

**ON THE INTEGER TRANSLATES  
OF A COMPACTLY SUPPORTED FUNCTION:  
DUAL BASES AND LINEAR PROJECTORS**

by

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dual bases and linear projectors

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ABSTRACT

Given a multivariate compactly supported function  $\phi$ , we discuss here linear projectors to the space  $S(\phi)$  spanned by its integer translates. These projectors are constructed with the aid of a dual basis for the integer translates of  $\phi$ , hence under the assumption that these translates are linearly independent. Our main result shows that the linear functionals of the dual basis are local, hence makes it possible to construct local linear projectors onto  $S(\phi)$ . We then discuss, for a general compactly supported function, a scheme for the construction of such local projectors.

In the second part of the paper we apply these observations to piecewise-polynomials and piecewise-exponentials to obtain a necessary and sufficient condition for a quasi-interpolant to be a projector. The results of that part extend and refine recent constructions of dual bases and linear projectors for polynomial and exponential box splines.

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**1. Introduction**

Let  $\phi$  be a complex-valued compactly supported continuous function defined on  $\mathbb{R}^s$ , and  $E^\alpha$  the shift operator

$$E^\alpha f = f(\cdot + \alpha).$$

Associated with  $\phi$  is the semi-discrete convolution operator

$$(1.1) \quad \phi * : c \mapsto \phi * c := \sum_{\alpha \in \mathbb{Z}^s} c(\alpha) E^{-\alpha} \phi,$$

which acts from the domain

$$\mathcal{C} := \{c : \mathbb{Z}^s \rightarrow \mathbf{C}\}$$

of all complex-valued sequences defined on the multi-integers to the range

$$S(\phi) := \text{ran } \phi *$$

of the functions spanned by the integer translates of  $\phi$ .

In multivariate spline approximation,  $S(\phi)$  is of potential use as a space of approximants for a larger function space (e.g.,  $C(\mathbb{R}^s)$ ). The operator  $\phi *$  is exploited in the derivation of explicit approximation schemes from  $S(\phi)$ . Such a scheme may result in a linear projector onto  $S(\phi)$  (cf. [B<sub>1</sub>] and [BF] for the construction of linear projectors for *univariate* and *tensor product* splines). Since the construction of linear projectors in the multivariate case is usually quite involved, one is satisfied with the so-called quasi-interpolation schemes, which yield the same approximation order.

In case the integer translates of  $\phi$  are globally linearly independent (i.e.,  $\ker(\phi *) = 0$ ), a natural way to define a linear projector onto  $S(\phi)$  is with the aid of a linear functional  $\lambda$  that satisfies

$$\lambda(E^\alpha \phi) = \delta_{\alpha,0}, \quad \alpha \in \mathbb{Z}^s.$$

For then, the functionals  $\Lambda := \{\lambda E^\alpha\}_\alpha$  form a dual basis for  $\Phi := \{E^{-\alpha} \phi\}_\alpha$ , and a linear projector  $\Psi$  can then be defined in the usual way:

$$\Psi = \sum_{\alpha \in \mathbb{Z}^s} E^{-\alpha} \phi \lambda E^\alpha.$$

For the analysis of the approximation properties of the projector, its **localness** is important: a projector  $\Psi$  is termed “local” if for every compact  $A \subset \mathbb{R}^s$  there exists a compact  $B \subset \mathbb{R}^s$  such that  $\Psi(f)|_A$  is determined by  $f|_B$ . The existence of a local projector is guaranteed in case the generator  $\lambda$  of the dual basis  $\Lambda$  is local. The construction of local linear projectors is facilitated if one assumes that the integer translates of  $\phi$  are **locally linearly independent** i.e., the condition

$$(1.2) \quad (\phi * c)|_A = 0 \text{ and } \text{supp } E^{-\alpha} \phi \cap A \neq \emptyset \implies c(\alpha) = 0,$$

for every open  $A \subset \mathbb{R}^s$ . It is one of the main themes of this paper to show that local projectors can be constructed even under the weaker assumption of global linear independence. As a matter of fact, that observation neither makes use of the shift-invariance of the space  $S(\phi)$  nor of the fact that  $\phi$  is a function:

**(1.3) Theorem.** *Let  $\Phi = \{\phi_\alpha\}_{\alpha \in \mathbb{Z}^s}$  be a locally finite collection of (globally) linearly independent compactly supported distributions in  $\mathcal{D}'(\mathbb{R}^s)$ . Then each functional  $\lambda_\alpha$  in the (algebraic) dual basis  $\Lambda = \{\lambda_\alpha\}_{\alpha \in \mathbb{Z}^s}$  of  $\Phi$  is local. Precisely, for every  $\alpha \in \mathbb{Z}^s$  there exists a compact  $B_\alpha \subset \mathbb{R}^s$  such that  $\lambda_\alpha(f) = 0$  whenever  $\text{supp } f \cap B_\alpha = \emptyset$ .*

In section 2 we prove (1.3)Theorem and employ this result to show that the assumption of linear independence of the integer translates of  $\phi$  is already sufficient to allow the existence of a dual basis based on a linear functional  $\lambda$  of point-evaluation at a finite set of  $\mathbb{R}^s$ . The proof of the theorem gives also information about the diameter of  $\text{supp } \lambda$  which in the case of local linear independence coincides with the standard result. Finally, we comment about the connection of these results to the area of cardinal interpolation by translates of a compactly supported function.

The construction of local projectors is then discussed in section 3. There we take  $\phi$  to be an arbitrary function whose translates are linearly independent, and, based on a new extended notion of a quasi-interpolant, provide a necessary and sufficient condition for a quasi-interpolant to be a linear projector. With the aid of this observation, we then describe a general scheme for the derivation of local projectors onto  $S(\phi)$ .

In the two last sections we examine the piecewise-exponential case (which contains the piecewise-polynomial case). Section 4 is devoted to a brief discussion of some known methods for constructing quasi-interpolants for piecewise-exponentials. These results, together with observations from section 3, are used in the last section where we show that for piecewise-exponentials a slightly stronger sense of local linear independence is sufficient to imply that every quasi-interpolant is also a linear projector. We conclude that section with a review of the constructions of linear projectors in [DM<sub>1,2</sub>] and [J<sub>1,2</sub>] providing thereby new proofs and extensions to these results.

## 2. Linear projectors are local

In these section we prove (1.3)Theorem and discuss some of its applications.

The following is an equivalent form of (1.3)Theorem, which is slightly more convenient for the proof employed:

**(2.1) Theorem.** *Let  $\Phi = \{\phi_\alpha\}_{\alpha \in \mathbb{Z}^s}$  be a locally finite collection of globally linearly independent compactly supported distributions in  $\mathcal{D}'(\mathbb{R}^s)$ . Then there exists a ball*

$$(2.2) \quad B := \{x : \|x\| \leq L\}$$

*such that if  $f = \sum_{\alpha \in \mathbb{Z}^s} c(\alpha)\phi_\alpha$  satisfies  $\text{supp } f \cap B = \emptyset$  then  $c(0) = 0$ .*

Indeed, (1.3)Theorem readily follows from (2.1)Theorem: assume (without loss) that  $\alpha = 0$  in (1.3)Theorem. For every  $f = \sum_{\alpha \in \mathbb{Z}^s} c(\alpha)\phi_\alpha \in \text{span } \Phi$ ,  $\lambda_0(f) = c(0)$ . So, if we assume (2.1)Theorem and choose  $B_0$  of (1.3)Theorem to be  $B$  in (2.2), then whenever  $\text{supp } f \cap B = \emptyset$ , (1.3)Theorem implies  $\lambda_0(f) = c(0) = 0$ .

We postpone the proof of (2.1)Theorem to the end of this section, and discuss now some of its applications.

Suppose that  $\Phi \subset C(\mathbb{R}^s)^\dagger$  and choose  $\alpha \in \mathbb{Z}^s$ . Let  $B_\alpha$  be the ball associated with  $\alpha$  (by (1.3)Theorem). Define

$$(2.3) \quad \nu_\alpha := \{\beta \in \mathbb{Z}^s : \text{supp } \phi_\beta \cap B_\alpha \neq \emptyset\}.$$

Since the elements of  $\Phi$  are locally finite, the set  $\nu_\alpha$  is finite. Defining

$$S(\Phi) = \text{span } \Phi, \quad S_\alpha(\Phi) := \text{span}\{\phi_\beta|_{B_\alpha} : \beta \in \nu_\alpha\},$$

(1.3)Theorem implies that any extension  $\mu_\alpha \in C(\mathbb{R}^s)^*$  of the restricted linear functional  $\lambda_\alpha|_{S_\alpha(\Phi)}$  is also an extension of  $\lambda_\alpha$ , provided that  $\text{supp } \mu_\alpha \subset B_\alpha$ . Now,  $S_\alpha(\Phi)$  is a finite-dimensional subspace of  $C(B_\alpha)$ ,  $B_\alpha$  being compact, and hence there are various available ways to represent the restriction of  $\lambda_\alpha$  to  $S_\alpha(\Phi)$ ; e.g., one may choose a set  $b_\alpha \subset B_\alpha$  of cardinality  $\#\nu_\alpha$ , which is total for  $S_\alpha(\Phi)$  (i.e., no element in  $S_\alpha(\Phi) \setminus 0$  vanishes on  $b_\alpha$ ). Then there is a unique linear combination  $\mu_\alpha = \sum_{x \in b_\alpha} c(x)\delta_x$  satisfying

$$(2.4) \quad \mu_\alpha(f) = \lambda_\alpha(f), \quad \forall f \in S(\Phi),$$

where  $\delta_x$  is the functional of point-evaluation at  $x$ . Thus we conclude

**(2.5) Corollary.** *Assume that  $\Phi \subset C(\mathbb{R}^s)$ . Then there exists a projector*

$$(2.6) \quad \Psi : C(\mathbb{R}^s) \rightarrow S(\Phi) : f \mapsto \sum_{\alpha \in \mathbb{Z}^s} \mu_\alpha(f)\phi_\alpha,$$

such that each  $\mu_\alpha$  is supported on a finite set.

In the special case of interest, viz. when  $\phi_\alpha = E^{-\alpha}\phi$ , we have

$$\sum_{x \in b} c(x)\phi(x + \alpha) = \mu(E^\alpha\phi) = \begin{cases} 1, & \alpha = 0, \\ 0, & \alpha \neq 0. \end{cases}$$

(with  $\phi := \phi_0$ ,  $\mu := \mu_0 = \sum_{x \in b} c(x)\delta_x$ , and  $b := b_0$ ). This proves the following

**(2.7) Corollary.** *Let  $\phi$  be a compactly supported continuous function whose integer translates are globally linearly independent. Then there exists a finite linear combination  $\psi$  of (real) translates of  $\phi$  satisfying*

$$\psi|_{\mathbb{Z}^s} = \delta_\alpha.$$

Such  $\psi$  is usually referred to as “a fundamental solution”. In cardinal interpolation, one looks for a fundamental solution spanned by (infinitely many) *integer* translates of  $\phi$ . We note that the existence of the latter fundamental solution does not require the global linear independence of the translates, yet if one imposes decay conditions on  $\psi$  then global linear independence is, in general, an insufficient condition for the existence of such  $\psi$  (cf. [Ri] for details).

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$\dagger$   $C(\mathbb{R}^s)$  is chosen only for convenience; any superspace  $V \supset S(\phi)$  in which point-evaluation is well defined will do.

**Proof of (2.1) Theorem:** Assume, on the contrary, that such a ball  $B$  does not exist. For every positive integer  $n$ , let  $B_n$  be the open ball centered at the origin with radius  $n$  and define

$$(2.8) \quad \nu_n := \{\alpha \in \mathbb{Z}^s : \text{supp } \phi_\alpha \cap B_n \neq \emptyset\}.$$

Let  $M_n$  be the linear space of all sequences  $c$  defined on  $\nu_n$  and satisfying

$$(2.9) \quad \text{supp}\left(\sum_{\alpha \in \nu_n} c(\alpha)\phi_\alpha\right) \cap B_n = \emptyset.$$

Since  $\Phi$  is locally finite, every  $\nu_n$  is finite, and hence every  $M_n$  is finite-dimensional. On the other hand, no  $M_n$  is trivial, since by our assumption each  $M_n$  must contain at least one element satisfying  $c(0) \neq 0$ . To obtain the desired contradiction, we will show that there is a non-trivial sequence in  $\mathcal{C}$  whose restriction to each  $\nu_n$  lies in  $M_n$ . Since the union of the sets  $\{\nu_n\}_n$  is  $\mathbb{Z}^s$ , such sequence  $c$  induces a non-trivial vanishing combination of  $\Phi$  thus contradicting the linear independence of the elements in  $\Phi$ .

For this purpose, define  $M_0 := \mathbb{C}$  and for all non-negative integers  $m \geq n > 0$ , let  $R_n^m$  be the restriction map from  $M_m$  to  $M_n$  (with  $R_0^n : c \mapsto c(0)$  and  $R_0^0$  the identity mapping). Clearly,

$$(2.10) \quad R_0^m \neq 0, \quad \forall m.$$

Defining  $K_n \subset M_n$  by

$$K_n := \bigcap_{m \geq n} \text{ran } R_n^m,$$

we note that the condition

$$K_n \neq 0$$

is necessary (but apparently not sufficient) for the existence of a sequence  $c$  whose restriction to each  $\nu_n$  lies in  $M_n$ . Indeed, we claim

**(2.11) Lemma.** *For each  $n \geq 0$ ,  $K_n \neq 0$ .*

**Proof:** Since for every  $n \leq k \leq m$

$$(2.12) \quad R_n^m = R_n^k R_k^m,$$

$\{\text{ran } R_n^m\}_{m \geq n}$  is a decreasing sequence of finite-dimensional vector spaces. Therefore, for all sufficiently big  $k$  and  $m$ ,  $R_n^m = R_n^k$ , and hence  $K_n = \text{ran } R_n^m$  for all sufficiently big  $m$ . Finally, since  $R_0^m = R_0^n R_n^m$ , then (2.10) implies that  $R_n^m \neq 0$ , hence so is  $K_n$ .  $\spadesuit$

In the following we will prove the existence of a sequence  $c \in \mathcal{C}$  whose restriction to each  $\nu_n$  is in  $K_n$ , hence in  $M_n$ . For this we first note that the proof of the previous lemma shows that  $K_n = \text{ran } R_n^m$ , for sufficiently big  $m$ . Therefore, we can find  $m$  such that

$$K_j = \text{ran } R_j^m, \quad j = n, n+1,$$

and thus

$$(2.13) \quad K_n = R_n^{n+1} R_{n+1}^m M_m = R_n^{n+1} K_{n+1}.$$

We can now complete the proof of (2.1) Theorem as follows: let  $n$  be arbitrary. By (2.11) Lemma there exists a non-trivial  $c_n \in K_n$ . Invoking (2.13) we may choose  $c_{n+1} \in K_{n+1}$  whose restriction to  $\nu_n$  is  $c_n$ . Again, (2.13) can be employed to provide  $c_{n+2} \in K_{n+2}$  whose restriction to  $\nu_{n+1}$  is  $c_{n+1}$ . Proceeding in this manner we obtain  $(c_m \in K_m)_{m=n}^{\infty}$  such that  $R_n^m c_m = c_n$ . This gives rise to a sequence  $c \in \mathcal{C}$  satisfying

$$\text{supp} \left( \sum_{\alpha \in \nu_m} c(\alpha) \phi_\alpha \right) \cap B_m = \emptyset, \quad \forall m,$$

while  $c$  is non-trivial since its restriction  $c_n$  to  $\nu_n$  is non-trivial. We therefore conclude that the distributions in  $\Phi$  are globally linearly dependent, in contradiction to the assumptions of the theorem.

♠

### 3. Linear projectors and quasi-interpolation

Throughout this section we assume that  $\phi$  is a compactly supported continuous function whose integer translates are globally linearly independent, and  $A \subset \mathbb{R}^s$  is an open bounded set which satisfies for every  $c \in \mathcal{C}$  the condition

$$(3.1) \quad (\phi * c)|_A = 0 \implies c(0) = 0.$$

The existence of such an  $A$  was proved in (1.3) Theorem.

Given a linear functional  $\lambda : C(\mathbb{R}^s) \rightarrow \mathbb{C}$ , we examine here conditions that guarantee that the operator

$$Q_\lambda := \phi * \Lambda(\cdot)$$

is a projector. Here

$$\Lambda : C(\mathbb{R}^s) \rightarrow \mathcal{C} : f \mapsto (\lambda E^\alpha f)_{\alpha \in \mathbb{Z}^s}.$$

Our aim is to generate a space  $F \subset S(\phi)$  (which replaces the missing polynomial space usually associated with a piecewise-polynomial  $\phi$ ) that is of help in the identification of a projector  $Q_\lambda$ . We use the notation  $f| := f|_{\mathbb{Z}^s}$ , and let

$$\phi *'$$

stand for the semidiscrete convolution operator from  $C(\mathbb{R}^s)$  to  $S(\phi)$  defined by

$$(3.2) \quad \phi *' f := \phi * f| = \sum_{\alpha \in \mathbb{Z}^s} f(\alpha) E^{-\alpha} \phi.$$

Finally,

$$\phi '| *' f$$

denotes the discrete convolution  $\phi| * f|$ .



**(3.3) Theorem.** Let  $F$  be a subspace of  $S(\phi)$  satisfying

$$(3.4) \quad F|_A = S(\phi)|_A,$$

and assume that  $\text{supp } \lambda \subset A$ . Then the following conditions are equivalent:

$$(a) \quad Q_\lambda(f) = f, \quad \forall f \in F.$$

(b)  $Q_\lambda$  is a projector, i.e.,

$$Q_\lambda(f) = f, \quad \forall f \in S(\phi).$$

**Proof:** The implication (b)  $\implies$  (a) is trivial. For the converse, it is necessary and sufficient to prove that

$$\lambda(E^{-\alpha}\phi) = \delta_\alpha(0).$$

Fix  $\alpha$  and let  $f \in F$  be such that  $f|_A = (E^{-\alpha}\phi)|_A$ ; then, since  $Q_\lambda(f) = f$ ,

$$Q_\lambda(f)|_A = E^{-\alpha}\phi|_A.$$

Now,  $E^{-\alpha}\phi = \phi * \delta_\alpha$ , while  $Q_\lambda(f) = \phi * \Lambda(f)$ , and therefore, by (3.1), we must have  $\lambda(f) = \Lambda(f)(0) = \delta_\alpha(0)$ . On the other hand,  $\lambda$  is supported on  $A$  and therefore since  $E^{-\alpha}\phi$  and  $f$  coincide on  $A$  we conclude  $\lambda(E^{-\alpha}\phi) = \lambda(f) = \delta_\alpha(0)$ .  $\spadesuit$

The assumption  $\text{supp } \lambda \subset A$  in the theorem is essential, as shown by the following simple example:

**Example.** Let  $\phi$  be the univariate hat function supported on  $[0, 2]$ . Let  $F = \pi_1$ ,  $\lambda = \frac{1}{2}(\delta_{\frac{1}{2}} + \delta_{1\frac{1}{2}})$ . Then  $Q_\lambda$  reproduces  $\pi_1$ , and  $F = \pi_1$  also satisfies (3.4) with  $A$  being any subset of  $[0, 1]$  or  $[1, 2]$ . Yet,  $\lambda$  is supported in no one of these  $A$ 's and therefore  $Q_\lambda$  is not guaranteed to be a projector. Indeed,

$$\lambda(E^\alpha\phi) = \begin{cases} \frac{1}{2}, & \alpha = 0, \\ \frac{1}{4}, & \alpha = \pm 1, \\ 0, & \text{otherwise.} \end{cases}$$

Needless to say, there exist projectors whose corresponding  $\lambda$  is supported in no admissible  $A$ ; e.g.,  $\lambda = \frac{1}{2}(-\delta_{\frac{1}{3}} + 2\delta_{\frac{2}{3}} + 2\delta_{\frac{4}{3}} - \delta_{\frac{5}{3}})$ .

We now employ the above theorem, to show that, with an appropriate choice of the space  $F$ , the task of constructing linear projectors is reduced to the construction of the so-called quasi-interpolants. For that, assume that  $F$  is a shift-invariant (i.e., closed under integer translates) subspace of  $S(\phi)$ . Then it follows ([B<sub>2</sub>],[Ro]) that  $F$  is an invariant subspace for  $\phi *'$ . If we further assume that  $F$  is finite-dimensional and  $\phi *'$  is 1-1 on  $F$ , then  $\phi *'$  induces an automorphism on  $F$ . We may then follow [BH], call this automorphism  $T$  and define a functional on  $F$  by

$$[0]T^{-1} : f \mapsto T^{-1}f(0).$$

For  $\mu \in F^*$  rather than  $[0]T^{-1}$ , the linear independence of the integer translates of  $\phi$  would imply that  $Q_\mu$  is not the identity mapping on  $F$  and therefore no extension of such  $\mu$  would yield a projector. On the other hand, for  $\mu = [0]T^{-1}$  it follows [BH], that  $\mu(E^\alpha f) = T^{-1}f(\alpha)$  and therefore (with  $Q_\mu$  defined only on  $F$ )

$$Q_\mu(f) = f, \forall f \in F.$$

In case  $f \mapsto f|_A$  is 1-1 on  $F$ , this functional can be extended to a functional  $\lambda : C(\mathbb{R}^s)$  that is supported on  $A$  and then (3.3)Theorem would imply that  $Q_\lambda$  is a projector.

We summarize all these observations in the following

**(3.5) Corollary.** *Let  $F$  be a finite-dimensional shift-invariant subspace of  $S(\phi)$  which satisfies (3.4). Assume that the operator  $T := \phi *' |_F$  is injective and consider the following conditions for a linear functional  $\lambda \in C(\mathbb{R}^s)^*$ :*

(a)  $Q_\lambda$  is a projector.

(b)  $\lambda(f) = [0]T^{-1}f, \forall f \in F$ .

Then (a)  $\implies$  (b), and the converse implication holds provided that  $\lambda$  is supported in  $A$ .

The construction of a quasi-interpolant using the inverse of the map  $T$  appears first in [BH] in the context of the approximation order for box splines (hence for a polynomial  $F$ ), without however making the connection to linear projectors. The discussion above emphasizes the fact (which is by now well-known) that this is (essentially) the only way to construct quasi-interpolants. In particular, other constructs (cf. e.g., [SF], [DM<sub>1</sub>], [CD], [CL] for piecewise-polynomials and [DR] for piecewise-exponentials) are as a matter of fact special ways to extend  $[0]T^{-1}$ .

Our next task is to prove the existence of a space  $F$  which satisfies all these conditions. This purpose can be accomplished without appeal to linear independence. First, we associate with  $A$  the set

$$(3.6) \quad \nu(A) := \nu_\phi(A) := \{\alpha \in \mathbb{Z}^s : A - \alpha \cap \text{supp } \phi \neq \emptyset\},$$

which consists of all integers  $\alpha$  whose corresponding  $E^{-\alpha}\phi$  has some support on  $A$ . Furthermore, we assume without loss that  $\phi|_A \neq 0$ ; otherwise  $\phi$  can be replaced by one of its non-integer translates.

We now look for a shift-invariant space  $P$  which on the one hand interpolates correctly on  $\nu(A)$  (that is,  $\dim P = \#\nu(A)$  and no  $p \in P \setminus 0$  vanishes identically on  $\nu(A)$ ), while on the other hand has trivial intersection with  $\ker \phi *'$ . Any space satisfying these two conditions will do here. In particular, we may first choose  $P$  to be a homogeneous translation-invariant polynomial space that interpolates correctly on  $\nu(A)$ , (cf. [BR<sub>1</sub>] for construction of such  $P$  of least degree). If  $\phi *'$  is not 1-1 on  $P$ , it may be replaced by a space  $P_\theta := e_\theta P$ , where the exponential  $e_\theta : x \mapsto e^{\theta \cdot x}$  is chosen such that the discrete convolution  $\phi *' e_\theta \neq 0$ . This readily implies that the operator  $\phi *'$  is 1-1 on  $e_\theta P$  (and hence so is  $\phi *'$ ). Since for every  $\theta \in \mathbb{C}^s$ , the space  $P_\theta$  also interpolates correctly on  $\nu(A)$ ,  $P_\theta$  satisfies the required conditions.

Define now

$$F := \phi *' P_\theta.$$

Since  $P_\theta$  is translation-invariant, hence shift-invariant, so is  $F$ .  $F$  is also finite-dimensional; in fact, since  $\phi *'$  is 1-1 on  $P_\theta$ ,  $\dim F = \#\nu(A)$ . Moreover, by the definition of  $\nu(A)$ ,  $S(\phi)|_A = \text{span}\{E^{-\alpha}\phi|_A\}_{\alpha \in \nu(A)}$ , and since  $P_\theta$  interpolates correctly on  $\nu(A)$ , we have

$$F|_A = S(\phi)|_A.$$

Finally, the discrete convolution  $\phi *'$  is injective on  $P_\theta$ , thus induces an automorphism on that space and consequently  $F$  coincides with  $P_\theta$  on  $\mathbb{Z}^s$ . We then conclude

**(3.7) Theorem.** *Let  $\phi$  be a compactly supported continuous function, and assume that  $\phi|_A \neq 0$ . Let  $A$  be an open subset of  $\mathbb{R}^s$ . Then there exists a shift-invariant subspace  $F \subset S(\phi)$  satisfying*

- (a)  $\dim F = \#\nu(A)$ ;
- (b)  $\phi *'|_F$  is an automorphism;
- (c)  $F|_A = S(\phi)|_A$ .
- (d)  $F|_A$  is, up to multiplication by an exponential, a homogeneous (sequence) polynomial space.

The following scheme sketches the various steps required in the construction of linear projectors using the approach above.

**(3.8) Scheme.** *Let  $\phi$  be a compactly supported function whose integer translates are globally linearly independent. Check whether  $\phi|_A = 0$ ; if so replace  $\phi$  by a translate of it. Find a subset  $A \subset \mathbb{R}^s$  satisfying (3.1); then*

- (a) Compute the finite set  $\nu(A)$ .
- (b) Find a polynomial space  $P$  which interpolates correctly on  $\nu(A)$ . For that you may apply the algorithm given in [BR<sub>1</sub>; §4].
- (c) Find an exponential  $e_\theta$  such that  $\sum_{\alpha \in \mathbb{Z}^s} e_\theta(\alpha)\phi(-\alpha) \neq 0$ . Define  $F = \phi *' e_\theta P$ . At this point you may wish to replace  $F$  by a shift-invariant subspace of it which still satisfies (3.4).
- (d) For a given basis  $f_1, \dots, f_n$  for  $F$ , find the basis  $g_1, \dots, g_n$  for  $F$  that satisfies

$$\phi *' g_j = f_j, \quad j = 1, \dots, n.$$

(This step can also be executed with discrete convolution, i.e., with  $\phi|_A$  replacing  $\phi$ ).

- (e) Define a linear functional on  $F$  by  $\mu(f_j) = g_j(0)$ ,  $j = 1, \dots, n$ , and extend  $\mu$  to your favorite choice of a functional  $\lambda$  defined on some superspace of  $S(\phi)$  and supported in  $A$ . (In case the extension is to  $C(\mathbb{R}^s)$ , you may choose  $\lambda$  to be supported on an appropriate finite subset of  $A$  with cardinality  $\leq \#\nu(A)$ ).
- (f) The resulting  $Q_\lambda$  is a linear projector.

#### 4. Piecewise-polynomials and piecewise-exponentials: quasi-interpolation

In this section we review some methods concerning the construction of quasi-interpolants for piecewise-exponentials (and in particular piecewise-polynomials). Most of the results here are known, and the approach taken follows that of [BR<sub>2</sub>]. The discussion also makes use of various observations from [BAR], [B<sub>2</sub>] and [Ro].

Let  $H$  be a finite-dimensional exponential space. This means, by definition, that each function in  $H$  admits the form

$$(4.1) \quad \sum_{j=1}^n e_{\theta_j} p_j, \quad \theta_j \in \mathbf{C}^s, \quad p_j \in \pi, \quad j = 1, \dots, n.$$

We refer to the elements of  $H$  simply as “exponentials”. Furthermore, we assume hereafter that  $H$  is translation-invariant, which implies that a basis for  $H$  is given in terms of functions of the form

$$e_{\theta} p, \quad p \in \pi.$$

The **spectrum** of  $H$  is the set of frequencies of all exponentials in  $H$ :

$$(4.2) \quad \text{spec } H := \{\theta \in \mathbf{C}^s : e_{\theta} \in H\}.$$

Now let  $\phi$  be a (compactly supported) piecewise- $H$  function for which

$$(4.3) \quad H \subset S(\phi).$$

A **quasi-interpolant** here means any linear map  $Q$  from some superspace  $V$  of  $S(\phi)$  into  $S(\phi)$  which satisfies

$$(4.4) \quad Q(f) = f, \quad \forall f \in H.$$

Under the regularity assumption

$$(4.5) \quad \widehat{\phi}(-i\theta) \neq 0, \quad \forall \theta \in \text{spec } H,$$

the operator  $T := \phi *' |_H$  is an automorphism.

**(4.6) Proposition.** *For  $\lambda \in V^*$ , the condition*

$$\lambda(f) = [0]T^{-1}(f), \quad \forall f \in H,$$

*is sufficient for  $Q_{\lambda}$  to be a quasi-interpolant. Furthermore, this condition is also necessary in case  $\phi*$  is injective.*

This last proposition shows that a careful study of the map  $T^{-1}$ , hence also of  $T$ , is essential for construction of a quasi-interpolant. This study is facilitated by the following

**(4.7) Result[BR<sub>2</sub>].** *Let  $e_{\theta} p$  be a function in  $S(\phi)$  (with  $\theta \in \mathbf{C}^s$  and  $p \in \pi$ ). Then*

$$\phi *' (e_{\theta} p) = \phi * (e_{\theta} p),$$

*where the right hand side convolution is the usual convolution between functions (or distributions).*

The above result suggests that in order to find a preimage of  $e_{\theta} p \in H$ , we may solve the convolution equation

$$\phi * (e_{\theta}?) = e_{\theta} p.$$

Dividing both sides by  $e_\theta$  and applying Fourier transform we get

$$(E^{-i\theta}\widehat{\phi})^\widehat{?} = \widehat{p}.$$

Since  $\text{supp } \widehat{p} = 0$  and  $\widehat{\phi}(-i\theta) \neq 0$ , we may divide both sides of (4.7) by  $E^{-i\theta}\widehat{\phi}$  to conclude

$$\widehat{?} = \frac{\widehat{p}}{E^{-i\theta}\widehat{\phi}} = \sum_{\alpha \geq 0} \frac{D^\alpha \psi(\theta)}{\alpha!} \widehat{D^\alpha p}, \quad \psi(x) := \frac{1}{\widehat{\phi}(-ix)},$$

hence

$$? = \sum_{\alpha \geq 0} \frac{D^\alpha \psi(\theta)}{\alpha!} D^\alpha p.$$

With  $\frac{D^\alpha \psi(\theta)}{\alpha!}$  denoted by  $a_{\theta, \alpha}$ , we conclude

**(4.8) Proposition.** For  $e_\theta p \in H$

$$(4.9) \quad [0]T^{-1}(e_\theta p) = [0] \sum_{\alpha \geq 0} a_{\theta, \alpha} D^\alpha p.$$

Combining the last proposition with (4.6) Proposition we conclude

**(4.10) Corollary.** Assume that the integer translates of  $\phi$  are globally linearly independent. Then the condition

$$\lambda(e_\theta p) = [0] \sum_{\alpha} a_{\theta, \alpha} D^\alpha p, \quad \forall e_\theta p \in H$$

is necessary and sufficient for  $Q_\lambda$  to be a quasi-interpolant.

## 5. Piecewise-polynomials and piecewise-exponentials: linear projectors

Here we combine the results of the two previous sections in the derivation of linear projectors for the piecewise-exponential space  $S(\phi)$ .

Retaining the notations of section 4, (and especially the notation  $[\theta]$  for point-evaluation in  $\theta$ ), we assume throughout this section that  $H \subset S(\phi)$ , and that there exists an open bounded set  $A \subset \mathbb{R}^s$  for which

$$(5.1) \quad \#\nu(A) = \dim H,$$

(where  $\nu(A)$  is as in (3.6)). It follows that  $H$  satisfies the condition required of  $F$  in (3.4). Furthermore, the translates of  $\phi$  are locally linearly independent on  $A$  (in the sense of (1.2)), and in particular  $A$  satisfies (3.1). With the aid of [Ro; Lem. 2.2] we can also easily conclude that  $\phi_{*'} is 1-1 on  $H$ .$

Therefore, (3.5) Corollary reads here:

**(5.2) Corollary.** *Let  $\lambda$  be a linear functional defined on an extension  $V$  of  $S(\phi)$  and vanishing on all  $f \in S(\phi)$  supported in  $\mathbb{R}^s \setminus A$ . If*

$$Q_\lambda(f) = f, \quad \forall f \in H,$$

then  $Q_\lambda$  is a projector.

Equivalently,  $Q_\lambda$  is a projector if  $\lambda$  extends the linear functional  $[0]T^{-1} \in H^*$ . The precise values of  $[0]T^{-1}$  on  $H$  were determined in (4.10)Corollary, but of course many extensions (to various  $V$ 's) of  $[0]T^{-1}$  are available. To draw the connections between the results here and the constructions of dual bases for a box spline space in  $[DM_{1,2}]$  and  $[J_{1,2}]$ , we concentrate now on the case when  $[0]T^{-1}$  is represented (and thus extended) with the aid of differential operators. First, we associate with every  $q \in \pi$  a linear functional  $q^* \in H^*$  defined by

$$q^*(f) = q(D)f(0), \quad \forall f \in H.$$

Note that for  $e_\theta p \in H$

$$q^*(e_\theta p) = p^* E^\theta(q) = [0] \sum_{\alpha \geq 0} \frac{(D^\alpha q)(\theta)}{\alpha!} D^\alpha p,$$

while on the other hand, by (4.10)Corollary

$$[0]T^{-1}(e_\theta p) = [0] \sum_{\alpha \geq 0} a_{\theta, \alpha} D^\alpha p.$$

Hence we conclude

**(5.3) Corollary.** *Let  $q \in \pi$  satisfy*

$$(5.4) \quad D^\alpha q(\theta) = \alpha! a_{\theta, \alpha}, \quad \forall \theta \in \text{spec } H, \quad |\alpha| \leq \max\{\deg p : e_\theta p \in H\},$$

with  $\{a_{\theta, \alpha}\}$  as in (4.10)Corollary. For a space  $V \supset S(\phi)$ , let  $\lambda \in V^*$  be an extension of  $q^* \in H^*$  which vanishes on all functions in  $S(\phi)$  with support in  $\mathbb{R}^s \setminus A$ . Then  $Q_\lambda$  is a projector.

In case all the integer translates of  $\phi$  belong to  $H$  in a neighborhood of the origin, one may choose to extend  $q^*$  (at least on  $S(\phi)$ ) to the functional  $\lambda(f) = q(D)f(0)$ ,  $f \in S(\phi)$ . We note that in general condition (5.4) is sufficient but not necessary for the equality  $[0]T^{-1} = q^*$ .

Choosing  $\phi$  to be a polynomial or exponential box spline, (5.3)Corollary verifies  $[DM_1; \text{Thm. 5.1}]$  and  $[DM_2; \text{Thm. 5.1}]$ . Note that the approach taken here avoids the application of Poisson's summation formula, hence we need not to impose any further restrictions on the polynomial  $q$  (see  $[J_2]$  for a discussion of the difficulty in the application of Poisson's summation formula). Poisson's formula is implicitly used here, since this is the key tool in the proof of (4.7)Result. Nevertheless, that latter result holds for any compactly supported distribution  $\phi$ .

In case  $\phi$  is not smooth enough at the origin, one may wish to represent  $[0]T^{-1}$  by  $q_\theta^* E^\theta$  with  $\theta \in \mathbb{R}^s$  chosen such that  $S(\phi)$  is locally in  $H$  in a neighborhood of  $\theta$ . To find the connection between the various  $q_\theta$ 's, let  $P \subset \pi$  be a space dual to  $H$  in the sense that the map

$$P \rightarrow H^* : q \mapsto q^*$$

is bijective (hence every  $\mu \in H^*$  is uniquely represented by some  $q^*$  with  $q \in P$ ). With  $\{f_j\}_{j=1}^n$  and  $\{p_j\}_{j=1}^n$  dual bases for  $H$  and  $P$  respectively, one has

$$f = \sum_{j=1}^n p_j^*(f) f_j, \quad \forall f \in H.$$

Let  $q$  be the unique polynomial in  $P$  satisfying  $q^* = [0]T^{-1}$  and let  $\theta \in \mathbb{R}^s$ . Then

$$\begin{aligned} [0]T^{-1}(f) &= q^*(f) = (q^* E^{-\theta})(E^\theta f) \\ &= \sum_{j=1}^n p_j^*(E^\theta f) q^*(E^{-\theta} f_j) \\ &= [\theta] \left( \sum_{j=1}^n (q(D) f_j) (-\theta) p_j(D) f \right). \end{aligned}$$

Since  $H$  is translation-invariant and  $q$  does not vanish on  $\text{spec } H$ ,  $q(D)$  is injective on  $H$ , hence  $\{q(D) f_j\}_{j=1}^n$  is also a basis for  $H$ . We have proved

**(5.5) Corollary.** *Given a basis  $\{g_j\}_{j=1}^n$  for  $H$  and a dual space  $P \subset \pi$  for  $H$ , there exists a unique basis  $\{p_j\}_{j=1}^n$  for  $P$  satisfying for all  $\theta \in \mathbb{R}^s$*

$$[0]T^{-1} = [\theta] \left( \sum_{j=1}^n g_j(-\theta) p_j(D) \right).$$

Moreover,  $\{p_j\}_{j=1}^n$  is the basis for  $P$  which is dual to  $\{f_j\}_{j=1}^n$ , where  $\{f_j\}_{j=1}^n$  are defined by

$$f_j \in H, \quad q(D) f_j = g_j, \quad j = 1, \dots, n,$$

with  $q \in P$  the unique polynomial satisfying  $q^* = [0]T^{-1}$ .

For an exponential box spline  $\phi$  there are two natural choices for a dual  $P$  for  $H$  (cf. [BDR; §4] for details).

For a polynomial  $H$ , one may write each of the polynomials  $\{g_j\}_{j=1}^n$  in the above corollary in power form and then use summation by parts to obtain

**(5.6) Corollary.** *Assume that  $H \subset \pi_k$  for some non-negative integer  $k$ . Then there exist polynomials  $\{p_\alpha\}_{|\alpha| \leq k}$  such that*

$$[0]T^{-1} = [\theta] \left( \sum_{|\alpha| \leq k} \theta^\alpha p_\alpha(D) \right).$$

The sequence  $\{p_\alpha\}_{|\alpha| \leq k}$  is unique in case we impose the restriction  $\{p_\alpha\} \subset P$  for a space  $P$  dual to  $H$ . We mention that in case  $H$  is invariant under the complex involution, it is self-dual and we then may choose  $\{p_\alpha\} \subset H$ .

(5.6) Corollary captures the construction in [J<sub>1</sub>]. There  $\phi$  was a polynomial box spline and  $P$  was chosen as a specific known dual of  $H$ .

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