotonicity and the case $b > 1, s < 1$ by a Lévy perpetuity argument involving the hypergeometric series. We also observe that $\beta_{a,b}^{-s}$ is self-decomposable if and only if $2a + b + s + bs \geq 1$, and that in this case it is not necessarily a generalized Gamma convolution. On the other hand, we prove that all negative powers of the Gamma distribution are generalized Gamma convolutions, answering to a recent question of L. Bondesson.

Keywords: Beta distribution, Gamma distribution, Generalized Gamma convolution, Hyperbolically complete monotonicity, Hypergeometric series, Lévy perpetuity, Self-decomposability, Stieltjes transform.

1. Introduction and statement of the results

This paper is a sequel to our previous article Bosch and Simon (2013), where we have established the infinite divisibility of all negative powers γ_a^{-s} of the Gamma distribution γ_a, with density

$$\frac{x^{a-1}}{\Gamma(a)} e^{-x} \mathbf{1}_{(0,+\infty)}(x).$$

More precisely, in Bosch and Simon (2013) we had completed the already known situation $s \geq 1$ by an argument involving the exponential functional of a spectrally negative Lévy process, valid in the case $s \leq 1$ and in this case only. We consider here the same problem for the Beta distribution $\beta_{a,b}$, with density

$$\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1} \mathbf{1}_{(0,1)}(x).$$

Recall that a positive random variable X is infinitely divisible, which we will denote by $X \in \mathcal{I}$, if and only if its Laplace exponent $\varphi(\lambda) = -\log \mathbb{E}[e^{-\lambda X}]$ is a Bernstein function, viz.

$$\varphi(\lambda) \;=\; a\lambda \;+\; \int_0^\infty (1 - e^{-\lambda x})\,\nu(dx)$$

with $a \geq 0$ (the drift coefficient) and ν a non-negative measure on $(0, +\infty)$ (the Lévy measure) whose integral along $1 \wedge x$ is finite. When ν is absolutely continuous with a density of the type $x^{-1}k(x)$ for some non-increasing function k, this means that X is self-decomposable ($X \in \mathcal{S}$ for short) in other words that for all $c \in (0,1)$ there is a decomposition

$$X \;\stackrel{d}{=}\; cX \;+\; X_c,$$

where X_c is an independent random variable. The distribution of X is called a generalized Gamma convolution, which we will denote by $X \in \mathcal{G}$, when X is self-decomposable and its above spectral function k is completely monotone (CM). From the probabilistic point of view, the $\mathcal{G}-$property means that X can be written as a Wiener-Gamma integral, that is the improper integral of some deterministic function along the Gamma subordinator - see Proposition 1.1 in James et al. (2008). From the analytical viewpoint, the complete monotonocity of k means that the Laplace exponent of X is a Thorin-Bernstein function. We refer to Bondesson (1992), James et al. (2008), Sato (1999), Steutel and Van Harn (2003) for various accounts on infinite divisibility, self-decomposability and generalized Gamma convolutions, and to the recent monograph Schilling et al. (2010), which is devoted to Bernstein functions.

The positive powers of $\beta_{a,b}$ cannot be infinitely divisible because of their bounded support - see Theorem 24.3 in Sato (1999). On the other hand, it is well-known - see Example VI.12.21 in Steutel and Van Harn (2003) and the proof of Theorem 1 thereafter for details - that $-\log(\beta_{a,b}) \in \mathcal{I}$ with an explicit Lévy measure. In the present paper, it will be shown that all negative powers of $\beta_{a,b}$ belong to $\mathcal{I}$. This allows to retrieve the $\mathcal{I}-$property for

$$-\log(\beta_{a,b}) \;=\; \lim_{s\to 0+} s^{-1}(\beta_{a,b}^{-s} - 1), \tag{1.1}$$

and also for all the negative powers γ_a^{-s} in view of the convergence in law

$$b^{-s}\beta_{a,b}^{-s} \;\stackrel{d}{\longrightarrow}\; \gamma_a^{-s} \qquad \text{as } b \to +\infty. \tag{1.2}$$

The infinite divisibility of $\beta_{a,b}^{-s}$ is equivalent to that of $\beta_{a,b}^{-s} - 1$, whose support is $\mathbb{R}^+$. In the case $s = 1$, the latter random variable is known as the Beta random variable of the second kind, and its infinite divisibility appears in the list of examples of Appendix B in Steutel and Van Harn (2003). So far, the problem of infinite divisibility for other negative powers of the Beta distribution seems to have escaped investigation. Having independent interest, these random variables appear as multiplicative factors because of their moments of the Gamma type - see Janson (2010). For instance, they are connected to real stable densities via the Kanter random variable - see (2.4) and (7.1) in Simon (2014). It was conjectured in Jedidi and Simon (2013) that $\beta_{a,b}^{-s} \in \mathcal{G}$ for all $s \geq 1$ - see Conjecture 3.2. therein.

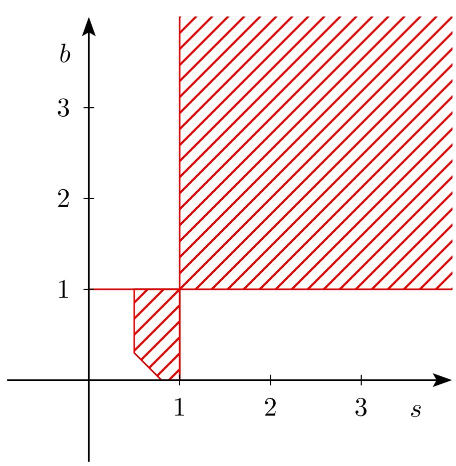
It does not seem possible to express the Laplace transform of $\beta_{a,b}^{-s}$ or $\beta_{a,b}^{-s} - 1$ in a sufficiently explicit way in order to show that it is a Bernstein function. We will hence proceed via different methods, characterizing properties which are more informative than the sole infinite divisibility. Let us first introduce the class $\mathcal{M}$ of positive random variables having a CM density on $(0, +\infty)$. This class is included in $\mathcal{I}$ by Goldie's criterion - see e.g. Theorem 51.6 in Sato (1999).

Theorem 1. *One has $\beta_{a,b}^{-s} - 1 \in \mathcal{M}$ if and only if $b \leq 1$.*

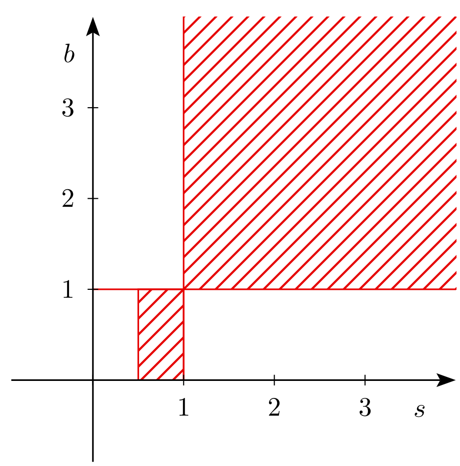
Second, let us consider the class $\mathcal{H}$ of positive random variables having a hyperbolically completely monotone (HCM) density on $(0, +\infty)$. A function $f : (0, +\infty) \to (0, +\infty)$ is said to be HCM if for every $u > 0$ the function $f(uv)f(u/v)$ is CM in the variable $v + 1/v$. The fact that $\mathcal{H} \subset \mathcal{G}$ is a key-result in showing the infinite divisibility of a probability measure on $\mathbb{R}^+$ with explicit density but without explicit Laplace transform. It allows to prove the $\mathcal{I}-$property of many classical or less classical positive distributions, for which sometimes no other kind of argument is known. We refer to Chapters 4-6 in Bondesson (1992) for a complete account, and to Theorem 5.1.2 therein for a proof of the inclusion $\mathcal{H} \subset \mathcal{G}$.

Theorem 2. *One has $\beta_{a,b}^{-s} - 1 \in \mathcal{H}$ if and only if one of the three following conditions is verified.*

1. $b \wedge s > 1$.
2. $b = 1$ *or* $s = 1$.
3. $b < 1, s \in [1/2, 1)$ *and* $a + b + s \geq 1$.



The case $a < 1/2$.



The case $a \geq 1/2$.

So far, we can deduce that $\beta_{a,b}^{-s} \in \mathcal{I}$ if $b \leq 1$ or $s \geq 1$. In order to handle the situation $b > 1, s < 1$, let us introduce the class of Lévy perpetuities

$$\mathcal{E} \;=\; \left\{ I(Z) = \int_0^\infty e^{-Z_s}\, ds,\;\; Z \;=\; \{Z_t,\, t \geq 0\} \text{ is a Lévy process with } Z_t \to +\infty \text{ a.s.} \right\}.$$

Recall that $I(Z)$ is an a.s. convergent integral if and only if Z drifts towards $+\infty$ - see Theorem 1 in Bertoin and Yor (2005). Observe also that $I(Z)$ has bounded support when Z is a subordinator with positive drift, and hence may not be in $\mathcal{I}$. However, more can be said when Z is spectrally negative (Z is an SNLP for short), in other words that it has no positive jumps. Introduce the subclass

$$\mathcal{E}_- \;=\; \left\{ I(Z) = \int_0^\infty e^{-Z_s}\, ds,\;\; Z \;=\; \{Z_t,\, t \geq 0\} \text{ is an SNLP with positive mean} \right\}.$$

Recall - see the introduction to Chapter 7 in Bertoin (1996) - that an SNLP is characterized by its moment generating function, whose logarithm reads

$$\log \mathbb{E}[e^{\lambda Z_1}] \;=\; a\lambda \,+\, b\lambda^2 \,+\, \int_{-\infty}^0 (e^{\lambda x} - 1 - \lambda x)\, \nu(dx), \quad \lambda \geq 0,$$

with $a \in \mathbb{R}, b \in \mathbb{R}^+$ and ν a non-negative measure on $(-\infty, 0)$ whose integral along $1 \wedge x^2$ is finite, and that an SNLP drifts towards $+\infty$ if and only if it has a positive mean. The fact that $\mathcal{E}_- \subset \mathcal{S}$ follows from the Markov property at the a.s. finite stopping time $T_x = \inf\{t > 0, Z_t = x\}$ with $x > 0$, which implies the decomposition

$$I(Z) \;\stackrel{d}{=}\; e^{-x} I(Z) \;+\; \int_0^{T_x} e^{-\tilde{Z}_s} ds,$$

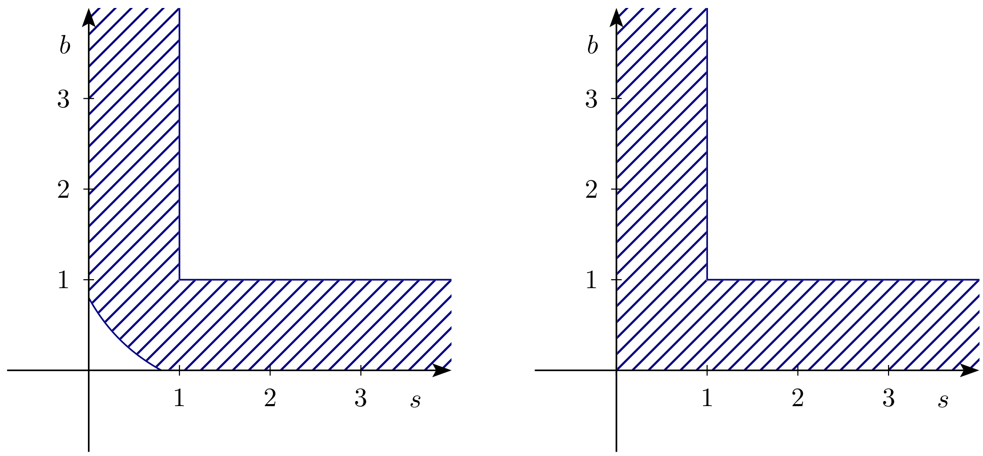
where $\tilde{Z}$ is an independent copy of Z. This observation, which is folklore and may be traced back to Vervaat (1979) in a more general framework, was used in Bosch and Simon (2013) to prove the self-decomposability of $\gamma_a^{-s}, s < 1$.

Theorem 3. *One has*

$$\beta_{a,b}^{-s} \in \mathcal{E}_- \iff b \wedge s \;\leq\; 1 \;\leq\; 2a + b + s + bs.$$

As for Theorem 2, the domain characterizing the $\mathcal{E}_-$−property takes a different shape according as $a < 1/2$ or $a \geq 1/2$ - see the following figure. Combining the three above theorems and an asymptotic analysis which will be performed in the next section, entails the main result of the present paper.

Corollary. *One has $\beta_{a,b}^{-s} \in \mathcal{I}$ for all $a, b, s > 0$. Moreover, $\beta_{a,b}^{-s} \in \mathcal{S}$ if and only if* $2a + b + s + bs \geq 1$.



The case $a < 1/2$. The case $a \geq 1/2$.

Notice that by (1.1), the second statement of the corollary allows to recovers the criterion for $-\log(\beta_{a,b}) \in \mathcal{S}$ which is known to be $2a + b \geq 1$ - see again Example VI.12.21 in Steutel and Van Harn (2003). Towards the end of this paper we will observe that there are situations where $\beta_{a,b}^{-s} \in \mathcal{S} \cap \mathcal{G}^c$. It is also worth mentioning that the set of parameters where $\beta_{a,b}^{-s} \in \mathcal{E}_- \cap \mathcal{H}$, which is characterized by the cases 2. and 3. of Theorem 2, is thicker than the set of parameters for $\gamma_a^{-s} \in \mathcal{E}_- \cap \mathcal{H}$ which is the only line $\{s = 1\}$ - see Remark 1 (c) and Section 3.2 in Bosch and Simon (2013).

Our previous paper Bosch and Simon (2013) had left unanswered the question whether $\gamma_a^{-s} \in \mathcal{G}$ for $s < 1$. This problem was motivated by the fact that $\gamma_a^{-s} \in \mathcal{H}$ for $s \geq 1$ but not for $s < 1$ since otherwise γ_a^s, which is not infinitely divisible, would be also in $\mathcal{H}$ - see Bosch and Simon (2013) for details and references. This question is also mentioned as an open problem in Section 4 (vi) of Bondesson (2014). Our last result provides a positive answer.

Theorem 4. *One has $\gamma_a^{-s} \in \mathcal{G}$ for all $a, s > 0$.*

The proof of this theorem is actually a simple consequence of the main result of Bondesson (2014), which states the important property that the class $\mathcal{G}$ is stable by independent multiplication. Let us also give a few words about the proofs of the three other theorems. For Theorem 1, we use Steutel's characterization of exponential mixtures and a logarithmic transformation. The proof of Theorem 2, which is not as immediate as for the HCM characterization of γ_a^{-s}, relies upon Stieltjes transforms and the maximum principle for harmonic functions. For Theorem 3 we appeal to Bertoin-Yor's characterization of $\mathcal{E}_-$, as well as several properties of the classical hypergeometric series, old ones from Klein (1890) and recent ones from Anderson et al. (2007). All these proofs are given in the next section. We conclude the paper with several remarks.

2. Proofs

2.1. Proof of Theorem 1

The density of $\beta_{a,b}^{-s} - 1$ reads

$$f_{a,b,s}(x) \;=\; \frac{\Gamma(a+b)}{s\Gamma(a)\Gamma(b)}\,(x+1)^{\frac{1-a-b}{s}-1}((x+1)^{\frac{1}{s}}-1)^{b-1}\mathbf{1}_{(0,+\infty)}(x)$$

and is not log-convex if $b > 1$ because

$$(\log f_{a,b,s})'(x) \;=\; \frac{1}{s(x+1)}\left(\frac{b-1}{(x+1)^{\frac{1}{s}}-1} \,-\, a \,-\, s\right) \;\sim\; \frac{b-1}{x} \quad \text{as } x \to 0.$$

By the Schwarz inequality - see the proof of Theorem 51.6 in Sato (1999), this shows the only if part. The same computation shows that $f_{a,b,s}$ is log-convex for $b \le 1$, which is known to be sufficient for infinite divisibility - see Theorem 51.4 in Sato (1999). However, it does not seem easy to show directly the reinforcement that $f_{a,b,s}$ is actually CM whenever $b \le 1$.

To do so, we first observe that by the proof of Corollary 3.2. in Jedidi and Simon (2013), we have $X \in \mathcal{M} \Rightarrow e^X - 1 \in \mathcal{M}$ for every positive random variable X. Hence, it is enough to show that $-\log(\beta_{a,b}) \in \mathcal{M}$. This latter property follows from Steutel's theorem as had already been noticed in Example VI.12.21 in Steutel and Van Harn (2003), but we will sketch the argument for the sake of completeness. We first compute, using Malmsten's formula for the Gamma function - see e.g. Formula 1.9(1) p. 21 in Erdélyi et al. (1953),

$$\begin{aligned}\mathbb{E}[e^{-\lambda(-\log(\beta_{a,b}))}] \;=\; \mathbb{E}[\beta_{a,b}^{\lambda}] \;&=\; \frac{\Gamma(a+\lambda)}{\Gamma(a)} \times \frac{\Gamma(a+b)}{\Gamma(a+b+\lambda)}\\ &=\; \exp\left(-\int_0^\infty (1-e^{-\lambda x})\,\frac{e^{-ax}-e^{-(a+b)x}}{x(1-e^{-x})}\,dx\right).\end{aligned}$$

A further computation using an integration by parts similar to the proof of Lemma 3.2. in Jedidi and Simon (2013) - see also the expression of the function $v(\lambda)$ in Example VI.12.21 of Steutel and Van Harn (2003) - shows that

$$\frac{e^{-ax}-e^{-(a+b)x}}{x(1-e^{-x})} \;=\; \int_0^\infty \sharp\{n \in \mathbb{N},\; a+n \le t < a+b+n\}\, e^{-xt}\,dt, \qquad x > 0.$$

The counting function inside the integral clearly satisfies the requirement of Theorem 51.12 in Sato (1999) (Steutel's theorem) if and only if $b \le 1$. This completes the proof. □

2.2. Proof of Theorem 2

Introduce the function

$$g_{a,b,s}(x) \;=\; \frac{s\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\left(\frac{x}{s}\right)^{1-b} f_{a,b,s}(x).$$

Since the density of $\beta_{a,b}^{-s} - 1$ is $f_{a,b,s}(x)$ and since $g_{a,b,s}(0+) = 1$, it follows from Property (iv) p.68 and Theorem 5.4.1 in Bondesson (1992) that $\beta_{a,b}^{-s} - 1 \in \mathcal{H}$ if and only if

$$G_{a,b,s} \;=\; -\log(g_{a,b,s})$$

is a Thorin-Bernstein function. By Theorem 8.2 (ii) in Schilling et al. (2010), this is equivalent to $G'_{a,b,s}$ being a Stieltjes transform in the sense of Definition 2.1 in Schilling et al. (2010). We will use the characterization of Stieltjes transforms given by Corollary 7.4 in Schilling et al. (2010). Compute

$$G'_{a,b,s}(x) \;=\; \frac{a+s}{s(x+1)} \;+\; (b-1)\left(\frac{1}{x} \;-\; \frac{1}{s(x+1)\left((x+1)^{1/s}-1\right)}\right),$$

which is clearly a Stieltjes transform if $b = 1$ or $s = 1$. We henceforth suppose $b \neq 1$ and $s \neq 1$. If $s < 1/2$, then $G'_{a,b,s}$ has at least two poles at $e^{\pm 2\mathrm{i}\pi s} - 1$ and hence cannot be a Stieltjes transform. On the other hand, if $s \geq 1/2$ a computation shows that $G'_{a,b,s}$ has an analytic continuation on $\mathbb{C} \setminus (-\infty, -1]$, with

$$G'_{a,b,s}(0) \;=\; \frac{s+a}{s} \;+\; \frac{(b-1)(s+1)}{2s}\cdot$$

We henceforth suppose $s \geq 1/2$. Setting $x = re^{\mathrm{i}\theta} - 1$ with $r > 0$ and $\theta \in (0, \pi)$, compute

$$\begin{aligned}\mathrm{Im}(G'_{a,b,s}(x)) \;&=\; \frac{-(a+s)\sin(\theta)}{sr} \\ &+\; (1-b)\left(\frac{r\sin(\theta)}{|re^{\mathrm{i}\theta}-1|^2} + \frac{\sin(\theta)}{sr|r^{1/s}e^{\mathrm{i}\theta/s}-1|^2} - \frac{\sin((1+1/s)\theta)}{sr^{1-1/s}|r^{1/s}e^{\mathrm{i}\theta/s}-1|^2}\right).\end{aligned}$$

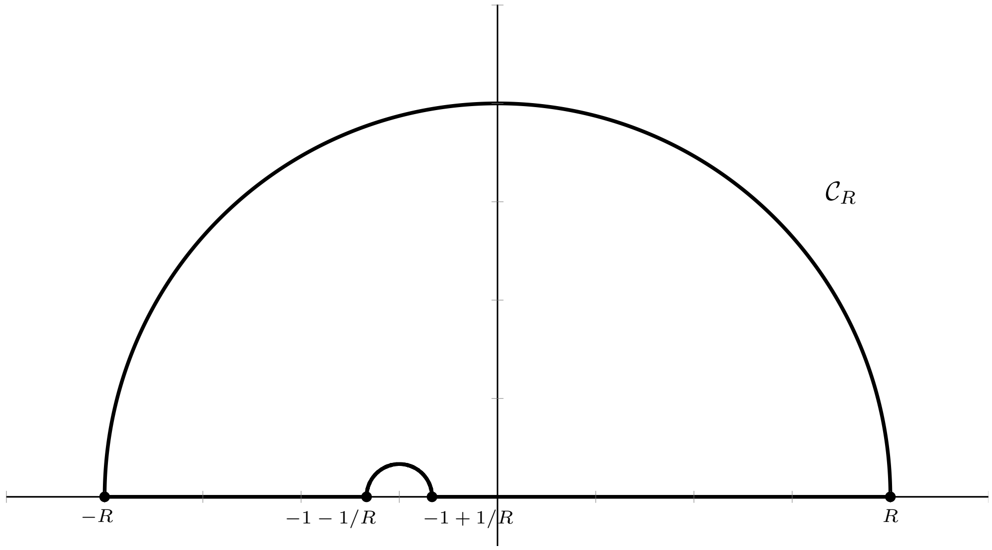
Letting $r \to 0$ and $\theta \to \theta_0 \in [0, \pi)$ we obtain

$$\mathrm{Im}(G'_{a,b,s}(x)) \;=\; \frac{(1-(a+b+s))\sin(\theta)}{sr} \;+\; o(\sin(\theta)r^{-1}),$$

which shows the necessity of the condition $a + b + s \geq 1$. Letting now $r \to 0$ and $\theta \to \pi$ we obtain

$$\mathrm{Im}(G'_{a,b,s}(x)) \;=\; \frac{(1-b)\sin(\pi/s)}{sr^{1-1/s}} \;+\; \frac{(1-(a+b+s))\sin(\theta)}{sr} \;+\; o(\sin(\theta)r^{-1}),$$

which shows the necessity of the condition $(b-1)\sin(\pi/s) \geq 0$. All in all we have shown that $\beta_{a,b}^{-s} - 1 \in \mathcal{H}$ only if 1. 2. or 3. is satisfied, and to finish the proof it remains to prove that $G'_{a,b,s}$ is a Stieltjes transform whenever 1. or 3. holds. To do so, we will use the same argument as in the main theorem of Bosch (2014). The function $\mathrm{Im}(G'_{a,b,s})$ is harmonic on the open upper half-plane as the imaginary part of an analytic function. Suppose there exists x_0 such that $\mathrm{Im}(x_0) > 0$ and $\mathrm{Im}(G'_{a,b,s}(x_0)) > 0$. For any integer $R \geq 2$, let $\mathcal{C}_R$ be the following contour



and $\mathcal{D}_R$ be the domain surrounded by the curve $\mathcal{C}_R$. Note that $x_0 \in \mathcal{D}_R$ for a sufficiently large R. Moreover $G'_{a,b,s}$ extends to a continuous function on $\overline{\mathcal{D}_R}$. From the maximum principle, there exits $x_R \in \mathcal{C}_R$ such that $\mathrm{Im}(G'_{a,b,s}(x_R)) \geq \mathrm{Im}(G'_{a,b,s}(x_0))$. Since $G'_{a,b,s}(x) \to 0$ uniformly as $|x| \to +\infty$, a compactness argument allows to suppose $x_R \to x_\infty \in \mathbb{R}$ as $R \to +\infty$. Set $l = x_\infty + 1$. Necessarily $l \leq 0$ since $G'_{a,b,s}$ is real on $(-1, +\infty)$. If $l < 0$, a computation shows that

$$\mathrm{Im}(G'_{a,b,s}(x_R)) \xrightarrow[R\to+\infty]{} \frac{(1-b)\sin(\pi/s)|l|^{1/s-1}}{s||l|^{1/s}e^{\mathrm{i}\pi/s}-1|^2}$$

and is non-positive if 1. or 3. holds. This shows that $l = 0$. Writing $x_R = \rho_R e^{i\theta_R} - 1$, one has $\rho_R \to 0$ as $R \to +\infty$, and $\theta_R \in [0, \pi]$. Again, a compactness argument allows to suppose that $\theta_R \to \theta_\infty \in [0, \pi]$. The above analysis for the only if part shows clearly that $\limsup_{R\to+\infty} \mathrm{Im}(G'_{a,b,s}(x_R))$ is also non-positive, which contradicts the hypothesis $\mathrm{Im}(G'_{a,b,s}(x_0)) > 0$.

□

2.3. Proof of Theorem 3

We will use the following criterion, which is an easy consequence of Proposition 2 in Bertoin and Yor (2002) and its proof.

Fact (Bertoin-Yor). *Let X be a positive random variable. Then $X \in \mathcal{E}_-$ if and only if there exists $t > 0$ such that $\mathbb{E}[e^{t/X}] < \infty$ and an* SNLP *with positive mean Z whose*

Laplace exponent Ψ *is such that*

$$\Psi(u) \;=\; \frac{u\mathbb{E}[X^{-(u+1)}]}{\mathbb{E}[X^{-u}]}$$

for all $u > 0$. *Moreover, one has then* $X \stackrel{d}{=} I(Z)$.

To be more precise, the if part and the identity in law are obtained directly from the statement of Proposition 2 in Bertoin and Yor (2002), whereas the only if part follows in replacing the integer k by a positive real number u in equation (7) and the two following equations of Bertoin and Yor (2002).

Notice that since $1/\beta_{a,b}^{-s} = \beta_{a,b}^{s}$ has bounded support, the first condition in the above fact is always fulfilled. Computing

$$\frac{u\mathbb{E}[\beta_{a,b}^{s(u+1)}]}{\mathbb{E}[\beta_{a,b}^{su}]} \;=\; \frac{u\Gamma(a+s+su)\Gamma(a+b+su)}{\Gamma(a+b+s+su)\Gamma(a+su)}$$

and applying the above criterion, we deduce that $\beta_{a,b}^{-s} \in \mathcal{E}_-$ if and only if

$$\Psi : u \;\mapsto\; u \times \frac{\Gamma(a+s+u)}{\Gamma(a+b+s+u)} \times \frac{\Gamma(a+b+u)}{\Gamma(a+u)}$$

is the Laplace exponent of an SNLP with positive mean. Observe that the expression of $\Psi(u)$ is symmetric in b and s. Using Gauss' summation formula - see e.g. Formula 2.8.(46) p.104 in Erdélyi et al. (1953) - we obtain the two transformations

$$\Psi(u) \;=\; u \times {}_2F_1(b, -s; a+b+u; 1) \;=\; u \times {}_2F_1(s, -b; a+s+u; 1),$$

where

$${}_2F_1(\lambda, \mu; \nu; z) \;=\; \sum_{n \geq 0} \frac{(\lambda)_n (\mu)_n}{(\nu)_n \, n!} \, z^n$$

is the classical hypergeometric series, which is defined via the Pochhammer symbols $(x)_0 = 1$ and $(x)_n = x(x+1)\ldots(x+n-1)$ if $n \geq 1$. These transformations imply

$$\Psi(u) \;=\; u \,-\, bsu \sum_{n \geq 1} \frac{(1+b)_{n-1}(1-s)_{n-1}}{(a+b+u)_n n!} \tag{2.1}$$

$$\;=\; u \,-\, bsu \sum_{n \geq 1} \frac{(1+s)_{n-1}(1-b)_{n-1}}{(a+s+u)_n n!}. \tag{2.2}$$

Using (2.1) and the partial fraction decomposition

$$\frac{(n-1)!}{(a+b+u)_n} \;=\; \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{(-1)^k}{a+b+k+u} \;=\; \int_0^\infty e^{-(a+b+u)x}(1-e^{-x})^{n-1} dx,$$

we deduce from Fubini's theorem

$$\begin{aligned}\Psi(u) &= u - bsu\int_0^\infty e^{-(a+b+u)x}\left(\sum_{n\ge 1}\frac{(1+b)_{n-1}(1-s)_{n-1}}{n!(n-1)!}(1-e^{-x})^{n-1}\right)dx\\ &= u - u\int_0^\infty e^{-ux}\,\rho_{a,b,s}(x)\,dx\end{aligned}$$

with the notation

$$\rho_{a,b,s}(x) \;=\; bse^{-(a+b)x}\,{}_2F_1(1+b,1-s;2;1-e^{-x}).$$

Above, the use of Fubini's theorem is justified by the fact that the summands are all positive as soon as $n > s$. Similarly, (2.2) entails

$$\Psi(u) \;=\; u - u\int_0^\infty e^{-ux}\,\rho_{a,s,b}(x)\,dx,$$

from which we deduce $\rho_{a,b,s} = \rho_{a,s,b}$ by Laplace inversion. The latter identity, in accordance with the initial symmetry in (b,s) of our problem, can also be obtained from one of Kummer's formulæ for the hypergeometric function - see e.g. 2.1.4(23) p.64 in Erdélyi et al. (1953). In the following, we hence may and will suppose that $s \ge b$. If $s > b$, it follows again from Gauss' summation formula that

$${}_2F_1(1+b,1-s;2;1-e^{-x}) \;\to\; \frac{\Gamma(s-b)}{\Gamma(1+s)\Gamma(1-b)}, \qquad x\to+\infty$$

and this formula extends by continuity to the case $s = b \in \mathbb{N}$, the right-hand side being replaced by $(-1)^b b^{-1}$. If $s = b \notin \mathbb{N}$, the asymptotic expansion 2.3.1(2) p.74 in Erdélyi et al. (1953) yields

$${}_2F_1(1+b,1-b;2;1-e^{-x}) \;\sim\; \frac{x}{\Gamma(1+b)\Gamma(1-b)}, \qquad x\to+\infty.$$

In all cases, we deduce that $\rho_{a,b,s}\to 0$ as $x\to+\infty$ and since $\rho_{a,b,s}$ is clearly smooth on $[0,+\infty)$, an integration by parts entails finally

$$\Psi(u) \;=\; u + \int_0^{+\infty}(1-e^{-ux})\,\rho'_{a,s,b}(x)\,dx \;=\; u - \int_{-\infty}^0 (e^{ux}-1)\,\rho'_{a,s,b}(-x)\,dx, \quad u\ge 0.$$

It is now clear from the above that Ψ is the Laplace exponent of an SNLP if and only if $\rho'_{a,s,b}$ is non-positive on $[0,+\infty)$. We first compute

$$\rho'_{a,s,b}(0) \;=\; -\frac{bs}{2}\,(2a+b+bs+s-1),$$

which shows the necessity of the condition $2a+b+bs+s\ge 1$. Moreover, if $b = b\wedge s > 1$, a theorem of F. Klein (see Formel (4), (18), Satz p.587 in Klein (1890), and also the subsequent paper Van Vleck (1902) for an explanation of Klein's results in the English language and further results in the same vein) shows that $x \mapsto_2 F_1(1+b, 1-s; 2; 1-e^{-x})$ vanishes $[b]$ times on $(0, +\infty)$, where $[x]$ denotes the integer part of a real number $x \notin \mathbb{N}$ and means $x-1$ if $x \in \mathbb{N}$. Since $\rho_{a,b,s} \to 0$ at infinity, this entails that $\rho'_{a,s,b}$ takes positive values, concluding the proof of the only if part.

We now suppose $b \le 1 \le 2a+b+bs+s$ and show that $\rho'_{a,b,s}$ is non-positive. If $b = 1$, we have ${}_2F_1(1+s, 0; 2; 1-e^{-x}) = 1$ for all $x > 0$ and this leads to

$$\rho_{a,1,s}(x) \;=\; se^{-(a+s)x},$$

a decreasing function. If $b, s < 1$, we have $(s+1)(1-b) > 0$ together with $1-s > 0$, and a direct consequence of Theorem 1.3(2) in Anderson et al. (2007) is the log-concavity of $\rho_{a,b,s}$. In particular, this function is non-increasing on $(0, +\infty)$ because $\rho'_{a,b,s}(0) \le 0$ by assumption. Last, if $b < 1 \le s$, it is clearly enough to prove that the non-negative function $z \mapsto_2 F_1(1+b, 1-s; 2; z)$ is non-increasing on $(0, 1)$. If $s = 1$ this function is constant, whereas if $s > 1$ its derivative equals

$$\frac{(1+b)(1-s)}{2}\, {}_2F_1(2+b, 2-s; 3; z)$$

and is negative on $(0, 1)$ since, by Klein's theorem, the function $z \mapsto_2 F_1(2+b, 2-s; 3; z)$ does not vanish on $(0, 1)$.

□

2.4. Proof of the Corollary

By Theorems 1, 2 and 3, it is enough to show that $\beta_{a,b}^{-s} - 1 \notin \mathcal{S}$ as soon as $2a+b+s+bs < 1$. Supposing $\beta_{a,b}^{-s} - 1 \in \mathcal{S}$, there exists a non-increasing function $k_{a,b,s}$ such that

$$\Phi(\lambda) = \mathbb{E}\left(e^{-\lambda(\beta_{a,b}^{-s}-1)}\right) = \exp\left(-\int_0^\infty (1-e^{-\lambda x})\, \frac{k_{a,b,s}(x)}{x}\, dx\right).$$

With the notation of Section 2.2, we write

$$-\frac{\Phi'(\lambda)}{\Phi(\lambda)} \;=\; \frac{\displaystyle\int_0^\infty e^{-\lambda x} x^b g_{a,b,s}(x) dx}{\displaystyle\int_0^\infty e^{-\lambda x} x^{b-1} g_{a,b,s}(x) dx}.$$

Then, noticing that $g_{a,b,s}(x) = 1 + c_{a,b,s}\, x + o(x)$ as $x \to 0$ with $c_{a,b,s} = (1-2a-b-s-bs)/2s$, for every $\alpha > 0$ we obtain by dominated convergence

$$\int_0^\infty e^{-\lambda x} x^\alpha g_{a,b,s}(x) dx \;=\; \frac{1}{\lambda^{\alpha+1}}\left(\Gamma(\alpha+1) + \frac{c_{a,b,s}\,\Gamma(\alpha+2)}{\lambda} + o(\lambda^{-1})\right) \quad \text{as } \lambda \to +\infty.$$

This shows the asymptotic expansion

$$-\frac{\Phi'(\lambda)}{\Phi(\lambda)} \;=\; \frac{b}{\lambda}\left(1+\frac{c_{a,b,s}}{\lambda}+o(\lambda^{-1})\right) \quad \text{as } \lambda\to+\infty,$$

and since

$$\lambda\frac{\Phi'(\lambda)}{\Phi(\lambda)} + b \;=\; \int_0^\infty e^{-x}\left(b-k_{a,b,s}\left(\frac{x}{\lambda}\right)\right)dx,$$

we first deduce $k_{a,b,s}(0+)=b$. In the same way, writing

$$\int_0^\infty xe^{-x}\left(b-k_{a,b,s}\left(\frac{x}{\lambda}\right)\right)\frac{\lambda}{x}\,dx \;=\; \lambda\left(\lambda\frac{\Phi'(\lambda)}{\Phi(\lambda)}+b\right) \xrightarrow[\lambda\to+\infty]{} -b\,c_{a,b,s}$$

shows that

$$k'_{a,b,s}(0+) \;=\; \frac{b(1-(2a+b+s+bs))}{2s}.$$

Hence, $k_{a,b,s}$ is a non-increasing function only if $2a+b+s+bs\geq 1$.

□

2.5. Proof of Theorem 4

Since the class $\mathcal{G}$ is closed with respect to weak convergence, taking (1.2) along integers it is enough to show that $\beta_{a,b}^{-s}\in\mathcal{G}$ for all $a,s>0$ and $b\in\mathbb{N}$. By Theorem 3 in Bondesson (2014), the latter is a consequence of $-\log(\beta_{a,b})\in\mathcal{G}$. The aforementioned Malmsten's formula for the Gamma function entails

$$\mathbb{E}[\beta_{a,b}^{\lambda}] \;=\; \exp\left(-\int_0^\infty (1-e^{-\lambda x})\,\frac{e^{-ax}-e^{-(a+b)x}}{x(1-e^{-x})}\,dx\right),$$

and we see that the $\mathcal{G}$ property for $-\log(\beta_{a,b})$ is equivalent to the complete monotonicity of the function

$$x \;\mapsto\; \frac{e^{-ax}-e^{-(a+b)x}}{1-e^{-x}} \;=\; e^{-ax}(1-e^{-bx})\sum_{k\geq 0}e^{-kx},$$

which is easily characterized - see again Example VI.12.21 in Steutel and Van Harn (2003) - by $b\in\mathbb{N}$.

□

3. Further remarks

3.1. On the $\mathcal{G}$−property for $\beta_{a,b}^{-s}$

Let us first observe that contrary to those of the Gamma distribution, the negative powers of the Beta distribution might belong to $\mathcal{S}\cap\mathcal{G}^c$. Indeed, if $\beta_{a,b}^{-s}\in\mathcal{G}$ then so does $\beta_{a,b}^{-s}-1$, whose density is

$$f_{a,b,s}(x) \;\sim\; \frac{\Gamma(a+b)}{s\Gamma(a)\Gamma(b)}\left(\frac{x}{s}\right)^{b-1} \qquad \text{as } x \to 0+.$$

Hence, by Theorem 4.1.4 in Bondesson (1992) - see also the above proof of the Corollary - the Thorin mass of $\beta_{a,b}^{-s}-1$ is then equal to b and, by Theorem 4.1.1 in Bondesson (1992), the function $g_{a,b,s}$ introduced during the proof of Theorem 2 must be CM. However, a computation shows the first order expansion $(\log(g_{a,b,s}))'(x) = C_1 + C_2 x + o(x)$ at zero, with

$$C_1 \;=\; \frac{1}{2}\left(\frac{1-2a}{s}-1\right) - \frac{b}{2}\left(\frac{1}{s}+1\right) \quad \text{and} \quad C_2 \;=\; 1 + \frac{a}{s} + \frac{(b-1)}{12}\left(\frac{1}{s}+1\right)\left(\frac{1}{s}+5\right).$$

The coefficient C_2 is positive for $b \vee s \geq 1$, but for all $a > 0$ there are some $b, s \in (0,1)$ such that $C_2 < 0$. In these cases the function $g_{a,b,s}$ is not log-convex and hence not CM, so that $\beta_{a,b}^{-s} \notin \mathcal{G}$. Observe finally that these b, s can be chosen such that $2a+b+s+bs \geq 1$. At first sight, it does not seem easy to characterize the $\mathcal{G}-$property for $\beta_{a,b}^{-s}$, as we did for the $\mathcal{M}, \mathcal{H}, \mathcal{E}_-$ and $\mathcal{S}-$properties.

3.2. On the SNLP's associated with $\beta_{a,b}^{-s}$

Under the condition of Theorem 3, it follows from the Fact that $\beta_{a,b}^{-s}$ is the perpetuity of a certain compound Poisson process with unit drift and negative jumps. More precisely, one has

$$\beta_{a,b}^{-s} \;\stackrel{d}{=}\; \int_0^\infty e^{-(t-N_t^{a,b,s})}\,dt \tag{3.1}$$

where, using the notation of Section I.1 p. 12 in Bertoin (1996), the process $\{N_t^{a,b,s},\, t \geq 0\}$ is compound Poisson with Lévy measure $\nu_{a,b,s}(dx) = -s^{-2}\rho'_{a,b,s}(s^{-1}x)dx$ on $\mathbb{R}^+$. If $s=1$, we have

$$\nu_{a,b,1}(dx) \;=\; b(a+b)e^{-(a+b)x}dx$$

and this is a slight extension of Example 4 p. 36 in Bertoin and Yor (2002), erroneously attributed to Gjessing and Paulsen (1997) - we could not locate this special case therein. If $b=1$, we have

$$\nu_{a,1,s}(dx) \;=\; (1+a/s)e^{-(1+a/s)x}dx,$$

providing a representation, which we could not find in the literature, of each negative power $\mathbf{U}^{-c}$ of the uniform law $\mathbf{U}$ on $(0,1)$ as the perpetuity of a compound Poisson process with unit drift and exponential Lévy measure $(1+1/c)e^{(1+1/c)x}$ on $\mathbb{R}^-$. Notice that $\nu_{a,1,s}$ has a mass always equals to one (and a mean strictly less than one, in accordance with the positive mean of the SNLP whose perpetuity is $\mathbf{U}^{-s/a}$), whereas $\nu_{a,b,1}$ is not a probability when $b \neq 1$.

When b or s is an integer, the hypergeometric series defining $\rho_{a,b,s}$ is a Jacobi polynomial and the Lévy measure $\nu_{a,b,s}$ takes a simpler form. For every $n \geq 2$, some computations using Formula 10.8(16) in Erdélyi et al. (1953) show that the density of $\nu_{a,n,s}$ on $\mathbb{R}^+$ equals

$$\sum_{k=0}^{n-1} (1 + (a+k)/s)\, c_{k,n,s}\, e^{-(1+(a+k)/s)x}$$

with

$$c_{k,n,s} \;=\; \prod_{\substack{p\,=\,0\\ p\,\neq\,k}}^{n-1} (1 - s/(p-k)).$$

It can also be checked from the number of roots of Jacobi polynomials that this density is indeed non-negative if and only if $s \leq 1$. Similarly and with the same notation, the density of $\nu_{a,b,n}$ on $\mathbb{R}^+$ equals

$$\sum_{k=0}^{n-1} bn^{-2}(a+b+k)\, c_{k,n,b}\, e^{-(a+b+k)x/n}.$$

These two examples compute the perpetuities of certain compound Poisson processes with unit drift and hyperexponential Lévy measures on $\mathbb{R}^-$.

Changing the variable in (3.1), we obtain

$$b^{-s}\beta_{a,b}^{-s} \;\stackrel{d}{=}\; \int_0^\infty e^{-(b^s t - N^{a,b,s}_{b^s t})}\, dt$$

for all $b \geq 1, s \leq 1$, which is the perpetuity of an SNLP with Laplace exponent

$$\Psi_{a,b,s}(\lambda) \;=\; b^s \lambda \left(1 \,-\, s^{-1} \int_0^\infty e^{-\lambda x}\, \rho_{a,b,s}(s^{-1}x)\, dx \right), \qquad \lambda \geq 0.$$

If $s = 1$ and $b \to +\infty$ one has immediately $\Psi_{a,b,s}(\lambda) \to \lambda(a+\lambda)$, in accordance with (1.2) and Dufresne's result - see Example 3 p.36 in Bertoin and Yor (2002). If $s < 1$ and $b \to +\infty$, we can also recover the SNLP with infinite variation whose perpetuity is distributed as γ_a^{-s}. Indeed, rewriting

$$\Psi_{a,b,s}(\lambda) \;=\; b^s \left(1 - \int_0^\infty \rho_{a,b,s}(x)\, dx \right) \lambda \;+\; s^{-1}\lambda \int_0^\infty (1 - e^{-\lambda x})\, b^s \rho_{a,s,b}(s^{-1}x)\, dx,$$

we see on the one hand from Formula 2.3.2(14) p.77 in Erdélyi et al. (1953) and dominated convergence that

$$s^{-1}\lambda \int_0^\infty (1 - e^{-\lambda x})\, b^s \rho_{a,s,b}(s^{-1}x)\, dx \quad \to \quad \frac{\lambda}{\Gamma(1-s)} \int_0^\infty (1 - e^{-\lambda x})\, \frac{e^{-(1+a/s)x}}{(1 - e^{-x/s})^{1+s}}\, dx$$

as $b \to +\infty$. The right-hand side transforms, after an integration by parts similar to the one used for Theorem 3, into

$$\int_{-\infty}^{0} (e^{\lambda x} - 1 - \lambda x) \, \frac{e^{(1+a/s)x}(s + e^{x/s} + a(1 - e^{x/s}))}{s\Gamma(1-s)(1-e^{x/s})^{2+s}} \, dx.$$

On the other hand, a change of variable and another integration by parts yields

$$\begin{aligned} b^s \left(1 - \int_0^\infty \rho_{a,b,s}(x)\, dx\right) &= b^s \left(1 - bs \int_0^1 (1-z)^{a+b-1} \,{}_2F_1(1+b, 1-s; 2; z)\, dz\right) \\ &= (a+b-1) b^s \int_0^1 (1-z)^{a+b-2} \,{}_2F_1(b, -s; 1; z)\, dz \\ &= (a+b-1) b^s \int_0^1 (1-z)^{a+s-1} \,{}_2F_1(1+s, 1-b; 1; z)\, dz \\ &= \frac{(a+b-1)b^s}{a+s} \,{}_2F_1(1+s, 1-b; a+s+1; 1) \\ &= \frac{\Gamma(a+s)\Gamma(a+b)b^s}{\Gamma(a)\Gamma(a+b+s)}, \end{aligned}$$

where the third equality follows from 2.9(2) p.105 in Erdélyi et al. (1953), the fourth equality from 2.4(2) p.78 in Erdélyi et al. (1953), and the fifth equality from Gauss' summation formula. Alternatively, this identity can also be seen from the proof of Theorem 3 - see the beginning of Section 3.3 thereafter. Putting everything together, we see that

$$\Psi_{a,b,s}(\lambda) \;\to\; \left(\frac{\Gamma(a+s)}{\Gamma(a)}\right) \lambda \;+\; \int_{-\infty}^{0} (e^{\lambda x} - 1 - \lambda x) \, \frac{e^{(1+a/s)x}(s + e^{x/s} + a(1 - e^{x/s}))}{s\Gamma(1-s)(1-e^{x/s})^{2+s}} \, dx$$

as $b \to +\infty$, and the limit on the right-hand side is precisely the Laplace exponent of the Lemma in Bosch and Simon (2013).

When the conditions of Theorem 3 are not fulfilled, that is when $\rho'_{a,b,s}$ takes positive values, it is easily seen that the function $\lambda \mapsto \Psi(\mathrm{i}\lambda)$ is no more the Lévy-Khintchine exponent of a Lévy process. By Proposition 2 in Bertoin and Yor (2002), this shows that $\beta_{a,b}^{-s}$ is not distributed as the perpetuity of a Lévy process having positive jumps and finite exponential moments, and we believe that it is not distributed as the perpetuity of any Lévy process at all. In a different direction, let us recall that spectrally positive Lévy processes might have infinitely divisible perpetuities. It follows indeed from Theorem 2.1(j) in Gjessing and Paulsen (1997) that for every $a > 0, b > 1$ one has

$$\beta_{a,b}^{-1} \;\stackrel{d}{=}\; 1 \;+\; \int_0^\infty e^{-(N_t^{a,b} - t)}\, dt$$

where $\{N_t^{a,b},\, t \geq 0\}$ is a compound Poisson process with Lévy measure $(a+b-1)(b-$

$1)e^{(1-b)x}dx$ on $\mathbb{R}^+$. From this example of Gjessing and Paulsen (1997) let us finally observe, with our previous notation, the mysterious identity

$$\int_0^\infty e^{-(t-N_t^{a,b,1})}dt \stackrel{d}{=} 1 + \int_0^\infty e^{-(N_t^{a,b}-t)}dt.$$

3.3. Other remarks

The proof of Theorem 3 shows that for every $a, b, s > 0$ the function

$$\begin{aligned}\lambda \mapsto \frac{\Gamma(a+b+\lambda)\Gamma(a+s+\lambda)}{\Gamma(a+\lambda)\Gamma(a+b+s+\lambda)} &= \left(1 - \int_0^\infty \rho_{a,b,s}(x)\right) + \int_0^\infty (1-e^{-\lambda x})\rho_{a,b,s}(x)\,dx \\ &= \frac{\Gamma(a+b)\Gamma(a+s)}{\Gamma(a)\Gamma(a+b+s)} + \int_0^\infty (1-e^{-\lambda x})\rho_{a,b,s}(x)\,dx\end{aligned}$$

is Bernstein (without drift but with an additional murder coefficient) if and only if $\rho_{a,b,s}$ is non-negative, which, by the aforementioned theorem of F. Klein, occurs if and only if $b \wedge s \leq 1$. It follows then from Theorem 3.6 (ii) in Schilling et al. (2010) that the function

$$\lambda \mapsto \frac{\Gamma(a+\lambda)\Gamma(a+b+s+\lambda)}{\Gamma(a+b+\lambda)\Gamma(a+s+\lambda)}$$

is CM. The fact that the latter function is CM for all $a, b, s > 0$ is an easy consequence of Malmsten's formula for the Gamma function - see Theorem 6 in Bustoz and Ismail (1986).

The related random variables of the type

$$\left(\beta_{a,b}^{-1} - 1\right)^s \stackrel{d}{=} \left(\frac{\gamma_b}{\gamma_a}\right)^s, \qquad a, b, s > 0,$$

with an independent quotient on the right-hand side, appear in the literature as generalized Beta random variables of the second kind, or GB2 random variables. Their density function is

$$\frac{\Gamma(a+b)}{s\Gamma(a)\Gamma(b)}\,(x^{\frac{1}{s}}+1)^{-(a+b)}x^{\frac{b}{s}-1}\mathbf{1}_{(0,+\infty)}(x)$$

and easily seen to be HCM if and only if $s \geq 1$. Let us mention in passing that this contrasts with the independent product $(\gamma_a \times \gamma_b)^s$, which belongs to $\mathcal{H}$ if $|b-a| \leq 1/2 \leq s$ - see Section 2.2 in Bosch (2014). It is also clear that the GB2 random variables do not have negative exponential moments and hence cannot belong to $\mathcal{E}_-$. It would be interesting to know whether the GB2 random variables belong to $\mathcal{I}$ notwithstanding, when $s < 1$. It can be shown rather easily by the γ_2-criterion - see e.g. Theorem VI.4.5 in Steutel and Van Harn (2003) - that this is indeed the case for $a \leq 1/2 \leq s \leq b = 1-a$. We postpone the remaining cases to future research.

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