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On the infinitesimal Torelli problem of elliptic surfaces

By

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Introduction.

Let X be a compact complex manifold of dimension two. By virture of a result due to Kodaira, the second cohomology of X carries a Hodge structure $H^2(X, \boldsymbol{C})=\bigoplus_{p+q=2} H^{p,q}(X)$, $H^{p,q}(X)=H^q(X, \Omega_X^p)$, where Ω_X^p denotes a sheaf of holomorphic p-forms on X, even if X is non-Kähler. Let Θ_X denote the tangent sheaf of X. The contraction $\Theta_X \otimes \Omega_X^2 \to \Omega_X^1$ induces a cup-product

$$H^0(X, \Omega_X^2) \otimes H^1(X, \Theta_X) \longrightarrow H^1(X, \Omega_X^1),$$

and moreover we obtain *an infinitesimal period map of two forms of* X

$$\delta\colon H^1(X, \Theta_X) \longrightarrow \mathrm{Hom}_{\boldsymbol{C}}(H^{2,0}(X), H^{1,1}(X)).$$

We say that the infinitesimal Torelli theorem holds for X if the map δ is injective.

An infinitesimal period map is closely related to a differential of a (global) period map. In fact, the local Torelli theorem in the sense of Griffiths (3) is reduced to the infinitesimal one.

The purpose of this paper is to study the infinitesimal Torelli problem of elliptic surfaces. The Main Theorem of this paper is as follows.

Main Theorem. *Let φ: $X \to C$ be an elliptic surface with a base curve C. Assume that*: (i) *there exist no multiple fibres and the exceptional curves of the first kind, and* (ii) *the geometric genus $p_g(X)$ of X is positive. Then the infinitesimal Torelli theorem holds for X if one of the following conditions is satisfied.*

(A): *The functional invariant $J(X)$ of X is not constant.*

(B): *$J(X)$ is constant, but not equal to* 0 *or* 1, *and the base curve is a rational curve.*

(C): *$J(X)$ is constant, but not equal to* 0 *or* 1, *and*

$$\chi(X, \mathcal{O}_X)=p_g(X)-q(X)+1 \geqq 3.$$

We shall give counter-examples of the infinitesimal Torelli theorem (Remark 6.2, and §7).

The global Torelli problem for general elliptic surface still remains open[*)] even if base curves of elliptic surfaces are projective lines.

In Appendix, *we shall study the global Torelli problem for "Kodaira surface"*. By definition, Kodaira surface is an elliptic surface whose first Betti number is three, and whose canonical bundle is trivial. Its second cohomology carries a polarized Hodge structure induced by the cup-product. We can construct the coarse moduli space $\mathcal{K}$ of Kodaira surface with a fixed degree, and we can define the period map associated with this polarized Hodge structure Φ: $\mathcal{K}\to D/\Gamma$, where D denotes the period domain and Γ is a discrete group. By the explicit calculation of this period map, we can prove that every fibre of Φ consists of infinite numbers of points. Hence *the global Torelli theorem does not hold for Kodaira surface, although the local Torelli theorem holds for them.*

The plan of this paper is as follows. Section 2 is a review of the necessary background in Hodge theory. In section 3, we reduce Main Theorem to studying some pairings. In section 4, we study the properties of the direct image sheaves $R^{\cdot}\varphi_*\Omega^{\cdot}_X$. After some remarks on deformation of elliptic surfaces in section 5, we complete the proof of Main Theorem in section 6. Section 7 is a study of period maps of elliptic bundles and we give conter-examples to the local Torelli theorem. Section 8 consists of some Tables which summarize our results in this paper.

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§ 1. Notations and Conventions.

In the present paper, we mean by a surface X (resp. a curve C) a compact complex manifold of dimension two (resp. of dimension one).

Let Ω^p_Y denote the sheaf of germs of holomorphic p-forms on a compact complex manifold Y. We use the following notations.

$B^k(Y)=\dim_C H^k(Y, \boldsymbol{C})$.

$h^{p,q}(Y)=\dim_C H^q(Y, \Omega^p_Y)$.

$p_g(X)=h^{2,0}(X)$: the geometric genus of a surface X.

$q(X)=h^{0,1}(X)$: the irregularity of a surface X.

$\chi(X, \mathcal{O}_X)=1-q(X)+p_g(X)$.

Let φ: $X\to C$ be a proper flat holomorphic map from a surface X onto a curve C. We call a surface X *an elliptic surface* with a base curve C, if a general fibres of φ are nonsingular elliptic curves. We often call the morphism $\varphi:X\to C$ above an elliptic surface. If we do not mention otherwise, *we always assume that an elliptic surface is relatively minimal and has no multiple singular fibre.* We use freely the theorey of elliptic surfaces due to Kodaira (cf. (9), (10), (11), (12)).

§ 2. Hodge structures of surfaces and the infinitesimal period map.

2.0. Let X be a surface which may be non-Kähler. We first prove the following theorem which is essentially due to Kodaira.

Theorem 2.1. The second cohomology $H^2(X, \boldsymbol{R})$ of a surface X carries a Hodge structure of weight 2, that is, $H^2(X, C)=\bigoplus_{p+q=2} H^{p,q}(X)$, $\overline{H^{p,q}}=H^{q,p}$, where $H^{p,q}=H^q(X, \Omega_X^p)$.

Proof. For every surface X, the Hodge spectral sequence

$$E_1^{p,q}=H^q(X, \Omega_X^p) \Longrightarrow H^{p+q}(X, C)$$

degenerates at E_1-terms (Kodaira (11), Theorem 3). Hence we have only to show that $\overline{H^{p,q}}=H^{q,p}$. By the definition of the Hodge spectral sequence, it is sufficient to show that $\overline{H^{2,0}}=H^{0,2}$. Let us consider $H^{p,q}$ as the Dolbeault cohomology. Then we can choose linearly independent holomorphic two forms $\omega_1, \omega_2, \ldots, \omega_h$ for basis of $H^{2,0}$ where $h=h^{2,0}$. Since the complex conjugate of holomorphic two forms are $\bar\partial$-closed form of the type (0, 2), it defines an element of $H^{0,2}$. Since we have $h^{2,0}=h^{0,2}$ by Serre duality, we have only to show that the forms $\bar\omega_1, \bar\omega_2, \ldots, \bar\omega_h$ are $\bar\partial$-cohomologically independent. Put $a_1\bar\omega_1+a_2\bar\omega_2+\ldots+a_h\bar\omega_h=\bar\partial\varphi$ where a_i are constants and φ is a C^∞ differential form of the type (0, 1). Since $\partial\bar\varphi$ is holomorphic and $\partial\varphi\Lambda\partial\bar\varphi=0$, we have $d(\varphi\Lambda\partial\bar\varphi)=\bar\partial\varphi\Lambda\partial\bar\varphi$. Hence, by Stokes' formula, we conclude that $\bar\partial\varphi=0$. This implies that $a_i=0$. Q.E.D.

Remark 2.2. For every surface X, there exists a decomposition

$$H^1(X, C)=H^0(X, \Omega_X^1)\oplus H^1(X, \mathcal{O}_X),$$

and an inclusion $\overline{H^0(X, \Omega_X^1)} \hookrightarrow H^1(X, \mathcal{O}_X)$ (Kodaira (11), Theorem 3). Moreover Kodaira showed that if $B^1(X)$ is even, $h^{1,0}=h^{0,1}$, if $B^1(X)$ is odd, $h^{1,0}=h^{0,1}-1$.

2.1. Let π: $X\to S$ be a proper smooth surjective holomorphic map from a complex manifold X to a connected complex manifold S. We assume that each fibre of π is a (connected) surface. We put

H_C^2: flat vector bundle associated with $R^2\,\pi_* C_X$,

$H^{p,q}=\bigcup_{t\in S} H^{p,q}(X_t)$: C^∞ subbundles of H_C^2,

$\mathbf{F}^p=\bigoplus_{i\geq p} H^{i,2-i}$: C^∞ subbundles of H_C^2,

H_R^2: the canonical real structure of H_C^2,

∇: a flat connection of H_C^2.

One can prove the following theorem in the same way as the case in which X is Kähler (cf. (2), (6)).

Theorem 2.3. *Let π: $X\to S$ be as above. Then the subbundles $\mathbf{F}^p$ are holomorphic subbundles of H_C^2. Moreover the data $\{H_C^2, \mathbf{F}^p, \nabla, S\}$ gives the variation of Hodge structure of weight two, that is,*

(i) $\mathbf{F}^r\oplus\overline{\mathbf{F}^{1-r}} \xrightarrow{\sim} H_C^2$, *where — denote the complex conjugation with respect to H_R^2,*

(ii) $\nabla\mathcal{O}(\mathbf{F}^p)\hookrightarrow\mathcal{O}(\mathbf{F}^{p-1})\otimes\Omega_S^1$.

Remark 2.4. The definition of a variation of Hodge structure above is sufficient for our purpose. See (4), (14).

2.2. Let us recall the definition of the infinitesimal period map and its relation to the differential of a period map.

Let X be an n-dimentional complact complex manifold. Assume that the n-th cohomology $H^n(X, \boldsymbol{C})$ carries a Hodge structure of weight n. Let Θ_X denote the sheaf of germs of holomorphic vector fields of X.
A contraction

$$\Theta_X \otimes \Omega_X^p \longrightarrow \Omega_X^{p-1} \tag{2.1}$$

induces a cup-product map

$$H^1(X, \Theta_X) \otimes H^p(X, \Omega_X^p) \longrightarrow H^{p+1}(X, \Omega_X^{p-1}) \tag{2.2}$$

for each pair of integers (p, q), $p+q=n$, $p \geq 0$, $q > 0$. By this cup-product (2.2), we can consider a cohomology class γ of $H^1(X, \Theta_X)$ as an element of Hom $(H^{p,q}(X), H^{p+1,q-1}(X))$. Hence we can define a $\boldsymbol{C}$-linear map

$$\delta: H^1(X, \Theta_X) \longrightarrow \bigoplus_{p+q=n} \mathrm{Hom}\,(H^{p,q}, H^{p+1,q-1}). \tag{2.3}$$

Definition 2.5. We call a $\boldsymbol{C}$-linear map δ in (2.3) an infinitesimal period map of holomorphic n-forms of X.

Let π: $X \rightarrow S$ be a proper, smooth, surjective holomorphic map between complex manifolds with connected fibres. Assume that *the n-th cohomology* $H^n(X_t, \boldsymbol{C})$ *of each fibre* $X_t (=\pi^{-1}(t))$ *carries a Hodge structure of weight n.* For each point o on S, we can take a small open neighborhood U of o, and we can define the period map

$$\Phi: U \longrightarrow D, \tag{2.4}$$

where D denote a period domain in the sense of (4) (see also (14)).

Then Griffiths (3) showed that the following diagram is commutative.

$$\begin{array}{ccc} T_o(U) & \xrightarrow{\quad d\Phi \quad} & T_{(o)}(D) \\ \rho \downarrow & & \uparrow \cup \\ H^1(X_o, \Theta_{X_o}) & \xrightarrow{\quad \delta \quad} & \bigoplus_{p+q=n} \mathrm{Hom}\,(H^{p,q}(X_o), H^{p+1,q-1}(X_o)) \end{array}$$

where ρ denote *the Kodaira-Spencer map at the point o.*

From this fact, we can easily see the following theorem.

Theorem 2.5. *Let* π: $X \rightarrow S$ *be as above. Anssume that the Kodaira-Spencer map is injective at the point o. Then the period map* Φ *in* (**2.4**) *is a local embedding at o if the infinitesimal period map of* X_0 *is injective.*

§ 3. Reduction of Main Theorem.

3.0. Let X be a surface. By (2.3), we have the infinitesimal period map of two forms of X:

$$\delta: H^1(X, \Theta_X) \longrightarrow \mathrm{Hom}\,(H^{2,0}(X), H^{1,1}(X)) \oplus \mathrm{Hom}\,(H^{1,1}(X), H^{0,2}(X)) \tag{3.1}$$

Taking the projection of the right hand side term to the first factor and dualizing this map by Serre duality, we have a cup-product

$$\mu: H^0(X, \Omega_X^2) \otimes H^1(X, \Omega_X^1) \longrightarrow H^1(X, \Omega_X^1 \otimes \Omega_X^2) \tag{3.2}$$

Note that this map is the cup-product in the sense of Godement (18).

Lemma 3.1. *The infinitesimal Torelli theorem holds for a surface X if and only if a cup-product* (**3.2**) *is surjective.*

Proof. The "if" part is obvious. For each element $\gamma \in H^1(X, \Theta_X)$ put $\delta(\gamma) = (f_1, f_2) \in \mathrm{Hom}\,(H^{2,0}(X), H^{1,1}(X)) \oplus \mathrm{Hom}\,(H^{1,1}(X), H^{0,2}(X))$. Then f_2 is the dual map of f_1 by the Serre duality. Hence $\delta(\gamma)=0$ if and only if $f_1=0$. This implies Lemma 3.1. Q.E.D.

From Lemma 3.1, Main Theorem is equivalent to the following theorem.

Theorem 3.2. *Let φ: $X \to C$ be an elliptic surface with a base curve C of genus g. Assume that the conditions* (i) *and* (ii) *in Main Theorem hold. Then the cup-product* (3.2) *is surjective if one of the conditions (A), (B), and (C) in Main Theorem is satisfied.*

3.1. Let φ: $X \to C$ be an elliptic surface with a base curve C and let $\mathcal{F}$ denote a coherent $\mathcal{O}_X$-sheaf. Since C is a curve, the Leray spectral sequence

$$E_2^{p,q} = H^p(C, R^q\varphi_*\mathcal{F}) \Rightarrow E_\infty^{p+q} = H^{p+q}(X, \mathcal{F}) \tag{3.3}$$

always degenerates at E_2-terms. Hence we have the following exact sequences

$$0 \longrightarrow H^1(C, \varphi_*\Omega_X^1) \longrightarrow H^1(X, \Omega_X^1) \longrightarrow H^0(C, R^1\varphi_*\Omega_X^1) \longrightarrow 0. \tag{3.4}$$

$$0 \longrightarrow H^1(C, \varphi_*\Omega_X^1 \otimes \Omega_X^2) \longrightarrow H^1(X, \Omega_X^1 \otimes \Omega_X^2) \longrightarrow H^0(C, R^1\varphi_*(\Omega_X^1 \otimes \Omega_X^2)) \longrightarrow 0. \tag{3.5}$$

Moreover we have an isomorphism

$$H^0(C, \varphi_*\Omega_X^2) \simeq H^0(X, \Omega_X^2). \tag{3.6}$$

Lemma 3.3. *A cup-product μ in* (**3.2**) *is compatible with the Leray spectral sequence.*

For a proof, see E.G.A. III (19), (12.2.6.1) and (12.2.6.2).

By Lemma 3.3, μ is surjective if and only if the following cup-products are surjective.

$$\mu_1: H^0(C, \varphi_*\Omega_X^2) \otimes H^1(C, \varphi_*\Omega_X^1) \longrightarrow H^1(C, \varphi_*(\Omega_X^1 \otimes \Omega_X^2)) \tag{3.7}$$

$$\mu_2: H^0(C, \varphi_*\Omega_X^2) \otimes H^0(C, R^1\varphi_*\Omega_X^1) \longrightarrow H^0(C, R^1\varphi_*(\Omega_X^1 \otimes \Omega_X^2)). \tag{3.8}$$

Hence we can reduce the proof of Theorem 3.3 to a study of the cup-product μ_1 and μ_2.

§ 4. Properties of sheaves $R^{\cdot}\varphi_*\Omega_X$.

4.0. Let $\varphi: X \to C$ be a minimal elliptic surface without multiple fibres. The following proposition is well-known. Proofs can be found in (10), (16).

Proposition 4.1. *Let $\varphi: X \to C$ be as above and let $\omega_{X/C}$ denote the relative canonical sheaf. Then we have*

(i) $\varphi_*\mathcal{O}_X \cong \mathcal{O}_C$,

(ii) $R^1\varphi_*\mathcal{O}_X$ is invertible sheaf on C,

(iii) $\deg R^1\varphi_*\mathcal{O}_X = -\chi(\mathcal{O}_X) = -(1-q(X)+p_g(X))$,

(iv) $\omega_{X/C} = \varphi^*(f^\vee)$, where f denote the invertible sheaf $R^1\varphi_*\mathcal{O}_X$,

(v) $\Omega_X^2 = \omega_X = \varphi^*(\Omega_C^1 \otimes f^\vee)$.

Remark 4.1. The statements (i), (ii) and (iii) in Proposition 4.1 hold for an elliptic surface without assumption on minimality and multiple fibres.

4.1. A canonical homomorphism.

Let $\varphi: X \to C$ be a minimal elliptic surface without multiple fibres and let $\Omega_{X/C}^1$ denote the sheaf of relative one forms of X over C. Since a base curve is nonsingular, we have an exact sequence

$$0 \longrightarrow \varphi_*\Omega_C^1 \longrightarrow \Omega_X^1 \longrightarrow \Omega_{X/C}^1 \longrightarrow 0. \tag{4.1}$$

Since X is a surface, there exists a canonical homomorphism

$$\Omega_X^1 \otimes \Omega_X^1 \longrightarrow \Omega_X^2 \cong \omega_X. \tag{4.2}$$

Considering a sheaf $\varphi^*\Omega_C^1$ as a subsheaf of Ω_X^1 by (4.1), from (4.2) we have a *canonical homomorphism*

$$\iota: \Omega_X^1 \otimes \varphi^*\Omega_C^1 \longrightarrow \omega_X. \tag{4.3}$$

4.2. A fundamental exact sequence.

Let $\varphi: X \to C$ be as in 4.1. The singular fibres of φ were classified by Kodaira (10). Let $a_1, a_2, \dots, a_r$ denote supports of all singular fibres of φ and we put

$$\Theta^i = \varphi^{-1}(a_i), \quad \Phi = \sum_{i=1}^r \Theta^i. \tag{4.4}$$

If a singular fibre Θ^i is of type ${}_1I_b$ or type II, we put $\bar{\Theta}^i = \phi$. In other case, a singular fibre Θ^i can be written in the form

$$\Theta^i = \sum_k m_k^i \cdot \Theta_k^i \tag{4.5}$$

where each irreducible component Θ_k^i is a nonsingular rational curve which intersects each other transeversely. Then for a singular fibre Θ^i in (4.5) we define a divisor $\bar{\Theta}^i$ by

$$\bar{\Theta}^i = \sum_k (m_k^i - 1) \cdot \Theta_k^i. \tag{4.6}$$

Moreover we put

$$\bar{\Theta}=\sum_{i=1}^{r}\bar{\Theta}^i. \tag{4.7}$$

Now we shall determine the kernel and cokernel of the canonical homomorphism ι in (4.3).

Let $\bar{\Theta}^i_{\mathrm{red}}$ (resp. $\bar{\Theta}_{\mathrm{red}}$) denote the underlying reduced subspace of $\bar{\Theta}^i$ (resp. $\bar{\Theta}$). If p is a cusp of a singular fibre of type II, we can take a local coordinate (X, Y) around the point p, and a local parameter t around $\varphi(p)$ such that $Y^2+X^3=t$. We define a ideal J_p of $\mathcal{O}_{X,p}$ by

$$J_p=(X^2, Y)\mathcal{O}_{X,p}.$$

Moreover for each point p on X, we denote by m_p a ideal sheaf of p. Then we define an ideal sheaf I of X whose stalk at each point p is given as follows.

$$I_{\mathrm{p}}=\begin{cases}\mathcal{O}_{X,p}, \text{ if } \varphi \text{ is smooth at } p,\\ \mathcal{O}_X(-(m^i_k-1)\,\Theta^i_k)_p, \text{ if } m^i_k\geqq 2,\ p\in\Theta^i_k-\text{(intersections)}\\ m_p\otimes\mathcal{O}_X(-(m^i_k-1)\,\Theta^i_k-(m^i_j-1)\Theta^i_j)_p, \text{ if } \Theta^i_k\neq\Theta^i_j \text{ and } p\in\Theta^i_k\cap\Theta^i_j,\\ m_p\otimes\mathcal{O}_{X,p} \text{ if } p \text{ is an ordinary double point of a singular fibre of type } {}_1I_b,\\ J_p\otimes\mathcal{O}_{X,p} \text{ if } p \text{ is a cusp of a singular fibre of type II.}\end{cases} \tag{4.8}$$

Then we have the following lemma.

Lemma 4.2. *Let us use the same notation as above. The canonical homomorphism* ι *in* (**4.3**) *induces exact sequences*

$$0\longrightarrow(\varphi^*\Omega^1_C\otimes\mathcal{O}(\bar{\Theta}))\otimes\varphi^*\Omega^1_C\longrightarrow\Omega^1_X\otimes\varphi^*(\Omega^1_C)\stackrel{\iota}{\longrightarrow}I\otimes\Omega^2_X\longrightarrow 0 \tag{4.9}$$

$$0\longrightarrow\varphi^*\Omega^1_C\otimes\mathcal{O}(\bar{\Theta})\longrightarrow\Omega^1_X\longrightarrow I\otimes\omega_{X/C}\longrightarrow 0 \tag{4.10}$$

where $\mathcal{O}(\bar{\Theta})$ denotes an invertible sheaf associated with the divisor $\bar{\Theta}$.

Proof. If the morphism φ is smooth at p, the canonical homomorphism is surjective and its kernel is isomorphic to $\varphi^*\Omega^1_C\otimes\varphi^*\Omega^1_{C,p}$. Hence there is nothing to prove in this case. Take a point p where φ is not smooth. Then we can choose a local coordinate (X, Y) around p and a local parameter t around $\varphi(p)$ such that

(i) $X^{m^i_k}=t$, if $p\in\Theta^i_k-$(intersections),

(ii) $X^{m^i_k}Y^{m^i_j}=t$, if p lies on a intersection of Θ^i_k and Θ^i_j,

(iii) $XY=t$, if p is an ordinary double point of a singular fibre of type ${}_1I_b$,

(iv) $Y^2+X^3=t$, if p is a cusp of a singular fibre of type II.

Since the map ι is locally given by

$$\xi\otimes\varphi^*(dt)\longrightarrow\xi\wedge\varphi^*(dt),\ \xi\in\Omega^1_{X,p},$$

the image of the map ι is generated by the following element in each case. (i) $X^{(m^i_k-1)}\,dX\wedge dY$, (ii) $X^{(m^i_k-1)}Y^{m^i_j}\,dX\wedge dY$ and $X^{m^i_k}Y^{(m^i_j-1)}\,dX\wedge dY$, (iii) $X\,dX\wedge dY$ and $Y\,dX\wedge dY$, (iv) $Y\,dX\wedge dY$ and $X^2\,dX\wedge dY$. Moreover the kernel of the map ι is generated by the following elements.

(i) $\frac{\varphi^*(dt)}{X^{(m_k^i-1)}}\otimes\varphi^*(dt)$, (ii) $\frac{\varphi^*(dt)}{X^{(m_k^i-1)}Y^{(m_j^i-1)}}\otimes\varphi^*(dt)$, (iii) and (iv) $\varphi^*(dt)\otimes\varphi^*(dt)$.

Hence we can easily see that the sequence (4.9) is exact at the point p. Hence the sequence (4.9) is exact. Tensoring the dual sheaf of $\varphi^*\Omega_C^1$ to (4.9), we obtain an exact sequence (4.10). Q.E.D.

4.3. Let us use the same notation as in 4.0, 4.1 and 4.2. From (4.6), (4.7) and (4.8), we get an exact sequence

$$0 \longrightarrow \mathrm{I} \longrightarrow \mathcal{O}(-\bar{\Theta}) \longrightarrow T \longrightarrow 0 \tag{4.11}$$

where T denote a torsion sheaf on X whose supports lies on the intersections of singular fibres. Now we shall prove the following proposition.

Proposition 4.3. *Let φ: $X\to C$ be a minimal elliptic surface without multiple singular fibres and let us use the same notation as above. Then*

(i) $\varphi_*I \cong \mathcal{O}(-\sum_{i=1}^r a_i)$

(ii) $R^1\varphi_*I \cong R^1\varphi_*\mathcal{O}_X\oplus T_1$, where T_1 is a torsion sheat on C

(iii) $\varphi_*\mathcal{O}(\bar{\Theta}) \cong \mathcal{O}_C$

(iv) $R^1\varphi_*\mathcal{O}(\bar{\Theta}) \cong \mathcal{O}(\sum_{i=1}^r a_i)\otimes f$.

Proof. Since φ_*I is invertible and $\varphi_*(\mathcal{O}(-\sum_{i=1}^r a_i))\subset I\subsetneqq\mathcal{O}_X$, the assertion (i) is obvious. Since the supports of the sheaf T in (4.11) are 0-dimentional, from (4.11) we get an exact sequence

$$\longrightarrow \varphi_*T \longrightarrow R^1\varphi_*I \longrightarrow R^1\varphi_*\mathcal{O}(-\bar{\Theta}) \longrightarrow 0. \tag{4.12}$$

Since C is a curve, every coherent sheaf is decomposed into a locally free part and a torsion part uniquely. Hence we can conclude from (4.12) that locally free part of $R^1\varphi_*I$ and $R^1\varphi_*\mathcal{O}(-\bar{\Theta})$ are isomorphic to each other. Then we shall prove the locally free part of $R^1\varphi_*\mathcal{O}(-\bar{\Theta})$ is isomorphic to $R^1\varphi_*\mathcal{O}_X$. Since the problem is local, we can assume that a divisor in (4.4) consists of only one singular fibre, that is $\Theta=\Theta_1$. If a singular fibre Θ is one of the types ${}_1\mathrm{I}_b$, II, III and IV, the multiplicity of each irreducible component is equal to one. This implies that the linear system $|\bar{\Theta}|$ is empty. Hence in this case there is nothing to prove. If a singular fibre Θ is one of the type I_b^*, II*, III* and IV*, we claim the following.

Claim. There exist a sequence $\{\Theta_k\}_{k=0}^s$ of divisors which satisfies the following conditions;

(a) $\bar{\Theta}_0=\phi$, $\bar{\Theta}_s=\bar{\Theta}$, *where Θ is one of the types I_b^*, II^*, III^* , and IV^*.*

(b) *For each integer k, $1\leqq k\leqq s$, $D_k=\bar{\Theta}_k-\bar{\Theta}_{k-1}$ is an irreducible nonsingular rational curve.*

(c) *For each integer k, $1\leqq k\leqq s$,* $\deg(\mathcal{O}_{D_k}(-\bar{\Theta}_{k-1}|_{D_k}))=(-\bar{\Theta}_{k-1})\cdot D_k\geqq -1$.

We assume that the cliam holds. Let us consider the following exact sequence for each integer k, $1 \leqq k \leqq s$,

$$0 \longrightarrow \mathcal{O}(-\bar{\Theta}_k) \longrightarrow \mathcal{O}(-\bar{\Theta}_{k-1}) \longrightarrow \mathcal{O}_{D_k}(-\bar{\Theta}_k|_{D_k}) \longrightarrow 0 \tag{4.13}$$

By the condition (b) and (c), we have $R^1\varphi_*\mathcal{O}_{D_k}(-\bar{\Theta}_{k-1}|D_k)=0$. Taking the direct images of the exact sequence (4.13), we have

$$\longrightarrow \varphi_*\mathcal{O}_{D_k}(-\bar{\Theta}_{k-1}|D_k) \longrightarrow R^1\varphi_*\mathcal{O}(-\bar{\Theta}_k) \longrightarrow R^1\varphi_*\mathcal{O}(-\bar{\Theta}_{k-1}) \longrightarrow 0.$$

Since the first term of above sequence is a torsion sheaf, the locally free part of $R^1\varphi_*\mathcal{O}(-\bar{\Theta}_k)$ and $R^1\varphi_*\mathcal{O}(-\bar{\Theta}_{k-1})$ are isomorphic to each other. By induction with respect to k, we conclude that the locally free part of $R^1\varphi_*\mathcal{O}(-\bar{\Theta}_k)$ is isomorphic to $R^1\varphi_*\mathcal{O}_X$. The proof of the claim is reduced to a calculations of the intersection numbers of divisors. One can find easily a sequence of divisors satisfying (a), (b) and (c). Hence the proof is left for readers. The proof of the assertion (iii) and (iv) is essentially same as in (i) and (ii). Hence we omit it.

4.4. Let φ: $X \to C$ be a minimal elliptic surface without multiple fibres. From the exact sequence (4.10), using (iv) in Proposition 4.1 and Proposition 4.3, we have the following exact sequence.

$$\begin{aligned} &0 \longrightarrow \Omega_C^1 \xrightarrow{g_1} \varphi_*\Omega_X^1 \xrightarrow{g_2} \mathcal{O}(-\sum_{i=1}^r a_i)\otimes f^\vee \\ &\xrightarrow{g_3} R^1\varphi_*\mathcal{O}(\bar{\Theta})\otimes\Omega_C^1 \xrightarrow{g_4} R^1\varphi_*\Omega_X^1 \xrightarrow{g_5} \mathcal{O}_C\oplus T_1 \longrightarrow 0. \end{aligned} \tag{4.14}$$

Let $J(X)$ denote *the functional invariant of an elliptic surface* φ: $X \to C$ (Kodaira (10), p. 572). If the functional invariant $J(X)$ is not constant, the moduli of general fibres changes corresponding to its values, and the elliptic surface has at least one singular fibre. If the functional invariant is constant, the moduli of general fibres does not change. From the exact sequence (4.14), we have the following proposition.

Proposition 4.4. *Let φ: $X \to C$ be as above. Then we have the followings.*
(*I*) *If the functional invariant $J(X)$ is not constant, then*

$$\varphi_*\Omega_X^1 \cong \Omega_C^1, \tag{4.15}$$

$$R^1\varphi_*\Omega_X^1 \cong \mathcal{O}_C\oplus T_2, \text{ where } T_2 \text{ denote a torsion sheaf.} \tag{4.16}$$

(*II*) *If the functional invariant $J(X)$ is constant, then*

$$0 \longrightarrow \Omega_C^1 \longrightarrow \varphi_*\Omega_X^1 \longrightarrow \mathcal{O}(-\sum_{i=1}^r a_i)\otimes f^\vee \longrightarrow 0 \quad (\textit{exact}) \tag{4.17}$$

$$0 \longrightarrow \Omega_C^1\otimes\mathcal{O}(\sum_{i=1}^r a_i)\otimes f \longrightarrow R^1\varphi_*\Omega_X^1 \longrightarrow \mathcal{O}_C\oplus T^1 \longrightarrow 0 \quad (\textit{exact}). \tag{4.18}$$

(III) If $J(X)$ is constant, but not equal to 0 or 1, we have $r=2\chi(\mathcal{O}_X)$, and

$$\mathcal{O}(-\sum_{i=1}^r a_i) \cong \mathcal{O}(2f). \tag{4.19}$$

Proof. The proof of (4.15) can be found in (5), Lemma 5.2. Hence we omit it. From (4.15), we conclude that the cokernel g_4 is a torsion sheaf. This implies (4.16). If $J(X)$ is constant, the rank of the sheaf $\varphi_*\Omega_X^1$ is two. Hence the cokernel of g_3 is a torsion sheaf. Since the sheaf $R^1\varphi_*\mathcal{O}(\overline{\Theta})$ is locally free (Proposition 4.3), g_3 is a zero map. This implies that (4.17) and (4.18). The proof of (III) is an easy execrise. Hence we omit it. Q.E.D.

§ 5. Remarks on deformations of elliptic surfaces.

5.0. Let X be a compact complex manifold. By deformations of X we mean a triple $(\mathcal{X}, T, \pi)$ where

(i) $\mathcal{X}$ and T are analytic sets,

(ii) π is locally trivial (in the sense of Kuranishi (8)) and proper homolophic map π: $\mathcal{X}\to T$, such that for some point of o on T, the fibre $\pi^{-1}(o)$ is biholomorphic to X.

Kuranishi (7) proved that for any compact complex manifold, the versal deformation always exists. This versal family is called Kuranishi family, and its base space T is called the Kuranishi space.

Kas (5) proved the followings. (theorem 6.2, (5))

Theorem 5.1. *Let φ: $X\to C$ be an elliptic surface satisfying the following conditions.*

(i) *Every fibre of φ is irreducible.*

(ii) $g=g(C)\geq 2$.

(iii) $4(1-q(X)+p_g(X))\geq 2g-2$.

(iv) *The functional invariant $J(X)$ is not constant.*

Then the Kuranishi space is smooth, and the number of moduli $\mu(X)$ is defined. Moreover we have

$$\mu(X)=\dim{}_C H^1(X, \Theta_X)=11(1-q(X)+p_g(X))+3g-3.$$

Remark 5.2. By using the Tyurina's theorem of simultaneous Brieskorn resolution of rational double points (15), we can show that the assumption (i) in Theorem 5.1 is not needed.

Moreover Kas (5) found the first example of a surface which has an obstruction to deformations, that is, an elliptic surface with a constant functional invariant.

Theorem 5.3. *Let φ: $X\to C$ be an elliptic surface with a constant functional invariant $J(X)$. Assume that:*

(i) $J(X)\neq 0$, 1.

(ii) $12(1-q(X)+p_g(X))>\frac{2+2g}{2}$

(iii) *the curve C is sufficiently general.*

Then the reduced structure of the Kuranishi space T of X is smooth and

$$\dim T=3(1-q(X)+p_g(X))+4g-3,$$

while we have $\dim{}_C H^1(X, \Theta_X)=11(1-q(X)+p_g(X))+4g-3$.

§ 6. Proof of Main Theorem.

6.0. In this section we complete the proof of Main Theorem. By the reduction of Main Teorem in §3, Main Theorem is equivalent to Theorem 3.2, and then we have only to prove that the cup-products μ_1 and μ_2 are surjective.

6.1. Let $\varphi: X \to C$ be a minimal elliptic surface without multiple singular fibres. Assume that the geometric genus $p_g(X)$ of X is positive. We put

$$N = \chi(X, \mathcal{O}_X) = 1 - q(X) + p_g(X),$$
$$g = \textit{the genus of a base curve } C.$$

When a base curve C is a rational curve, the geometric genus of X is positive if and only if N is greater than one. (Note that for any elliptic surface, N is non-negative.) If C is a rational curve and N=2, a surface X is a $K-3$ surface and the canonical bundle of X is trivial. Hence the cup-product (3.2) is clearly surjective. By this reason, we omit this case in the following proof. Hence *we assume that if C is a rational curve, N is greater than two.*

6.2. The case (A) of Main Theorem.

Let $\varphi: X \to C$ be as above. Assume that the functional invariant $J(X)$ of X is not constant. From (4.15), (4.16) and (v) in Propositon 4.1, we have isomorphisms:

$$H^0(C, \varphi_* \Omega_X^2) \simeq H^0(C, \Omega_C^1 \otimes f^\vee), \tag{6.1}$$

$$H^1(C, \varphi_*(\Omega_X^1 \otimes \Omega_X^2)) \simeq H^1(C, \Omega_C^1 \otimes \Omega_C^1 \otimes f^\vee), \tag{6.2}$$

$$H^0(C, R^1\varphi_* \Omega_X^1) \simeq H^0(C, \mathcal{O}_C) \oplus H^0(C, T), \tag{6.3}$$

$$H^0(C, R^1\varphi_*(\Omega_X^1 \otimes \Omega_X^2)) \simeq H^0(C, \Omega_C^1 \otimes f^\vee) \oplus H^0(C, T \otimes \Omega_C^1 \otimes f^\vee). \tag{6.4}$$

For an elliptic surface with non-constant functional invariant, the number N$=\chi(X, \mathcal{O}_X)$ is always positive (Kodaira (10)). By this reason and the assumption in 6.1, the degree of the invertible sheaf $\Omega_C^1 \otimes f^\vee$ is greater than zero. Hence we have

$$H^1(C, \Omega_C^1 \otimes \Omega_C^1 \otimes f^\vee) = 0.$$

From this fact, we conclude that the cup-product μ_1 in (3.7) is automatically surjective.

Next, we consider the cup-product μ_2 in (3.8). From (6.1), (6.3) and (6.4) the cup-product μ_2 is reduced to the following pairings.

$$H^0(C, \Omega_C^1 \otimes f^\vee) \otimes H^0(C, \mathcal{O}_C) \longrightarrow H^0(C, \Omega_C^1 \otimes f^\vee)$$

$$H^0(C, \Omega_C^1 \otimes f^\vee) \otimes H^0(C, T) \longrightarrow H^0(C, T \otimes \Omega_C^1 \otimes f^\vee).$$

The first pairing is clearly surjective. Since T is a torsion sheaf of a curve C, the second pairing is also surjective. This implies that μ_2 is surjective.

6.3. The case (B) and (C) of Main Theorem.

Let $\varphi: X \to C$ be as in 6.1 and let us assume that the functional invariant $J(X)$

of X is constant, but not equal to 0 or 1.

From the exact sequences (4.17) and (4.18), using the isomophism (4.19), we have the following exact sequences.

(6.5) $$0 \longrightarrow \Omega_C^1 \longrightarrow \varphi_*\Omega_X^1 \longrightarrow f \longrightarrow 0$$

(6.6) $$0 \longrightarrow \Omega_C^1 \longrightarrow \mathrm{R}^1\varphi_*\Omega_X^1 \longrightarrow \mathcal{O}_C \oplus T \longrightarrow 0$$

Since the degree of the invertible sheaf f is equal to $-\mathrm{N}<0$, from (6.5), we have an exact sequence

(6.7) $$0 \longrightarrow H^1(C, \Omega_C^1) \longrightarrow H^1(C, \varphi^*\Omega_X^1) \longrightarrow H^1(C, f) \longrightarrow 0.$$

Moreover, from the exact sequence (6.6), we obtain exact sequences;

(6.8) $$0 \longrightarrow H^0(C, \Omega_C^1 \otimes f^\vee) \longrightarrow H^0(C, R^1\varphi_*\Omega_X^1) \longrightarrow H^0(C, \mathcal{O}_C \oplus T)$$
$$\longrightarrow H^1(C, \Omega_C^1 \otimes f^\vee)=0 \ (deg f^\vee = N>0)$$

(6.9) $$0 \longrightarrow H^0(C, (\Omega_C^1)^2 \otimes (f^\vee)^2) \longrightarrow H^0(C, R^1\varphi_*(\Omega_X^1) \otimes \Omega_C^1 \otimes f^\vee)$$
$$\longrightarrow H^0(C, \Omega_C^1 \otimes f^\vee) + H^0(C, T_1 \otimes \Omega_C^1 \otimes f^\vee) \longrightarrow 0.$$

As we see in 6.2, we have $H^1(C, \Omega_C^1 \otimes \Omega_C^1 \otimes f^\vee)=0$. Hence, from (6.5), we have an isomorphism

(6.10) $$H^1(C, \varphi_*(\Omega_X^1) \otimes \Omega_C^1 \otimes f^\vee) \cong H^1(C, \Omega_C^1).$$

From (6.7), (6.8), (6.9) and (6.10), the cup-product μ_1 and μ_2 are reduced to the following pairings;

(6.11) $$\mu_1 \colon H^1(C, f) \otimes H^0(C, \Omega_C^1 \otimes f^\vee) \longrightarrow H^1(C, \Omega_C^1)$$

(6.12) $$\mu_2^1 \colon H^0(C, \Omega_C^1 \otimes f^\vee) \otimes H^0(C, \Omega_C^1 \otimes f^\vee) \longrightarrow H^0(C, (\Omega_C^1)^2 \otimes (f^\vee)^2)$$

(6.13) $$\mu_2^2 \ \colon H^0(C, \mathcal{O}_C \oplus T_1) \otimes H^0(C, \Omega_C^1 \otimes f^\vee)$$
$$\longrightarrow H^0(C, \Omega_C^1 \otimes f^\vee) \oplus H^0(C, T_1 \otimes \Omega_C^1 \otimes f^\vee).$$

The pairings μ_1 and μ_2^2 are clearly surjective. Hence the proof of (B) and (C) of Theorem 3.2 (or, equivalently, Main Theorem) is reduced to showing the pairing μ_2^1 is surjective.

If a base curve C is a rational curve, the pairing μ_2^1 is surjective. In fact, the degree of the sheaf $\Omega_C^1 \otimes f^\vee$ is positive. This implies the case (B) of Main Theorem.

If the genus of a base curve C is greater than zero, we apply the following lemma due to Mumford (cf. (13), Theorem 6).

Lemma 6.1. *Let C be a nonsingular complete curve of genus g and L is an invertiblesheaf on C such that* $\deg L \geqq 2g+1$. *Then the natural pairing*

$$H^0(C, L) \otimes H^0(C, L) \longrightarrow H^0(C, L^2)$$

is surjective.

From this lemma, we conclude that the pairing μ_2^1 is surjective if

$$\deg \Omega_C^1 \otimes f^\vee \geqq 2g+1.$$

Since $\deg f^\vee = \mathrm{N}$, the pairing μ_2^1 is surjective if N is greater than two. This implies the case (C) of Main Theorem.

Remark 6.2. *There exist a counter-example to the infinitesimal Torelli problem.* Let C be an elliptic curve and let P_1 and P_2 be two distinct points on C. There exists a branched double covering π: $\tilde{C} \to C$ whose branched points are P_1 and P_2. Let E be an elliptic curve. Then there are natural involtions;

$$\iota_1 : \tilde{C} \longrightarrow \tilde{C}$$
$$\iota_2 : E \longrightarrow E.$$

Put $X = \tilde{C} \times E$. Then we have an involution $\iota = (\iota_1, \iota_2)$: $X \to X$. The minimal resolution $\tilde{X}$ of the quotient variety $X/(1, \iota)$ has a natural elliptic fibration over C:

$$\begin{array}{ccc} X/(1,\iota) & \longleftarrow & \tilde{X} \\ \downarrow & & \downarrow \varphi \\ C & \overset{\sim}{=\!=\!=} & C. \end{array}$$

The general fibre of φ is a fixed elliptic curve E and it has exactly two singular fibres of the type I_0^* on the branched points. Hence this elliptic surface $\tilde{X}$ has a constant functional invariant and $\mathrm{N} = \chi(\tilde{X}, \mathcal{O}_X) = 1$.

We can apply the same argument as in 6.3 and the infinitesimal Torelli theorem holds for X if and only if the cup-product μ_2^1 in (6.12) is surjective.

But, since $\deg f^\vee = \mathrm{N} = 1$ and Ω_C^1 is trivial, we have $\deg \Omega_C^1 \otimes f^\vee = 1$. Hence, by Riemann-Roch theorem, we have

$$\dim {}_C H^0(C, \Omega_C^1 \otimes f^\vee) = 1, \ \dim {}_C H^0(C, (\Omega_C^1)^2 \otimes (f^\vee)^2) = 2.$$

This implies the cup-product μ_2^1 in (6.12) is not surjective.

§ 7. The infinitesimal period map of elliptic bundles. (Counter-examples to the Local Torelli theorem).

7.0. Let φ: X→C be an elliptic bundle with a base curve C, that is, a fibre bundle over a curve C whose typical fibre and structure group is an elliptic curve E. The following theorem is due to Kodaira (10).

Theorem 7.1. *Let φ: $X \to C$ be an elliptic surface without multiple singular fibres and the exceptional curves of the first kind. Then the following conditions are equivalent.*

(*a*) φ: *$X \to C$ is an elliptic bundle.*

(*b*) $\chi(X, \mathcal{O}_X) = \mathbf{0}$.

(*c*) $R^1\varphi_* \mathcal{O}_X \simeq \mathcal{O}_C$.

(*d*) $\Omega_X^2 \simeq \varphi^* \Omega_C^1$.

Let X be a relatively minimal surface. Assume that the first Betti number $B^1(X)$ of X is odd and greater than one. By the classification of surfaces (Kodaira (12), p790, Table I), X is an elliptic surface induced by the algebraic reduction. Moreover this elliptic surface is obtained from an elliptic bundle by means of a finite number of logarithmic transformations.

In other word, if the first Betti number $B^1(X)$ of X is odd and greater than one and its elliptic fibration has no multiple fibre, then it is an elliptic bundle.

7.1. Now we shall study the infinitesimal period map of elliptic bundles. (cf. Theorem 2.1.)

Let $\varphi: X \to C$ be an elliptic bundle. Assume that the geometric genus $p_g(X)$ is positive. Since φ is smooth, we have an exact sequence

$$0 \longrightarrow \varphi^*\Omega_C^1 \longrightarrow \Omega_X^1 \longrightarrow \Omega_{X/C}^1 \longrightarrow 0 \tag{7.1}$$

and *an isomorphism*

$$\Omega_{X/C}^1 \simeq \omega_{X/C}.$$

By Theorem 7.1, we have $\omega_{X/C} \simeq \mathcal{O}_X$. Hence, from (7.1) and this, we get the following exact sequences.

$$0 \longrightarrow \Omega_C^1 \longrightarrow \varphi_*\Omega_X^1 \longrightarrow \mathcal{O}_C \longrightarrow 0. \tag{7.2}$$

$$0 \longrightarrow \Omega_C^1 \longrightarrow R^1\varphi_*\Omega_X^1 \longrightarrow \mathcal{O}_C \longrightarrow 0. \tag{7.3}$$

Lemma 7.2. Let $\varphi: X \to C$ be as above. Then

(i) $R^1\varphi_*\Omega_X^1 \simeq \varphi_*\Omega_X^1$,

(ii) the exact sequences (7.2) and (7.3) split if and only if $B^1(X)$ is even.

Proof. The assertion (i) is easy. Hence we omit the proof. From (7.2), we have an long exact sequence

$$0 \longrightarrow H^0(C, \Omega_C^1) \longrightarrow H^0(C, \varphi_*\Omega_X^1) \xrightarrow{r} H^0(C, \mathcal{O}_C) \longrightarrow H^1(C, \Omega_C^1). \tag{7.4}$$

The splitting of the exact sequence (7.2) is equivalent to the surjectivity of the map r. Using the Leray spectral sequence and (c) of Theorem 7.1, we have $h^{1,0}(X) = \dim_C H^0(C, \varphi_*\Omega_X^1)$, $h^{0,1}(X) = q(X) = g(C)+1$. By Remark 2.2, $B^1(X)$ is even if and only if $h^{1,0}(X) = h^{0,1}(X)$. From (7.4), the map r is surjective if and only if $h^{1,0}(X) = g(C)+1 = h^{0,1}(X)$. Q.E.D.

Let $\varphi: X \to C$ be an elliptic bundle. By (d) of Proposition 7.1, we have an isomorphism

$$\varphi_*\Omega_X^2 \simeq \Omega_C^1.$$

By this and (i) of Lemma 7.2, the cup-products μ_1 and μ_2 in (3.7) and (3.8) are reduced to the following pairings;

$$\mu_1: H^0(C, \varphi_*\Omega_X^1) \otimes H^0(C, \Omega_C^1) \longrightarrow H^0(C, \varphi_*\Omega_X^1 \otimes \Omega_C^1). \tag{7.5}$$

$$\mu_2: H^1(C, \varphi_*\Omega_X^1) \otimes H^0(C, \Omega_C^1) \longrightarrow H^1(C, \varphi_*\Omega_X^1 \otimes \Omega_C^1). \tag{7.6}$$

From the isomorphism above, we have $p_g(X)=g(C)$. Since we assume that the geometric genus $p_g(X)$ is positive, the genus $g(C)$ is greater than zero.

If a base curve C is an elliptic curve, then Ω_C^1 is isomorphic to the structure sheaf $\mathcal{O}_C$. Hence the cup-products in (7.5) and (7.6) are always surjective. Hence *the infinitesimal Torelli theorem holds in this case.*

Now *we assume that $g(C)$ is greater than one.* From the exact sequence (7.2), we get an exact sequence

$$(7.7) \qquad 0 \longrightarrow \mathrm{H}^0(\mathrm{C}, \Omega_C^1 \otimes \Omega_C^1) \longrightarrow \mathrm{H}^0(\mathrm{C}, \varphi_*\Omega_X^1 \otimes \Omega_C^1) \longrightarrow \mathrm{H}^0(\mathrm{C}, \Omega_C^1) \longrightarrow 0,$$

and an isomorphism

$$(7.8) \qquad \mathrm{H}^1(\mathrm{C}, \varphi_*\Omega_X^1 \otimes \Omega_C^1) \xrightarrow{\sim} \mathrm{H}^1(\mathrm{C}, \Omega_C^1).$$

By the isomorphism (7.8) and *a natural surjection $H^1(C, \varphi_*\Omega_X^1) \longrightarrow H^1(C, \mathcal{O}_C)$,* we can easily see that the cup-production in (7.6) is reduced to a pairing

$$H^1(C, \mathcal{O}_C) \otimes H^0(C, \Omega_C^1) \longrightarrow H^1(C, \Omega_C^1).$$

Since this pairing is perfect and $H^1(C, \Omega_C^1) \simeq \boldsymbol{C}$, it is surjective. Hence the cup-product in (7.6) is always surjective.

Next we consider the cup-product μ_1 in (7.5). From the exact sequence (7.4), we have the following exact sequence and isomorphism.

$$(7.8) \quad 0 \longrightarrow H^0(C, \Omega_C^1) \longrightarrow H^0(C, \varphi_*\Omega_X^1) \longrightarrow H^0(C, \mathcal{O}_C) \longrightarrow 0, \text{ if } B^1(X) \text{ is even.}$$

$$(7.9) \quad H^0(C, \Omega_C^1) \cong H^0(C, \varphi_*\Omega_X^1), \text{ if } B^1(X) \text{ is odd.}$$

From these facts, we can reduce the cup-product μ_1 in (7.5) to the following pairings.

The case in which $B^1(X)$ is even.

$$(7.10) \quad H^0(C, \Omega_C^1) \otimes H^0(C, \Omega_C^1) \longrightarrow H^0(C, \Omega_C^1 \otimes \Omega_C^1).$$

$$(7.11) \quad H^0(C, \mathcal{O}_C) \otimes H^0(C, \Omega_C^1) \longrightarrow H^0(C, \Omega_C^1).$$

The case in which $B^1(X)$ is odd.

$$(7.12) \quad H^0(C, \Omega_C^1) \otimes H^0(C, \Omega_C^1) \longrightarrow H^0(C, \Omega_C^1 \otimes \Omega_C^1) \subsetneqq H^0(C, \varphi_*(\Omega_X^1) \otimes \Omega_C^1).$$

The pairing (7.11) *is clearly surjective. And the pairing* (7.10) *is surjective if* $g(C)=2$ *or if* $g(C)>2$ *and* C *is non-hyperelliptic.* (Noether).

The pairing (7.12) is equal to (7.10) and the image of this pairing is contained in *the proper subspace* $H^0(C, \Omega_C^1 \otimes \Omega_C^1)$ of $H^0(C, \varphi_*(\Omega_X^1) \otimes \Omega_C^1)$.

We summarize our results.

Theorem 7.3. *Let φ: $X \rightarrow C$ be an elliptic bundle. Assume that the genus of a base curve is greater than one. Then*

(i) *if $B^1(X)$ is even, the infinitesimal Torelli theorem holds for X, If and only if $g(C)=2$ or $g(C)>2$ and C is non-hyperelliptic,*

(ii) *if $B^1(X)$ is odd, the infinitesimal Torelli theorem does not hold for X. Moreover the infinitesimal period map has a kernel whose dimension is equal to or greater than $g(C)$.*

Remark 7.4. If a base curve C is an elliptic curve, the infinitesimal Torelli theorem always holds. In fact, *if $B^1(X)$ is even, X is a complex torus. And if $B^1(X)$ is odd, X is a Kodaira surface.* In each case, X has the trivial canonical bundle.

Remark 7.5. Maehara (17) showed that an elliptic bundle with odd $B^1(X)$ has the smooth Kuranishi space of dimension $4g(C)-2=\dim H^1(X, \Theta_X)$. Hence the period map is not a local embedding, if a genus of a base curve is greater than one. In this sense, *an elliptic bundle with odd $B^1(X)$ whose base curve has a genus $g(C)\geqq 2$ is a counter-example to Local Torelli theorem in the sense of Griffiths*(**3**).

Remark 7.6. If an elliptic surface has a multiple fibre, the infinitesimal Torelli theorem does not hold. (cf. Chakiris (1))

§ 8. Tables.

Let φ: $X \to C$ be a minimal elliptic surface without multiple fibres. Our results in this paper are summarized in the following Tables.

Put $N=1-q(X)+p_g(X)$ and let δ denote the infinitesimal period map (3.1).

(I) $N\geqq 1$.

(a) $g(C)=0$.

N	$p_g(X)$	$J(X)$	$\dim_{\mathbf{C}} H^1(X, \Theta_X)$	δ	X
1	0		10	X	rational
2	1		20	O	K-3
$\geqq 3$	$N-1$	not const.	$11N-3$	O	
		const. $\neq 0, 1$	$11N-3$	O	
		$=0, 1$.	?	?	

(b) $g(C)\geqq 1$.

N	$p_g(X)$	$J(X)$	$\dim_{\mathbf{C}} H^1(X, \Theta_X)$	δ
1 or 2	$N+g-1$	not const.	$11N+3g-3$	O
		const.	?	(?)*
$\geqq 3$	$N+g-1$	not const.	$11N+3g-3$	O
		const. $\neq 0, 1$.	$11N+4g-3$	O
		$=0, 1$.	?	?

(II) N=0. ($p_g(X)=g(C)$)

$g(C)$	$B^1(X)$	dim $cH^1(X, \Theta_X)$	C	δ	X
0	2	4		X	elliptic ruled
	1	4		X	Hopf
1	4	4		O	complex torus
	3	2		O	Kodaira
$\geqq 2$	$2g+2$	$4g-2$	$g(C)=2$ or non-hyperelliptic	O	
			$g(C)>2$ and hyperelliptic	X	
	$2g+1$	$4g-2$		X	

O...injective. X...not injective.

(?)*. There exists a counter-example to the infinitesimal Torelli theorem (cf. Remark 6.2).

Appendix. The global Torelli problem of Kodaira surfaces.

10.0. The coarse moduli space of Kodaira surfaces.

Definition 10.1. Let X be a relatively minimal surface. Then X is called Kodaira surface if $B^1(X)$ is equal to three and the canonical bundle of X is trivial.

Theorem 10.2. (*Kodaira*(**11**), *Theorem* **19**) *Let X be a Kodaira surface. Then X has an unique structure of an elliptic surface φ: $X \to C$ over an elliptic curve C. It is obtained as a quotient manifold of $\mathbf{C}^2$ by an affine transformation group generated by the following elements.*

$$g_1: (Z_1, Z_2) \longrightarrow (Z_1, Z_2+\frac{\omega+\sqrt{-1}}{k})$$

$$(10.1) \qquad g_2: (Z_1, Z_2) \longrightarrow (Z_1, Z_2+\frac{\omega+\sqrt{-1}}{k}\tau)$$

$$g_3: (Z_1, Z_2) \longrightarrow (Z_1+1, Z_2+Z_1)$$

$$g_4: (Z_1, Z_2) \longrightarrow (Z_1+\omega, Z_2-\sqrt{-1}\,Z_1)$$

Here, we denote by (Z_1, Z_2) a global coordinate of $\mathbf{C}^2$, by k a positive integer, by ω and τ elements of upper half plane $\mathrm{H}=\{\xi|\mathrm{Im}\,\xi>0\}$.

Definition 10.3. Let $\Gamma(\tau, \omega, k)$ denote a group generated by elements in (10.1) and we put

$$X_{(\tau,\omega,k)}=\mathbf{C}^2/\Gamma(\tau, \omega, k).$$

We call a Kodaira surface above "Kodaira surface of type (τ,ω,k)". The integer k is called a degree of a Kodaira surface.

Now we shall prove the following theorem.

Theorem 10.4. *Let k be a positive integer. The coarse moduli space of Kodaira surface of degree k is represented by the quotient analytic space*

$$\mathrm{H}\times\mathrm{H}/\begin{pmatrix} 1 & \mathbf{Z} \\ 0 & 1 \end{pmatrix}\times SL(2, \mathbf{Z}).$$

Proof. We put $\Gamma_k=\bigcup_{\tau,\omega}\Gamma(\tau, \omega, k)$. Then Γ_k acts on $\mathbf{C}^2\times\mathrm{H}^2$ in the obvious manner. Since this action is properly discontinuous, free from fixed points, and trivial on the factor H^2, we have a smooth fibration of Kodaira surface of degree k;

$$\pi\colon \mathbf{C}^2\times\mathrm{H}^2/\Gamma_k \longrightarrow \mathrm{H}^2. \tag{10.2}$$

It is easy to see that at each point of H^2 this family is complete and effectively parametrized. Hence we must only prove the following lemma.

Lemma 10.5. *Let (τ,ω) and (τ', ω') be two points on* H^2. *Kodaira surfaces of type (τ, ω, k) and (τ', ω', k) are mutually isomorphic to each other if and only if there exist an integer m and an element* $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of $SL(2, \mathbf{Z})$ such that

$$\tau=\tau'+m,\ \omega=\frac{a\omega'+b}{c\omega'+d}. \tag{10.3}$$

A proof is straightforward. Hence we omit it.

10.1. Explicit calculation of period map of Kodaira surfaces.

We first recall the following theorem due to Maehara (17).

Theorem 10.6. *Let $\varphi\colon X\to C$ be a Kodaira surface of degree k. We have a cohomological relation*

$$H^2(X, \mathbf{Z})=H^1(C, \mathbf{Z})\otimes H^1(E, \mathbf{Z})\oplus \mathbf{Z}/k\mathbf{Z} \tag{10.4}$$

where E denote a regular fibre.

Let P denote the cup-product on $H^2(X, \mathbf{Z})$. Then, by the index theorem, the symmetric bilinear form P defines a polarization in the sense of (14). By the Theorem 10.6, we can choose a base of $H^2(X, \mathbf{Z})/(\text{torsion})$

$$A_1,\ A_2,\ A_3,\ A_4,$$

such that

$$(P(A_i, A_j))=\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{10.5}$$

We put $H_{\mathbf{Z}}^2=H^2(X, \mathbf{Z})/(\text{torsion})$. Then $H_{\mathbf{Z}}^2$ is a free $\mathbf{Z}$-module of rank four. Moreover we put $H_{\mathbf{R}}^2=H_{\mathbf{Z}}^2\otimes\mathbf{R}$, $H_{\mathbf{C}}^2=H_{\mathbf{Z}}^2\otimes\mathbf{C}$. We define the orthogonal group with respect to a polarization P by

$$\mathrm{O}(H_{\mathbf{R}}^2, \mathrm{P})=\{a\in GL(H_{\mathbf{R}}^2)\mid {}^t a\ \mathrm{P}\ a=\mathrm{P}\}$$

Moreover, we put $\mathrm{SO(P)}=\{a\in\mathrm{O}(H_{\mathbf{R}}^2, \mathrm{P})\mid\det(a)=+1\}$. Then we can easily see the followings.

(I) $$\mathrm{SO}(\mathrm{P})=\mathrm{SO}^0(\mathrm{P})\amalg\mathrm{SO}^0(\mathrm{P})\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix}$$

where $\mathrm{SO}^0(\mathrm{P})$ denote the identity component of SO(P).

(II) There exists an isomorphism of Lie groups:

$$f\colon \mathrm{SL}(2, \boldsymbol{R})\times\mathrm{SL}(2, \boldsymbol{R}) \longrightarrow \mathrm{SO}^0(\mathrm{P}).$$

(III) $\mathrm{SO}(\mathrm{P}, H^2_{\boldsymbol{Z}})=\{a\in\mathrm{SL}(H^2_{\boldsymbol{Z}})|{}^t a\,\mathrm{P}\,a=\mathrm{P}\}$ is isomorphic to $\mathrm{SL}(2, \boldsymbol{Z})\times\mathrm{SL}(2, \boldsymbol{Z})$.

The classifying space of polarized Hodge structures $\{H^2_{\boldsymbol{Z}}, F^p, \mathrm{P}\}$ with the Hodge numbers $h^{2,0}=h^{0,2}=1$, $h^{1,1}=2$ is given by

$$D=\left\{\begin{matrix}[\lambda_1;\lambda_2;\lambda_3;\lambda_4]\in \boldsymbol{P}^3(\boldsymbol{C}),\ \lambda_1\lambda_4-\lambda_2\lambda_3=0\\ -\lambda_1\bar{\lambda}_4+\lambda_2\bar{\lambda}_3+\lambda_3\bar{\lambda}_2-\lambda_4\bar{\lambda}_1>0\end{matrix}\right\}$$

Then D has the two connected components D^+ and D^- which is given by

$$D^+(\text{resp. } D^-)=\{[\lambda_1;\lambda_2;\lambda_3;\lambda_4]\in D,\ \mathrm{Im}\,(\lambda_2/\lambda_1)>0\ (\text{resp. } <0)\}.$$

Then we can easily see that the Lie group $\mathrm{SO}^0(\mathrm{P})$ acts transitively on D^+, and D^+ is a symmetric bounded domain of type IV. Moreover there is an isomorphism (10.6)

(10.6) $$\begin{aligned}\Phi\colon H\times H &\longrightarrow D^+\\ (\tau,\omega)| &\longrightarrow [1;\tau;\omega;\tau\omega].\end{aligned}$$

Note that this isomorphism is equivariant for the group isomorphism f in (II) with respect to the natural action of $\mathrm{SL}(2, \boldsymbol{R})\times\mathrm{SL}(2, \boldsymbol{R})$ to $H\times H$.

Let us consider the complex analytic family of Kodaira surfaces of degree k in (10.2). Since the base space $H\times H$ is simply connected, we can trivialize the local system $R^2\pi_*\boldsymbol{Z}$. *If we choose a suitable trivialization, the period map associated with the variation of Hodge structures of Kodaira surfaces is given by the map* Φ *in* (10.6). *Since* Φ *is equivariant to the isomorphism* f *in (II), we have another period map*

$$\bar{\Phi}\colon H\times H/\begin{pmatrix}1&\boldsymbol{Z}\\0&1\end{pmatrix}\times\mathrm{SL}(2, \boldsymbol{Z}) \longrightarrow D^+/\mathrm{SO}^0(\mathrm{P}, H^2_{\boldsymbol{Z}})$$

from the coarse moduli space of Kodaira surface of degree k to the quotient analytic space.

We have the following commutative diagram;

(10.7) $$\begin{array}{ccccc} H\times H & & \xrightarrow{\quad\Phi\quad} & & D^+\\ \Big\downarrow & \searrow & & & \Big\downarrow\\ & & H\times H/\begin{pmatrix}1&\boldsymbol{Z}\\0&1\end{pmatrix}\times\mathrm{SL}(2, \boldsymbol{Z}) & & \\ & & & \searrow^{\bar{\Phi}} & \\ H\times H/\mathrm{SL}(2, \boldsymbol{Z})\times\mathrm{SL}(2, \boldsymbol{Z}) & & \xrightarrow{\quad\sim\quad} & & D^+/\mathrm{SO}(\mathrm{P}, H^2_{\boldsymbol{Z}})\end{array}$$

The global Torelli problem of Kodaira surfaces (of degree k) asks *whether the period map* $\bar{\Phi}$ *is injective.* By the commutative diagram (10.7) we have the following theorem.

Theorem 10.6. *Every fibre of $\bar{\Phi}$ consists of infinitely many points. Hence the global Torelli theorem does not hold for Kodaira surface, though the local Torelli theorem does hold.*

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*) Added in Proof: Recently, K. Chakiris [20] proved the weak global Torelli theorem.