ON THE INTEGER PART OF THE K-TH ROOT OF A POSITIVE INTEGER *

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Abstract For any positive integer m, let $a(m)$ denotes the integer part of the k-th root of m. That is, $a(m) = \left[m^{1/k}\right]$. In this paper, we study the asymptotic properties of

$$\sigma_{-\alpha}(f(a(m))),$$

where $0 < \alpha \leq 1$ be a fixed real number, $\sigma_{-\alpha}(n) = \sum_{l|n} \frac{1}{l^\alpha}$, $f(x)$ be a polynomial with integer coefficients. An asymptotic formula is obtained.

Keywords: Integer part sequence; k-th root; Mean value; Asymptotic formula.

§1. Introduction

For any positive integer m, let $a(m)$ denotes the integer part of the k-th root of m. That is, $a(m) = \left[m^{1/k}\right]$. For example, $a(1) = 1$, $a(2) = 1$, $a(3) = 1$, $a(4) = 1$, $\cdots$, $a(2^k) = 2$, $a(2^k + 1) = 2$, $\cdots$, $a(3^k - 1) = 2$, $a(3^k) = 3$, $\cdots$. In problem 80 of reference [1], Professor F. Smarandach asked us to study the asymptotic properties of the sequence $\{a(m)\}$. About this problem, it seems that none had studied it, at least we have not seen related paper before. In this paper, we shall use the elementary method to study the asymptotic properties

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of $\sigma_{-\alpha}(f(a(m)))$, and give an interesting asymptotic formula. That is, we shall prove the following:

Theorem. *Let $0 < \alpha \leq 1$ be a fixed real number, $f(x)$ be a polynomial with integer coefficients. Then for any real number $x > 1$, we have the asymptotic formula*

$$\sum_{m \leq x} \sigma_{-\alpha}(f(a(m))) = C_f(\alpha)x + O\left(x^{1-\alpha/k+\varepsilon}\right),$$

where

$$\sigma_{-\alpha}(n) = \sum_{l|n} \frac{1}{l^\alpha}, \quad C_f(\alpha) = \sum_{d=1}^{\infty} P_f(d)d^{-1-\alpha}, \quad P_f(d) = \sum_{f(n)\equiv 0(\bmod d), 0<n\leq d} 1,$$

and ε denotes any fixed positive number.

§2. Several lemmas

To complete the proof of the theorem, we need the following two simple lemmas:

Lemma 1. *Let $0 < \alpha \leq 1$ be a fixed real number, $f(x)$ be a polynomial with integer coefficients. Then for any real number $x > 1$, we have the asymptotic formula*

$$\sum_{n\leq x, f(n)\neq 0} \sigma_{-\alpha}(f(a(n))) = C_f(\alpha)x + O\left(x^{1-\alpha}\ln^\gamma x\right),$$

where γ is a certain constant, and

$$C_f(\alpha) = \sum_{d=1}^{\infty} P_f(d)d^{-1-\alpha}, \qquad P_f(d) = \sum_{f(n)\equiv 0(\bmod d), 0<n\leq d} 1.$$

Proof. (See reference [2]).

Lemma 2. *Let M be a fixed positive integer, $f(x)$ be a polynomial with integer coefficients. Then we have*

$$\sum_{t=1}^{M} t^{k-1}\sigma_{-\alpha}(f(t)) = \frac{C_f(\alpha)}{k}M^k + O\left(M^{k-\alpha}\ln^\gamma M\right).$$

Proof. Let $A(y) = \sum_{t\leq y} \sigma_{-\alpha}(f(t))$, by Abel's identity (see Theorem 4.2 of [3]) we have

$$\sum_{t=1}^{M} t^{k-1}\sigma_{-\alpha}(f(t))$$

$$
\begin{aligned}
&= M^{k-1}A(M) - A(1) - (k-1)\int_1^M y^{k-2}\, A(y)dy \\
&= M^{k-1}\left(C_f(\alpha)M + O\left(M^{1-\alpha}\ln^\gamma M\right)\right) \\
&\quad -(k-1)\int_1^M y^{k-2}\left(C_f(\alpha)y + O\left(y^{1-\alpha}\ln^\gamma y\right)\right)dy \\
&= C_f(\alpha)M^k + O\left(M^{k-\alpha}\ln^\gamma M\right) - \frac{C_f(\alpha)(k-1)}{k}M^k + O\left(M^{k-\alpha}\ln^\gamma M\right) \\
&= \frac{C_f(\alpha)}{k}M^k + O\left(M^{k-\alpha}\ln^\gamma M\right).
\end{aligned}
$$

This completes the proof of Lemma 2.

§3. Proof of Theorem

In this section, we shall complete the proof of Theorem. For any real number $x \geq 1$, let M be a fixed positive integer such that

$$M^k \leq x < (M+1)^k.$$

Let a_0 denotes the constant term of $f(x)$, from the definition of $a(m)$ and Lemma 2, we have

$$
\begin{aligned}
&\sum_{m\leq x}\sigma_{-\alpha}(f(a(m))) \\
&= \sum_{t=1}^{M}\sum_{(t-1)^k\leq m<t^k}\sigma_{-\alpha}(f(a(m))) + \sum_{M^k\leq m\leq x}\sigma_{-\alpha}(f(a(m))) \\
&= \sum_{t=1}^{M-1}\sum_{t^k\leq m<(t+1)^k}\sigma_{-\alpha}(f(t)) + \sigma_{-\alpha}(a_0) + \sum_{M^k\leq m\leq x}\sigma_{-\alpha}(f(M)) \\
&= \sum_{t=1}^{M-1}\left((t+1)^k - t^k\right)\sigma_{-\alpha}(f(t)) + O\left(\sum_{M^k\leq m\leq (M+1)^k}\sigma_{-\alpha}(f(M))\right) \\
&= \sum_{t=1}^{M-1}\left(C_k^1 t^{k-1} + C_k^2 t^{k-2} + \cdots + 1\right)\sigma_{-\alpha}(f(t)) \\
&\quad +O\left(\sum_{M^k\leq m\leq (M+1)^k}\sigma_{-\alpha}(f(M))\right) \\
&= k\sum_{t=1}^{M} t^{k-1}\sigma_{-\alpha}(f(t)) + O\left(M^{k-1}\sigma_{-\alpha}(f(t))\right) \\
&= C_f(\alpha)M^k + O\left(M^{k-\alpha+\varepsilon}\right),
\end{aligned}
$$

where we have used the fact that $\sigma_{-\alpha}(n) \ll n^\varepsilon$.

On the other hand, note that the estimate

$$\begin{aligned} 0 \le x - M^k &< (M+1)^k - M^k = kM^{k-1} + C_k^2 M^{k-2} + \cdots + 1 \\ &= M^{k-1}\left(k + C_k^2 \frac{1}{M} + \frac{1}{M^{k-1}}\right) \ll x^{\frac{k-1}{k}}. \end{aligned}$$

Now combining the above, we can immediately get the asymptotic formula

$$\sum_{m \le x} \sigma_{-\alpha}(f(a(m))) = C_f(\alpha)x + O\left(x^{1-\alpha/k+\varepsilon}\right).$$

This completes the proof of Theorem.

References

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