

On the Influence of Moments on Approximations by Portion of a Chebyshev Series in Central Limit Convergence

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1. Introduction

Let $X_i, i = 1, 2, 3, \dots$ be a sequence of independent and identically distributed random variables with $EX_i = 0$ and $\text{var } X_i = 1$. Write $F(x)$ for the distribution function and $f(t)$ for the characteristic function of X_i and put $S_n = \sum_{i=1}^n X_i$. Then,

$$F_n(x) = P(S_n \leq x\sqrt{n}) \rightarrow \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{1}{2}u^2} du$$

as $n \rightarrow \infty$. We shall herein be concerned with the influence of moments of X_i on the rate of convergence to zero of

$$A_{kn} = \sup_x |F_n(x) - G_{kn}(x)|$$

where

$$G_{kn}(x) = \Phi(x) + \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \sum_{s=1}^k Q_s(x) n^{-\frac{1}{2}s}$$

is a given portion of the Chebyshev series corresponding to the X_i (see for example Gnedenko and Kolmogorov [2], Section 38), the $Q_j(x)$ being polynomials of degree $3j - 1$ whose coefficients depend on the first $(j + 2)$ moments of X_i . Now, Cramér (see [2], Section 45) has shown that for distributions satisfying the condition (C) (that is, $\limsup_{t \rightarrow \infty} |f(t)| < 1$) and if $E|X_i|^{k+2} < \infty$ ($k \geq 1$), then $A_{kn} = o(n^{-k/2})$ as $n \rightarrow \infty$. Furthermore, Ibragimov [4] has produced necessary and sufficient conditions, under (C), for (i) $A_{kn} = o(n^{-k/2})$ ($k \geq 1$) and (ii) $A_{kn} = O(n^{-(k+\delta)/2})$, $0 < \delta \leq 1$, $k \geq 1$, but these conditions are not in general moment conditions. We shall provide, also under (C), some necessary and sufficient conditions in terms of moments on the rate of convergence of A_{kn} to zero.

In order to avoid presupposing the existence of moments of higher order than the second, we shall follow the formulation of [4] and prescribe an arbitrary numerical sequence $\beta_1 = 0, \beta_2 = 1, \beta_3, \beta_4, \dots$. On the basis of this sequence we form polynomials $Q_k(x)$ in such a way that their coefficients are expressed in terms of $\beta_1, \dots, \beta_{k+2}$ in the same way as the coefficients of the classical polynomials $Q_k(x)$ are expressed in terms of the cumulants $\kappa_1, \dots, \kappa_{k+2}$ of X_i . That is,

$$Q_k(x) = - \sum \frac{1}{j_1! \dots j_k!} \left(\frac{\beta_3}{3!} \right)^{j_1} \dots \left(\frac{\beta_{k+2}}{(k+2)!} \right)^{j_k} H_{3j_1 + \dots + (k+2)j_k - 1},$$

where the summation is over all non-negative solutions of $j_1 + 2j_2 + \dots + kj_k = k$ and $H_m(x)$ is the Hermite-Chebyshev polynomial

$$H_m(x) = (-1)^m e^{\frac{1}{2}x^2} \frac{d^m}{dx^m} e^{-\frac{1}{2}x^2}$$

(Petrov [8]). $Q_k(x)$ will henceforth be interpreted in this way.

Let $\alpha_1 = 0, \alpha_2 = 1, \alpha_3, \alpha_4, \dots$ be the ‘‘moment’’ sequence corresponding to the ‘‘cumulant’’ sequence $\beta_1 = 0, \beta_2 = 1, \beta_3, \beta_4, \dots$. We shall establish the following results.

Theorem 1. *In order that*

$$\sum_{n=1}^{\infty} n^{-1+(k+\delta)/2} \sup_x |F_n(x) - G_{kn}(x)| < \infty \tag{1}$$

where k is a non-negative integer and $0 < \delta < 1$, it is necessary and for $k=0$ or for distributions satisfying (C) also sufficient that

$$E|X_i|^{k+2+\delta} < \infty \quad \text{and} \quad \alpha_j = EX_i^j, \quad j = 1, 2, \dots, k+2. \tag{2}$$

Theorem 2. *In order that the relation (1) hold, where $0 < \delta < 1$, it is necessary and for distributions satisfying (C) or for $k=0$ also sufficient that*

$$f(t) = \exp \left\{ \sum_{s=2}^{k+2} \frac{(it)^s}{s!} \beta_s + |t|^{k+2} \gamma(t) \right\}, \tag{3}$$

where for $A > 0$,

$$\int_0^A \frac{|\gamma(t)|}{t^{1+\delta}} dt < \infty. \tag{4}$$

Unfortunately it has not been possible to treat the case $\delta = 0$ in general and then not without certain presuppositions on the existence of moments. In this case we find the following result.

Theorem 3. *Suppose $E|X_i|^{k+2} < \infty$ where k is a non-negative even integer and $\alpha_j = EX_i^j, j = 1, 2, \dots, k+2$. Then, for (1) to hold with $\delta = 0$ it is necessary and for $k=0$ or for distributions satisfying (C) also sufficient that $E|X_i|^{k+2} \log(1 + |X_i|) < \infty$.*

These theorems extend the work of Heyde [3] where the results for the case $k=0$ were obtained.

2. Preliminary Lemmas

Lemma 1. *Suppose $E|X_i|^r < \infty$ for some integer $r \geq 2$. Then $f(t)$ is representable in the form*

$$f(t) = \exp \left\{ \sum_{s=2}^r \frac{(it)^s}{s!} \kappa_s + |t|^r \gamma(t) \right\} \tag{5}$$

where $\gamma(t) = o(1)$ as $t \rightarrow 0$ and κ_s denotes the s -th cumulant of X_i . Furthermore, there exists an $\varepsilon > 0$ such that for $0 < t < \varepsilon, |\gamma(t)| > 0$ or $|\gamma(t)| \equiv 0$. If $|\gamma(t)| \equiv 0$ for $0 < t < \varepsilon, X_i$ has a normal distribution.

Proof. The representation in the form (5) with $\gamma(t)=o(1)$ as $t \rightarrow 0$ follows simply from a Taylor expansion of $\log f(t)$ (e.g. [2], p. 64).

Next, suppose $\gamma(t)=0$ for all $t \in \{t_k\}$ where $\{t_k\}$ is a sequence of non-zero real numbers converging to zero. Then,

$$\exp \left\{ \sum_{s=2}^r \frac{(it)^s}{s!} \kappa_s + |t|^r \gamma(t) \right\} = \exp \left\{ \sum_{s=2}^r \frac{(it)^s}{s!} \kappa_s \right\} \quad (6)$$

for all $t \in \{t_k\}$ and applying Theorem 4.2.1 of Linnik [5], we have that (6) holds for all real t . However, this is impossible unless $r=2$ since the left hand side of (6) represents a characteristic function and the right hand side does not, in view of Marcinkiewicz's Theorem (e.g. Lukacs [7], p. 147), unless $r=2$. Thus, if $r>2$ we must be able to choose $\varepsilon>0$ so that $\gamma(t)$ has no zeros in $(0, \varepsilon)$. If $r=2$, on the other hand, either (6) holds for all t in which case $\gamma(t) \equiv 0$ and X_i has a normal distribution or zero is not a limit point of a sequence of zeros of $\gamma(t)$ and hence we can choose an interval $(0, \varepsilon)$ containing no zeros of $\gamma(t)$.

Lemma 2. Suppose $E|X_i|^r < \infty$ for some integer $r \geq 2$. Then, $f(t)$ is representable in the form

$$f(t) = \sum_{s=0}^r \frac{(it)^s}{s!} \mu_s + |t|^r \beta(t) \quad (7)$$

where $\beta(t)=o(1)$ as $t \rightarrow 0$ and μ_s denotes the s -th moment of X_i . Furthermore, for any $A>0$ and $0 \leq \delta < 1$, the conditions $\int_0^A |\beta(t)| t^{-(1+\delta)} dt < \infty$ and $\int_0^A |\gamma(t)| t^{-(1+\delta)} dt < \infty$ are equivalent, $\gamma(t)$ being given by (5) and these conditions are in turn equivalent to $E|X_i|^{r+\delta} < \infty$ if $\delta > 0$, $E|X_i|^r \log(1+|X_i|) < \infty$ if r is even and $\delta=0$.

Proof. Firstly, the representation of $f(t)$ in the form (7) with $\beta(t)=o(1)$ as $t \rightarrow 0$ follows simply from a Taylor expansion of $f(t)$. Also, from Lemma 3 of [4] we find that as $t \rightarrow 0$,

$$\log f(t) - \sum_{s=2}^r \frac{(it)^s}{s!} \kappa_s = f(t) - \sum_{s=0}^r \frac{(it)^s}{s!} \mu_s + \frac{A_{r+1}}{(r+1)!} (it)^{r+1} + o(t^{r+1}),$$

where A_{k+1} is a constant, so that

$$|t|^r \gamma(t) = |t|^r \beta(t) + \frac{A_{r+1}}{(r+1)!} (it)^{r+1} + o(t^{r+1}) \quad (8)$$

and we readily deduce the equivalence of

$$\int_0^A |\beta(t)| t^{-(1+\delta)} dt < \infty \quad \text{and} \quad \int_0^A |\gamma(t)| t^{-(1+\delta)} dt < \infty, \quad 0 \leq \delta < 1.$$

Now suppose that

$$\int_0^A |\beta(t)| t^{-(1+\delta)} dt < \infty.$$

This implies

$$\int_0^A |\operatorname{Re} \beta(t)| t^{-(1+\delta)} dt < \infty$$

where Re denotes the real part and hence that

$$\left| \int_0^A \text{Re } \beta(t) t^{-(1+\delta)} dt \right| < \infty.$$

It is with this last condition that we shall work. We have

$$\begin{aligned} \left| \int_0^A \frac{\text{Re } \beta(t)}{t^{1+\delta}} dt \right| &= \left| \int_0^A \frac{1}{t^{r+\delta+1}} \left(\int_{-\infty}^{\infty} \text{Re} \left(e^{itx} - \sum_{s=0}^r \frac{(itx)^s}{s!} \right) dF(x) \right) dt \right| \\ &= \left| \int_{-\infty}^{\infty} dF(x) \int_0^A \frac{\text{Re} \left(e^{itx} - \sum_{s=0}^r \frac{(itx)^s}{s!} \right)}{t^{r+\delta+1}} dt \right| \\ &= \left| \int_{-\infty}^{\infty} dF(x) \int_0^A \frac{\left(\cos tx - \sum_{s=0}^R (-1)^s \frac{(tx)^{2s}}{(2s)!} \right)}{t^{r+\delta+1}} dt \right| \end{aligned} \tag{9}$$

where $R = [r/2]$, the integer part of $r/2$. But, after two integrations by parts,

$$\begin{aligned} &\int_0^A \frac{\left(\cos tx - \sum_{s=0}^R (-1)^s \frac{(tx)^{2s}}{(2s)!} \right)}{t^{r+\delta+1}} dt \\ &= \frac{-\left(\cos Ax - \sum_{s=0}^R (-1)^s \frac{(Ax)^{2s}}{(2s)!} \right)}{(r+\delta) A^{r+\delta}} - \frac{x \left(\sum_{s=0}^{R-1} (-1)^s \frac{(Ax)^{2s+1}}{(2s+1)!} - \sin Ax \right)}{(r+\delta)(r+\delta-1) A^{r+\delta-1}} \\ &\quad + \frac{x^2}{(r+\delta)(r+\delta-1)} \int_0^A \frac{\left(\sum_{s=0}^{R-1} (-1)^s \frac{(tx)^{2s}}{(2s)!} - \cos tx \right)}{t^{r+\delta-1}} dt, \end{aligned}$$

so that, recalling that $E|X_i|^r < \infty$, we must have using (9),

$$\left| \int_{-\infty}^{\infty} x^2 dF(x) \int_0^A \frac{\left(\cos tx - \sum_{s=0}^{R-1} (-1)^s \frac{(tx)^{2s}}{(2s)!} \right)}{t^{r+\delta-1}} dt \right| < \infty.$$

Continuing this reduction, we find ultimately that

$$\left| \int_{-\infty}^{\infty} x^{2R} dF(x) \int_0^A \frac{(\cos tx - 1)}{t^{r+\delta+1-2R}} dt \right| < \infty$$

which transforms to give

$$\int_{-\infty}^{\infty} |x|^{r+\delta} \left\{ \int_0^{|x|} \frac{1 - \cos u}{u^{r+\delta+1-2R}} du \right\} dF(x) < \infty.$$

Thus, if r is even, $R = r/2$ and

$$\int_{-\infty}^{\infty} |x|^{r+\delta} \left\{ \int_0^{|x|} \frac{1 - \cos u}{u^{1+\delta}} du \right\} dF(x) < \infty, \tag{10}$$

while if r is odd, $R=(r-1)/2$ and

$$\int_{-\infty}^{\infty} |x|^{r+\delta} \left\{ \int_0^{A|x|} \frac{1-\cos u}{u^{2+\delta}} du \right\} dF(x) < \infty. \tag{11}$$

(10) and (11) are clearly equivalent to $E|X_i|^{r+\delta} < \infty$ if $\delta > 0$. On the other hand, when $\delta = 0$, we have for $|x| > 1$

$$\int_0^{A|x|} \frac{1-\cos u}{u} du = \int_0^A \frac{1-\cos u}{u} du + \int_A^{A|x|} \frac{du}{u} - \int_A^{A|x|} \frac{\cos u}{u} du \sim \log|x|$$

as $|x| \rightarrow \infty$ so that (10) is equivalent to the condition $E|X_i|^r \log(1+|X_i|) < \infty$ when $\delta = 0$. We have thus shown that $\int_0^A |\beta(t)| t^{-(1+\delta)} dt < \infty$ implies $E|X_i|^{r+\delta} < \infty$ if $0 < \delta < 1$, $E|X_i|^r \log(1+|X_i|) < \infty$ if r is even and $\delta = 0$.

Finally, suppose that $E|X_i|^{r+\delta} < \infty$, $\delta > 0$, or that $E|X_i|^r \log(1+|X_i|) < \infty$ if r is even and $\delta = 0$. Then,

$$\begin{aligned} \int_0^A \frac{|\beta(t)|}{t^{1+\delta}} dt &= \int_0^A \left| \frac{f(t) - \sum_{s=0}^r \frac{(it)^s}{s!} \mu_s}{t^{r+1+\delta}} \right| dt \\ &= \int_0^A \frac{1}{t^{r+1+\delta}} \left| \int_{-\infty}^{\infty} \left(e^{itx} - \sum_{s=0}^r \frac{(itx)^s}{s!} \right) dF(x) \right| dt \\ &\leq \int_{-\infty}^{\infty} \left\{ \int_0^A \left| \frac{\left(e^{itx} - \sum_{s=0}^r \frac{(itx)^s}{s!} \right)}{t^{r+1+\delta}} \right| dt \right\} dF(x) \\ &= \int_{-\infty}^{\infty} |x|^{r+\delta} \left\{ \int_0^{A|x|} \frac{\left| e^{iu} - \sum_{s=0}^r \frac{(iu)^s}{s!} \right|}{u^{r+1+\delta}} du \right\} dF(x). \end{aligned} \tag{12}$$

But, using the inequalities

$$\begin{aligned} \left| e^{ix} - \sum_{s=0}^r \frac{(ix)^s}{s!} \right| &\leq \frac{|x|^{r+1}}{(r+1)!} \quad \text{for } |x| \leq 1, \\ \left| e^{ix} - \sum_{s=0}^r \frac{(ix)^s}{s!} \right| &\leq (1+e)|x|^r \quad \text{for } |x| \geq 1, \end{aligned}$$

(for the first of these see e.g. Lemma 1 of [4] while the second is obtained by taking the modulus of each of the terms and bounding this) we have

$$\begin{aligned} I(|x|) &= \int_0^{A|x|} \frac{\left| e^{iu} - \sum_{s=0}^r \frac{(iu)^s}{s!} \right|}{u^{r+1+\delta}} du \\ &\leq \begin{cases} \frac{1}{(r+1)!} \int_0^1 \frac{du}{u^\delta} & \text{for } |x| \leq A^{-1} \\ \frac{1}{(r+1)!} \int_0^1 \frac{du}{u^\delta} + (1+e) \int_1^{A|x|} \frac{du}{u^{1+\delta}} & \text{for } |x| > A^{-1}. \end{cases} \end{aligned}$$

Thus, $I(|x|) \leq c_1$ if $\delta > 0$, $I(|x|) \leq c_2 \log(|x| + 1)$ if $\delta = 0$, where c_1 and c_2 are positive constants and using these results in (12) we have

$$\int_0^A |\beta(t)| t^{-(1+\delta)} dt < \infty$$

if $\delta > 0$ or if r is even and $\delta = 0$. This completes the proof of the lemma.

Lemma 3. *Suppose (1) holds and that $E|X_i|^r < \infty$ for some integer $2 \leq r \leq k+2$. Then, $\alpha_j = EX_i^j$, $j = 1, 2, \dots, r$.*

Proof. The result of the lemma is true by specification for $r=2$ and we develop a proof by induction.

Suppose that $E|X_i|^s < \infty$, some $s \geq 2$ and $\alpha_j = EX_i^j$, $j = 1, 2, \dots, s$. Then, if $E|X_i|^{s+1} < \infty$, let $Q_j^*(x)$, $1 \leq j \leq s-1$ be the classical Chebyshev polynomials expressed in terms of the cumulants κ_j , $j = 1, 2, \dots, s+1$ of X_i and write

$$G_{s-1,n}^*(x) = \Phi(x) + \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \sum_{j=1}^{s-1} Q_j^*(x) \frac{1}{n^{j/2}}.$$

We have

$$\sup_x |G_{s-1,n}(x) - G_{s-1,n}^*(x)| \leq \sup_x |F_n(x) - G_{s-1,n}(x)| + \sup_x |F_n(x) - G_{s-1,n}^*(x)|, \quad (13)$$

and from Theorem 1 of [4],

$$n^{(s-1)/2} \sup_x |F_n(x) - G_{s-1,n}^*(x)| = o(1) \quad (14)$$

as $n \rightarrow \infty$. Also, from (1),

$$\liminf_{n \rightarrow \infty} n^{(k+\delta)/2} \sup_x |F_n(x) - G_{kn}(x)| = 0,$$

so that

$$\begin{aligned} & \liminf_{n \rightarrow \infty} n^{(s-1)/2} \sup_x |F_n(x) - G_{s-1,n}(x)| \\ & \leq \liminf_{n \rightarrow \infty} n^{(s-1)/2} \sup_x |F_n(x) - G_{kn}(x)| + \liminf_{n \rightarrow \infty} n^{(s-1)/2} \sup_x |G_{kn}(x) - G_{s-1,n}(x)| \quad (15) \\ & = 0 \end{aligned}$$

since

$$G_{kn}(x) - G_{s-1,n}(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \sum_{j=s}^k Q_j(x) n^{-j/2}.$$

Consequently, using (14) and (15) in (13),

$$\liminf_{n \rightarrow \infty} n^{(s-1)/2} \sup_x |G_{s-1,n}(x) - G_{s-1,n}^*(x)| = 0. \quad (16)$$

But,

$$G_{s-1,n}(x) - G_{s-1,n}^*(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} [Q_{s-1}(x) - Q_{s-1}^*(x)] \frac{1}{n^{(s-1)/2}}$$

since $\alpha_j = EX_i^j$, $j = 1, 2, \dots, s$ implies $Q_j(x) = Q_j^*(x)$, $1 \leq j \leq s-2$, and hence (16) can only hold if $Q_{s-1}(x) = Q_{s-1}^*(x)$. This gives $\kappa_{s+1} = \beta_{s+1}$ upon identifying coefficients and hence $\alpha_{s+1} = EX_i^{s+1}$ as required.

3. Proof of Theorems

We start by proving Theorems 1 and 2 simultaneously in the following three steps. Firstly we note that the equivalence of (2) and (3), (4) follows immediately from Lemmas 1 and 2. Next we shall show that (3), (4) ensures (1) under (C) or if $k=0$ and lastly that (1) ensures (2).

(3), (4) \Rightarrow (1) under (C) or if $k=0$.

Firstly we note that $G_{kn}(x)$ has a bounded first derivative; let $|G'_{kn}(x)| \leq B$. Then, using a bound due to Esseen ([2], Section 39), we have for any $T > 0$,

$$|F_n(x) - G_{kn}(x)| \leq \frac{1}{\pi} \int_{-T}^T \left| \frac{f_n(t) - g_{kn}(t)}{t} \right| dt + c \frac{B}{T} \tag{17}$$

where $f_n(t)$ and $g_{kn}(t)$ are the characteristic functions corresponding to $F_n(x)$ and $G_{kn}(x)$ respectively and c is a positive constant.

Now, using (3) we have

$$\begin{aligned} f_n(t) &= \left[f\left(\frac{t}{\sqrt{n}}\right) \right]^n \\ &= \exp \left\{ \sum_{s=2}^{k+2} \frac{(it)^s}{s!} \frac{\beta_s}{n^{(s-2)/2}} + n \left| \frac{t}{\sqrt{n}} \right|^{k+2} \gamma\left(\frac{t}{\sqrt{n}}\right) \right\}, \end{aligned}$$

while from Lemma 5 of [4],

$$g_{kn}(t) = \exp \left\{ \sum_{s=2}^{k+2} \frac{(it)^s}{s!} \frac{\beta_s}{n^{(s-2)/2}} + D(t) \right\},$$

where $D(t) = O(|n^{-\frac{1}{2}} t|^{k+1})$ as $t \rightarrow 0$ so that

$$\begin{aligned} |f_n(t) - g_{kn}(t)| &\leq \left| \exp \left\{ \sum_{s=2}^{k+2} \frac{(it)^s}{s!} \frac{\beta_s}{n^{(s-2)/2}} \right\} \right| \left| \exp \left(n \left| \frac{t}{\sqrt{n}} \right|^{k+2} \gamma\left(\frac{t}{\sqrt{n}}\right) \right) - 1 \right| \\ &\quad + \left| \exp \left\{ \sum_{s=2}^{k+2} \frac{(it)^s}{s!} \frac{\beta_s}{n^{(s-2)/2}} \right\} - g_{kn}(t) \right|. \end{aligned} \tag{18}$$

Now in view of Lemma 1, we may choose $a, 0 < a < 1$, so small that $\max_{|t| < a} |\gamma(t)| \leq \frac{1}{4}$.

Then for $|t| < a\sqrt{n}$, using the inequality $|e^x - 1| \leq |x| e^{|x|}$, the first term on the right hand side of (18) is bounded by

$$\begin{aligned} &\exp \left\{ \sum_{s=1}^{\lfloor \frac{1}{2}(k+2) \rfloor} (-1)^s \frac{t^{2s}}{(2s)!} \frac{\beta_{2s}}{n^{s-1}} \right\} \frac{|t|^{k+2}}{n^{k/2}} \left| \gamma\left(\frac{t}{\sqrt{n}}\right) \right| \exp \left\{ \frac{|t|^{k+2}}{n^{k/2}} \left| \gamma\left(\frac{t}{\sqrt{n}}\right) \right| \right\} \\ &\leq \exp \left\{ n \left(\sum_{s=1}^{\lfloor \frac{1}{2}(k+2) \rfloor} (-1)^s \frac{t^{2s}}{(2s)!} \frac{\beta_{2s}}{n^s} + \frac{1}{4} \left| \frac{t}{\sqrt{n}} \right|^{k+2} \right) \right\} n \left| \frac{t}{\sqrt{n}} \right|^{k+2} \left| \gamma\left(\frac{t}{\sqrt{n}}\right) \right| \\ &\leq \exp(-t^2/8) n \left| \frac{t}{\sqrt{n}} \right|^{k+2} \left| \gamma\left(\frac{t}{\sqrt{n}}\right) \right| \end{aligned} \tag{19}$$

for n sufficiently large since $\beta_2 = 1$. Also, from Lemma 5 of [4], the second term on the right hand side of (18) is bounded by

$$\frac{c}{n^{(k+1)/2}} (|t|^{3(k+1)} + |t|^{k+1}) e^{-t^2/4} \tag{20}$$

for $|t| < b\sqrt{n}$ when b is sufficiently small, c being a positive constant. Then, choosing $\alpha = \min(a, b)$ and using (18), (19) and (20), we have

$$\begin{aligned} & \sum_{n=1}^{\infty} n^{-1+(k+\delta)/2} \int_{-\alpha\sqrt{n}}^{\alpha\sqrt{n}} \left| \frac{f_n(t) - g_{kn}(t)}{t} \right| dt \\ & \leq c \sum_{n=1}^{\infty} n^{-1-(1-\delta)/2} \int_{-\alpha\sqrt{n}}^{\alpha\sqrt{n}} (|t|^{3k+2} + |t|^k) e^{-t^2/4} dt \\ & \quad + \sum_{n=1}^{\infty} n^{-1+\delta/2} \int_{-\alpha\sqrt{n}}^{\alpha\sqrt{n}} e^{-t^2/8} |t|^{k+1} \left| \gamma\left(\frac{t}{\sqrt{n}}\right) \right| dt \\ & \leq A + \sum_{n=1}^{\infty} n^{(k+\delta)/2} \int_{-\alpha}^{\alpha} |u|^{k+1} |\gamma(u)| e^{-nu^2/8} du, \end{aligned} \tag{21}$$

A being a finite constant. Furthermore, using a standard Abelian theorem (e.g. Feller [1], Vol. II, p. 423), we have

$$\lim_{t \uparrow 1} (1-t)^{(k+2+\delta)/2} \sum_{n=1}^{\infty} n^{(k+\delta)/2} t^n = \Gamma((k+2+\delta)/2)$$

so that for $u \neq 0$ it is possible to choose a constant $K_1 > 0$ such that

$$\sum_{n=1}^{\infty} n^{(k+\delta)/2} e^{-nu^2/8} \leq K_1 (1 - e^{-u^2/8})^{-(k+2+\delta)/2}$$

Also, $|u| \leq \alpha < 1$, so that

$$1 - e^{-u^2/8} > \frac{1}{8} u^2 (1 - \frac{1}{16} \alpha^2)$$

and hence

$$\sum_{n=1}^{\infty} n^{(k+\delta)/2} e^{-nu^2/8} \leq K_2 |u|^{-(k+2+\delta)}$$

for some $K_2 > 0$. Consequently, noting that $|\gamma(t)|$ is symmetric in t , we have

$$\begin{aligned} & \sum_{n=1}^{\infty} n^{(k+\delta)/2} \int_{-\alpha}^{\alpha} |u|^{k+1} |\gamma(u)| e^{-nu^2/8} du \\ & = 2 \int_0^{\alpha} u^{k+1} |\gamma(u)| \left\{ \sum_{n=1}^{\infty} n^{(k+\delta)/2} e^{-nu^2/8} \right\} du \tag{22} \\ & \leq 2K_2 \int_0^{\alpha} \frac{|\gamma(u)|}{u^{1+\delta}} du < \infty \end{aligned}$$

in view of (4).

Next write $T_\alpha = \{t: \alpha\sqrt{n} \leq |t| \leq \alpha n^{(k+1)/2}\}$, noting that T_α is empty if $k=0$. Then, using condition (C), $\max_{|t| \geq \alpha} |f(t)| = \theta < 1$, so that for t in T_α ,

$$|f_n(t)| = \left| f\left(\frac{t}{\sqrt{n}}\right) \right|^n \leq \theta^n$$

and hence

$$\int_{T_\alpha} \left| \frac{f_n(t)}{t} \right| dt \leq 2\theta^n \int_{\alpha\sqrt{n}}^{\alpha n^{(k+1)/2}} \frac{dt}{t} \sim \theta^n (k+1) \log n$$

as $n \rightarrow \infty$. Consequently,

$$\sum_{n=1}^{\infty} n^{-1+(k+\delta)/2} \int_{T_\alpha} \left| \frac{f_n(t)}{t} \right| dt < \infty. \tag{23}$$

Furthermore, by the rules for forming the polynomials $Q_j(x)$,

$$g_{kn}(t) = e^{-\frac{1}{2}t^2} \left[1 + \sum_{j=1}^k P_j(it) n^{-\frac{1}{2}j} \right]$$

where P_j is a polynomial of degree $3j$ determined from the formal identity

$$\exp \left\{ \sum_{j=3}^{\infty} \frac{\beta_j}{j!} \frac{(it)^j}{n^{(j-2)/2}} \right\} = 1 + \sum_{j=1}^{\infty} P_j(it) n^{-j/2}.$$

Thus,

$$|g_{kn}(t)| \leq e^{-\frac{1}{2}t^2} \left[1 + \sum_{j=1}^{3k} a_{nj} |t|^j \right]$$

where the a_{nj} are polynomials in $n^{-\frac{1}{2}}$ which tend to zero as $n \rightarrow \infty$. Consequently, we certainly have

$$\int_{T_\alpha} \left| \frac{g_{kn}(t)}{t} \right| dt \leq K e^{-\alpha^2 n/2} n^{3k(k+1)/2}$$

for K a positive constant which gives

$$\sum_{n=1}^{\infty} n^{-1+(k+\delta)/2} \int_{T_\alpha} \left| \frac{g_{kn}(t)}{t} \right| dt < \infty, \tag{24}$$

and from (23) and (24) we obtain

$$\int_{T_\alpha} \left| \frac{f_n(t) - g_{kn}(t)}{t} \right| dt \leq \int_{T_\alpha} \left| \frac{f_n(t)}{t} \right| dt + \int_{T_\alpha} \left| \frac{g_{kn}(t)}{t} \right| dt < \infty. \tag{25}$$

The required result (1) then follows, using (17) with $T = \alpha n^{\frac{1}{2}(k+1)}$, in view of (21), (22) and (25).

(1) \Rightarrow (2)

Firstly we symmetrize the X_i 's. Consider the sequence Y_i , $i=1, 2, 3, \dots$ of independent symmetrized random variables; each Y_i having the distribution of

the difference between two independent X_i 's. Clearly the Y_i have characteristic function $|f(t)|^2$ and the distribution function of $Z_n = n^{-\frac{1}{2}} \sum_{i=1}^n Y_i$ is $F_n(x) * (1 - F_n(-x - 0)) = F_n^*(x)$. Write $G_{kn}^*(x)$ for the convolution $G_{kn}(x) * (1 - G_{kn}(-x))$. Then,

$$\begin{aligned} & \sum_{n=1}^{\infty} n^{-1+(k+\delta)/2} \sup_x |F_n^*(x) - G_{kn}^*(x)| \\ &= \sum_{n=1}^{\infty} n^{-1+(k+\delta)/2} \sup_x |F_n(x) * (1 - F_n(-x - 0)) - G_{kn}(x) * (1 - G_{kn}(-x))| \\ &\leq \sum_{n=1}^{\infty} n^{-1+(k+\delta)/2} \sup_x |F_n(x) * (1 - F_n(-x - 0)) - G_{kn}(x) * (1 - F_n(-x - 0))| \\ &+ \sum_{n=1}^{\infty} n^{-1+(k+\delta)/2} \sup_x |G_{kn}(x) * (1 - F_n(-x - 0)) - G_{kn}(x) * (1 - G_{kn}(-x))| \\ &< \infty \end{aligned} \tag{26}$$

in view of (1).

Next we shall show that (26) implies $E|Y_i|^{k+2} < \infty$ and hence $E|X_i|^{k+2} < \infty$. We note that

$$\sum_{n=1}^{\infty} n^{-1+(k+\delta)/2} \{1 - G_{kn}^*(x_n)\} < \infty$$

where $x_n = \{(k + \delta + 1) \log n\}^{\frac{1}{2}}$, so that from (26),

$$\sum_{n=1}^{\infty} n^{-1+(k+\delta)/2} P\left(\left|\sum_{i=1}^n Y_i\right| > n^{\frac{1}{2}} x_n\right) < \infty. \tag{27}$$

But, for symmetric random variables,

$$P\left(\left|\sum_{i=1}^n Y_i\right| > n^{\frac{1}{2}} x_n\right) \geq \frac{1}{2} P\left(\max_{1 \leq k \leq n} |Y_k| > n^{\frac{1}{2}} x_n\right) \tag{28}$$

(e.g. [1], Vol. II, p. 147) and from Bonferroni's inequalities (e.g. [1], Vol. I, p. 100) we have

$$\begin{aligned} & n P(|Y_i| > n^{\frac{1}{2}} x_n) \{1 - \frac{1}{2}(n-1) P(|Y_i| > n^{\frac{1}{2}} x_n)\} \\ & \leq P\left(\max_{1 \leq k \leq n} |Y_k| > n^{\frac{1}{2}} x_n\right) \leq n P(|Y_i| > n^{\frac{1}{2}} x_n), \end{aligned}$$

while $n P(|Y_i| > n^{\frac{1}{2}} x_n) \rightarrow 0$ as $n \rightarrow \infty$ since $EY_i^2 = 2EX_i^2 < \infty$. Consequently,

$$P\left(\max_{1 \leq k \leq n} |Y_k| > n^{\frac{1}{2}} x_n\right) \sim n P(|Y_i| > n^{\frac{1}{2}} x_n) \tag{29}$$

as $n \rightarrow \infty$ and hence, from (27), (28) and (29),

$$\sum_{n=1}^{\infty} n^{(k+\delta)/2} P(|Y_i| > n^{\frac{1}{2}} x_n) < \infty. \tag{30}$$

But,

$$\begin{aligned}
 E|Y_i|^{k+2} &= - \int_0^\infty x^{k+2} dP(|Y_i| > x) \\
 &\leq (k+2) \int_0^\infty x^{k+1} P(|Y_i| > x) dx \\
 &= (k+2) \sum_{n=0}^\infty \int_{x_n n^{\frac{1}{2}}}^{x_{n+1} (n+1)^{\frac{1}{2}}} x^{k+1} P(|Y_i| > x) dx \\
 &\leq \sum_{n=0}^\infty P(|Y_i| > n^{\frac{1}{2}} x_n) \{((n+1) x_{n+1}^2)^{(k+2)/2} - (n x_n^2)^{(k+2)/2}\} \\
 &\leq c \sum_{n=0}^\infty n^{k/2} (\log n)^{k/2} P(|Y_i| > n^{\frac{1}{2}} x_n) < \infty
 \end{aligned}$$

in view of (30), c being a positive constant. It then follows that $E|X_i|^{k+2} < \infty$ and an appeal to Lemma 3 gives $\alpha_j = EX_i^j, j = 1, 2, \dots, k+2$.

Now we note that the characteristic functions of $F_n^*(x)$ and $G_{kn}^*(x)$ are $|f_n(t)|^2$ and $|g_{kn}(t)|^2$ respectively. Then, integrating by parts in the equation

$$|f_n(t)|^2 - |g_{kn}(t)|^2 = \int_{-\infty}^\infty e^{itx} d\{F_n^*(x) - G_{kn}^*(x)\},$$

we obtain

$$-\frac{|f_n(t)|^2 - |g_{kn}(t)|^2}{it} = \int_{-\infty}^\infty e^{itx} \{F_n^*(x) - G_{kn}^*(x)\} dx.$$

Also,

$$it e^{-t^2/2} = \int_{-\infty}^\infty e^{itx} \frac{x}{\sqrt{2\pi}} e^{-x^2/2} dx,$$

and we obtain from Parseval's identity

$$\int_{-\infty}^\infty \{|f_n(t)|^2 - |g_{kn}(t)|^2\} e^{-t^2/2} dt = \sqrt{2\pi} \int_{-\infty}^\infty \{F_n^*(x) - G_{kn}^*(x)\} x e^{-x^2/2} dx.$$

Thus from (26)

$$\begin{aligned}
 &\sum_{n=1}^\infty n^{-1+(k+\delta)/2} \left| \int_{-\infty}^\infty \{|f_n(t)|^2 - |g_{kn}(t)|^2\} e^{-t^2/2} dt \right| \\
 &= \sqrt{2\pi} \sum_{n=1}^\infty n^{-1+(k+\delta)/2} \left| \int_{-\infty}^\infty \{F_n^*(x) - G_{kn}^*(x)\} x e^{-x^2/2} dx \right| \tag{31} \\
 &\leq 2\sqrt{2\pi} \sum_{n=1}^\infty n^{-1+(k+\delta)/2} \sup_x |F_n^*(x) - G_{kn}^*(x)| < \infty.
 \end{aligned}$$

Furthermore, we note that for $0 < \alpha < \frac{1}{4}$,

$$\begin{aligned}
 &\sum_{n=1}^\infty n^{-1+(k+\delta)/2} \left| \int_{n^\alpha}^\infty e^{-t^2/2} \{|f_n(t)|^2 - |g_{kn}(t)|^2\} dt \right| \\
 &\leq 2 \sum_{n=1}^\infty n^{-1+(k+\delta)/2} \int_{n^\alpha}^\infty e^{-t^2/2} dt < \infty,
 \end{aligned}$$

so that in view of (31)

$$\sum_{n=1}^{\infty} n^{-1+(k+\delta)/2} \left| \int_0^{n^\alpha} e^{-t^2/2} \{ |f_n(t)|^2 - |g_{kn}(t)|^2 \} dt \right| < \infty. \tag{32}$$

Now from Lemma 5 of [4], letting $R = [\frac{1}{2}(k+2)]$, we have for $t < c'\sqrt{n}$, c' some suitably small positive constant,

$$\left| |g_{kn}(t)|^2 - \exp \left(\sum_{s=1}^R (-1)^s \frac{t^{2s} 2\kappa_{2s}}{(2s)! n^{s-1}} \right) \right| \leq \frac{c}{n^{(k+1)/2}} (|t|^{3(k+1)} + |t|^{k+1}) e^{-t^2/4}$$

for some $c > 0$ since $\alpha_j = EX_j^j, j = 1, 2, \dots, k+2$ ensures $\beta_j = \kappa_j, j = 1, 2, \dots, k+2$. Thus,

$$\begin{aligned} \sum_{n=1}^{\infty} n^{-1+(k+\delta)/2} \left| \int_0^{n^\alpha} \left\{ \exp \left\{ \sum_{s=1}^R (-1)^s \frac{t^{2s}}{(2s)!} \frac{2\kappa_{2s}}{n^{s-1}} \right\} - |g_{kn}(t)|^2 \right\} e^{-t^2/2} dt \right| \\ \leq c \sum_{n=1}^{\infty} n^{-1-(1-\delta)/2} \int_0^{n^\alpha} e^{-3t^2/4} (t^{3(k+1)} + t^{k+1}) dt < \infty. \end{aligned} \tag{33}$$

Also, from Lemma 1,

$$|f_n(t)|^2 = \left| f \left(\frac{t}{\sqrt{n}} \right) \right|^{2n} = \exp \left\{ \sum_{s=1}^R (-1)^s \frac{t^{2s}}{(2s)!} \frac{2\kappa_{2s}}{n^{s-1}} + 2 \frac{|t|^{k+2}}{n^{k/2}} \operatorname{Re} \gamma \left(\frac{t}{\sqrt{n}} \right) \right\},$$

and using this result in conjunction with (32) and (33),

$$\begin{aligned} \sum_{n=1}^{\infty} n^{-1+(k+\delta)/2} \\ \cdot \left| \int_0^{n^\alpha} e^{-t^2/2} \exp \left\{ \sum_{s=1}^R (-1)^s \frac{t^{2s}}{(2s)!} \frac{2\kappa_{2s}}{n^{s-1}} \right\} \left[1 - \exp \left(2 \frac{t^{k+2}}{n^{k/2}} \operatorname{Re} \gamma \left(\frac{t}{\sqrt{n}} \right) \right) \right] dt \right| < \infty. \end{aligned}$$

Now Lemma 1 tells us that for n large enough, $\operatorname{Re} \gamma \left(\frac{t}{\sqrt{n}} \right)$ will be of constant sign for $0 < t \leq n^\alpha$ and hence

$$\begin{aligned} \sum_{n=1}^{\infty} n^{-1+(k+\delta)/2} \\ \cdot \int_0^{n^\alpha} e^{-t^2/2} \exp \left\{ \sum_{s=1}^R (-1)^s \frac{t^{2s}}{(2s)!} \frac{2\kappa_{2s}}{n^{s-1}} \right\} \left| 1 - \exp \left(2 \frac{t^{k+2}}{n^{k/2}} \operatorname{Re} \gamma \left(\frac{t}{\sqrt{n}} \right) \right) \right| dt < \infty \end{aligned}$$

so that

$$\begin{aligned} \sum_{n=1}^{\infty} n^{-1+(k+\delta)/2} \\ \cdot \int_0^1 \exp \left\{ -3t^2/2 + \sum_{s=2}^R (-1)^s \frac{t^{2s}}{(2s)!} \frac{2\kappa_{2s}}{n^{s-1}} \right\} \left| 1 - \exp \left(2 \frac{t^{k+2}}{n^{k/2}} \operatorname{Re} \gamma \left(\frac{t}{\sqrt{n}} \right) \right) \right| dt < \infty \end{aligned}$$

which implies

$$\sum_{n=1}^{\infty} n^{-1+(k+\delta)/2} \int_0^1 \left| 1 - \exp \left(2 \frac{t^{k+2}}{n^{k/2}} \operatorname{Re} \gamma \left(\frac{t}{\sqrt{n}} \right) \right) \right| dt < \infty.$$

However, as $n \rightarrow \infty$,

$$\left| 1 - \exp\left(2 \frac{t^{k+2}}{n^{k/2}} \operatorname{Re} \gamma\left(\frac{t}{\sqrt{n}}\right)\right) \right| = \frac{2|t|^{k+2}}{n^{k/2}} \left| \operatorname{Re} \gamma\left(\frac{t}{\sqrt{n}}\right) \right| (1 + o(1)),$$

so that

$$\sum_{n=1}^{\infty} n^{-1+\delta/2} \int_0^1 t^{k+2} \left| \operatorname{Re} \gamma\left(\frac{t}{\sqrt{n}}\right) \right| dt < \infty,$$

and, upon making the transformation $u = t/\sqrt{n}$, this yields

$$\sum_{n=1}^{\infty} n^{(k+1+\delta)/2} \int_0^{1/\sqrt{n}} u^{k+2} |\operatorname{Re} \gamma(u)| du < \infty. \tag{34}$$

Now $\int_0^{1/\sqrt{x}} u^{k+2} |\operatorname{Re} \gamma(u)| du$ is monotone decreasing as x increases so for $X > 1$,

$$\begin{aligned} & \int_1^X x^{(k+1+\delta)/2} \left\{ \int_1^{1/\sqrt{x}} u^{k+2} |\operatorname{Re} \gamma(u)| du \right\} dx \\ & \leq \sum_{n=1}^{[X]} \int_n^{n+1} x^{(k+1+\delta)/2} \left\{ \int_1^{1/\sqrt{x}} u^{k+2} |\operatorname{Re} \gamma(u)| du \right\} dx \\ & \leq \sum_{n=1}^{[X]} (n+1)^{(k+1+\delta)/2} \int_1^{1/\sqrt{n}} u^{k+2} |\operatorname{Re} \gamma(u)| du \\ & \leq c \sum_{n=1}^{[X]} n^{(k+1+\delta)/2} \int_1^{1/\sqrt{n}} u^{k+2} |\operatorname{Re} \gamma(u)| du, \end{aligned}$$

c being a positive constant such that $(n+1)^{(k+1+\delta)/2} < c n^{(k+1+\delta)/2}$ for all positive integral n . Consequently, using (34), we have

$$\int_1^{\infty} x^{(k+1+\delta)/2} \left\{ \int_0^{1/\sqrt{x}} u^{k+2} |\operatorname{Re} \gamma(u)| du \right\} dx < \infty. \tag{35}$$

Now in view of (35) we must have

$$\int_{\omega}^{2\omega} x^{(k+1+\delta)/2} \left\{ \int_0^{1/\sqrt{x}} u^{k+2} |\operatorname{Re} \gamma(u)| du \right\} dx \rightarrow 0$$

as $\omega \rightarrow \infty$ and

$$\begin{aligned} & \int_{\omega}^{2\omega} x^{(k+1+\delta)/2} \left\{ \int_0^{1/\sqrt{x}} u^{k+2} |\operatorname{Re} \gamma(u)| du \right\} dx \\ & \geq \omega^{(k+3+\delta)/2} \int_0^{1/\sqrt{2\omega}} u^{k+2} |\operatorname{Re} \gamma(u)| du \geq 0, \end{aligned}$$

so that, putting $v = 1/\sqrt{2\omega}$, we conclude that

$$v^{-(k+3+\delta)} \int_0^v u^{k+2} |\operatorname{Re} \gamma(u)| du \rightarrow 0 \tag{36}$$

as $v \rightarrow 0$. Then, upon making the transformation $v = 1/\sqrt{x}$, (35) becomes

$$\int_0^1 \left\{ \int_0^v u^{k+2} |\operatorname{Re} \gamma(u)| du \right\} v^{-(k+4+\delta)} dv < \infty$$

and, in view of (36), the fact that $\int_0^1 |\operatorname{Re} \gamma(t)| t^{-(1+\delta)} dt < \infty$ holds follows immediately from an integration by parts. Lemma 2 then enables us to conclude that $E|Y_i|^{k+2+\delta} < \infty$ from which we deduce that $E|X_i|^{k+2+\delta} < \infty$. This completes the proof that (1) \Rightarrow (2) and hence the proof of Theorems 1 and 2.

For the proof of Theorem 3, we note firstly that the above proof that (3), (4) \Rightarrow (1) under (C) or if $k=0$ together with Lemmas 1 and 2 show that $E|X_i|^{k+2} \log(1+|X_i|) < \infty$ implies (1) with $\delta=0$ under (C) or if $k=0$. On the other hand, the above proof that (1) \Rightarrow (2) shows that (1) with $\delta=0$ implies $\int_0^1 |\operatorname{Re} \gamma(t)| t^{-1} dt < \infty$. Thus, from Lemmas 1 and 2 we have $E|Y_i|^{k+2} \log(1+|Y_i|) < \infty$ from which we deduce the required result that $E|X_i|^{k+2} \log(1+|X_i|) < \infty$.

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(Received July 15, 1970)