On the integrability of a generalized variable-coefficient Kadomtsev-Petviashvili equation

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Abstract: By considering the inhomogeneities of media, a generalized variable-coefficient Kadomtsev-Petviashvili (vc-KP) equation is investigated, which can be used to describe many nonlinear phenomena in fluid dynamics and plasma physics. In this paper, we systematically investigate complete integrability of the generalized vc-KP equation under a integrable constraint condition. With the aid of a generalized Bells polynomials, its bilinear formulism, bilinear Bäcklund transformations, Lax pairs and Darboux covariant Lax pairs are succinctly constructed, which can be reduced to the ones of several integrable equations such as KdV, cylindrical KdV, KP, cylindrical KP, generalized cylindrical KP, non-isospectral KP equations etc. Moreover, the infinite conservation laws of the equation are found by using its Lax equations. All conserved densities and fluxes are expressed in the form of accurate recursive formulas. Furthermore, an extra auxiliary variable is introduced to get the bilinear formulism, based on which, the soliton solutions and Riemann theta function periodic wave solutions are presented. And the influence of inhomogeneity coefficients on solitonic structures and interaction properties are discussed for physical interest and possible applications by some graphic analysis. Finally, a limiting procedure is presented to analyze in detail, asymptotic behavior of the periodic waves, and the relations between the periodic wave solutions and soliton solutions.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

It is important to investigate the integrability of nonlinear evolution equation (NLEE), which can be regarded as a pretest and the first step of its exact solvability. There are many significant properties, such as bilinear

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form, Lax pairs, infinite conservation laws, infinite symmetries, Hamiltonian structure, Painlevé test and bilinear Bäcklund transformation that can characterize integrability of nonlinear equations. Although there have been many methods proposed to deal with the NLEEs, e.g., inverse scattering transformation [1], Darboux transformation [2], Bäcklund transformation(BT) [3], Hirota method [4] and so on. By using the bilinear form for a given NLEE, one can not only construct its multisoliton solutions, but also derive the bilinear BT, and some other properties [4]-[7]. Unfortunately, one of the key steps of this method is to replace the given NLEE by some more tractable bilinear equations for new Hirota's variables. There is no general rule to find the transformations, nor for choice or application of some essential formulas (such as exchange formulas). During the early 1930s, Bell proposed the classical Bell polynomials, which are specified by a generating function and exhibiting some important properties [8]. Since then the Bell polynomials have been exploited in combinatorics, statistics, and other fields [11]-[13]. However, in recent years Lambert and co-workers have proposed an alternative procedure based on the use of the Bell polynomials to obtain parameter families of bilinear Bäcklund transformation and lax pairs for soliton equations in a lucid and systematic way [8]-[10]. The Bell polynomials are found to play an important role in the characterization of integrability of a nonlinear equation.

Recently, there has been growing interest in studying the variable-coefficient nonlinear evolution equations (NLEEs), which are often considered to be more realistic than their constant-coefficient counterparts in modeling a variety of complex nonlinear phenomena under different physical backgrounds [14]. Since those variable-coefficient NLEEs are of practical importance, it is meaningful to systematically investigate completely integrable properties such as bilinear form, Lax pairs, infinite conservation laws, infinite symmetries, Hamiltonian structure, Painlevé test, bilinear Bäcklund transformation, symmetry algebra and construct various exact analytic solutions, including the soliton solutions and periodic solutions. For describing the propagation of solitonic waves in inhomogeneous media, the variable-coefficient KP-type equations have been derived from many physical applications in plasma physics, fluid dynamics and other fields [15, 16].

In this paper, we will focus on a generalized variable-coefficient Kadomtsev-Petviashvili (vc-KP) equation with nonlinearity, dispersion and perturbed term

$$[u_t + h_1(y,t)u_{3x} + h_2(y,t)uu_x]_x + h_3(y,t)u_{2x} + h_4(y,t)u_{xy} + h_5(y,t)u_{2y} + h_6(y,t)u_x + h_7(y,t)u_y = 0, \tag{1.1}$$

where u is a differentiable function of x, y and t, $h_i(y,t)$ $i = 1, \ldots, 7$ are all analytic, sufficiently differentiable functions, may provide a more realistic model equation in several physical situations, e.g. in the propagation of (small-amplitude) surface waves in straits or large channels of (slowly) varying depth and width and nonvanishing vorticity. Eq. (1.1) can reduce to a series of integrable models or describe such physical phenomena as the electrostatic wave potential in plasma physics, the amplitude of the shallow-water wave and/or surface wave in fluid dynamics, etc [16]-[19]. Obviously, Eq. (1.1) contains quite a number of variable-coefficient KP models arising from various branches of physics, e.g. the KdV, cylindrical KdV, KP, cylindrical KP, generalized cylindrical KP and non-isospectral KP equations etc. Some currently important examples are given below:

• The celebrated, historic Korteweg-de Vries (KdV) equation [1, 20]

$$u_t + 6uu_{3x} + u_{3x} = 0, \tag{1.2}$$

has been found to model many physical, mechanical and engineering phenomena, such as ion-acoustic waves, geophysical fluid dynamics, lattice dynamics, the jams in the congested traffic etc.

• The Kadomtsev-Petviashvili (KP) equation [21]

$$(u_t + 6uu_{3x} + u_{3x})_x + \sigma_0 u_{2y} = 0, \tag{1.3}$$

where $\sigma_0 = \pm 1$, has been discovered to describe the evolution of long water waves, small-amplitude surface waves with weak nonlinearity, weak dispersion, and weak perturbation in the y direction, weakly relativistic soliton interactions in the magnetized plasma and some other nonlinear models.

• The cylindrical KdV equation [22, 23]

$$u_t + 6uu_{3x} + u_{3x} + \frac{1}{2t}u_x = 0, \tag{1.4}$$

was first proposed by Maxon and Viecelli in 1974 when they studied propagation of radically ingoing acoustic waves. And its counterpart in (2+1)-dimensional, the cylindrical KP equation [24, 25] and generalized cylindrical KP equation [17, 26]

$$(u_t + 6uu_{3x} + u_{3x})_x + \frac{\sigma_0^2}{t^2}u_{2y} + \frac{1}{2t}u_x = 0, \tag{1.5}$$

$$(u_t + h_2(t)uu_{3x} + h_1(t)u_{3x})_x + [f(t) + yg(t)]u_{2x} + r(t)u_{xy} + \frac{3\sigma_0^2}{t^2}u_{2y} + \frac{1}{2t}u_x = 0, \tag{1.6}$$

with $\sigma_0^2 = \pm 1$, have also been constructed to describe the nearly straight wave propagation which varies in a very small angular region [17], [24]-[26].

• The KP equation with time-dependent coefficients [18]

$$(u_t + uu_x + u_{3x})_x + \mu_3(t)u_x + \mu_4(t)u_{2y} = 0, \tag{1.7}$$

models the propagation of small-amplitude surface waves in straits or large channels of slowly varying depth and width and nonvanishing vorticity.

• Jacobi elliptic function solutions and integrability property for the following variable-coefficient KP equation

$$(u_t + h_1(t)uu_x + h_2(t)u_{3x})_x + h_3(t)u_{2y} + 6h_4(t)u_x = 0, \tag{1.8}$$

have been presented in Ref. [27].

• The following equation

$$(u_t + h_1(t)uu_x + h_2(t)u_{3x})_x + h_3(t)u_{2x} + h_4(t)u_{2y} = 0, \tag{1.9}$$

can be used to describe nonlinear waves with a weakly diffracted wave beam, internal waves propagating along the interface of two fluid layers, etc [19].

• Non-isospectral and variable-coefficient KP equations read [28]

$$(u_t + uu_x + u_{3x})_x + au_x + bu_y + cu_{2y} + du_{xy} + eu_{2x} = 0, \tag{1.10}$$

$$u_t + h_1(u_{3x} + 6uu_x + 3\sigma^2\partial_x^{-1}u_{yy}) + h_2(u_x - \sigma xu_y - 2\sigma\partial_x^{-1}u_y) - h_3(xu_x + 2u + 2yu_y) = 0, \tag{1.11}$$

where a, b, c, d, e are functions of y, t, and h_i ($i = 1, 2, 3$) are functions of t. Bilinear representations, bilinear Bäcklund transformations and Lax pairs for non-isospectral KP equations (1.10) and (1.11) are systematically investigated, respectively, in Refs. [28].

As we well known, the KdV, cylindrical KdV, KP, cylindrical KP, generalized cylindrical KP and non-isospectral KP equations belong to the integrable hierarchy of KP equation. In recent years, a large number of papers have been focusing on Painlevé property, dromion-like structures and various exact solutions of NLEE [29]-[48]. But their integrability, to the best of our knowledge, have not been studied in detail. The existence of infinite conservation laws can be considered as one of the many remarkable properties that deemed to characterize soliton equations. Under certain constraint conditions, the variable-coefficient models may be proved to be integrable and given explicit analytic solutions. The corresponding constraint conditions on Eq. (1.1) in this paper, which can be naturally found in the procedure of applying the Bell polynomials, will be

$$h_2 = c_0 h_1 e^{\int h_6 dt}, \quad \partial_y h_4 = h_6 + \partial_t \ln h_1 h_2^{-1}, \quad h_5 = 3\alpha^2 h_1, \quad \partial_y h_1 = \partial_y h_2 = h_7 = 0, \tag{1.12}$$

where c_0 and α being both arbitrary parameters.

The main purpose of this paper is extend the binary Bell polynomial approach to systematically construct bilinear formulism, bilinear Bäcklund transformations, Lax pairs and Darboux covariant Lax pairs of the generalized vc-KP equation (1.1) under conditions (1.12). To our knowledge, there have been no discussions about Eq. (1.1) under the conditions (1.12). Based on its Lax equations, the infinite conservation laws of the equation will be constructed. By using the bilinear formula, the soliton solutions and Riemann theta function periodic wave solutions are also presented.

The structure of the present paper is as follows. By virtue of the properties of the binary Bell polynomials, we systematically construct the bilinear representation, Bäcklund transformation, Lax pair and Darboux covariant Lax pairs of the generalized vc-KP equation (1.1) in Secs. 2-4, respectively. By means of its Lax equation, in Sec. 5, the infinite conservation laws of the equation also be constructed. In Sec. 6, based on the bilinear formula and the recently results in Ref.[51, 52], we present the soliton solutions and Riemann theta function periodic wave solutions of the generalized vc-KP equation (1.1) under the conditions (1.12) with $c_0 = 6$. And we also discuss the influence of inhomogeneity coefficients on solitonic structures and interaction properties for physical interest and possible applications by some graphic analysis. Finally, a limiting procedure is presented to analyze in detail, the relations between the periodic wave solutions and soliton solutions. And some introductions of multidimensional Bell polynomials and Riemann theta function wave are given in Appendix A, B, respectively.

2. Bilinear representation

In this section, we construct the bilinear representation of Eq. (1.1) by using an extra auxiliary variable instead of the exchange formulae.

Theorem 2.1. *Using the following transformation*

$$u = 12h_1h_2^{-1}(\ln f)_{xx}, \tag{2.1}$$

the generalized vc-KP equation (1.1) *can be bilinearized into*

$$\mathscr{D}(D_t, D_x, D_y) \equiv [D_xD_t + h_1D_x^4 + h_3D_x^2 + h_4D_xD_y + h_5D_y^2 + (h_6 + \partial_t \ln h_1h_2^{-1})\partial_x + h_7\partial_y - \delta]f \cdot f = 0, \tag{2.2}$$

where $\partial_x f \cdot f \equiv \partial_x f^2 = 2ff_x$, $\partial_y f \cdot f \equiv \partial_y f^2 = 2ff_y$, $\delta f \cdot f \equiv \delta f^2$, *and* $\delta = \delta(y, t)$ *is a constant of integration.*

Proof. To obtain the linearization of Eq. (1.1), a new variable q is introducing(q is called a potential field)

$$u = c(t)q_{2x}, \tag{2.3}$$

where c=$c(t)$ is a function to be determined. Substituting Eq. (2.3) into Eq. (1.1), one can write the resulting equation of the form

$$q_{2x,t} + h_1q_{5x} + ch_2q_{2x}q_{3x} + h_3q_{3x} + h_4q_{2x,y} + h_5q_{x,2y} + (h_6 + \partial_t \ln c)q_{2x} + h_7q_{xy} = 0, \tag{2.4}$$

where we will see that such decomposition is necessary to get bilinear form of Eq. (1.1). Moreover by the integration of Eq. (2.4) about x, one obtains

$$E(q) \equiv q_{xt} + h_1(q_{4x} + 3q_{2x}^2) + h_3q_{2x} + h_4q_{xy} + h_5q_{2y} + (h_6 + \partial_t \ln h_1h_2^{-1})q_x + h_7q_y = \delta, \tag{2.5}$$

by choosing the function $c(t) = 6h_1h_2^{-1}$ and using the formula (A.7), where $\delta = \delta(y, t)$ is a constant of integration. Based on the formula (A.7), Eq. (2.5) can be rewritten as the following form

$$E(q) = P_{xt}(q) + h_1P_{4x}(q) + h_3P_{2x}(q) + h_4P_{xy}(q) + h_5P_{2y}(q) + (h_6 + \partial_t \ln h_1h_2^{-1})q_x + h_7q_y = \delta. \tag{2.6}$$

Finally, according to the property (A.9) and changing the variable

$$q = 2\ln f \Longleftrightarrow u = c(t)q_{2x} = 12h_1h_2^{-1}(\ln f)_{xx}, \tag{2.7}$$

Eq. (2.6) produces the same bilinear representation $\mathscr{D}$ (2.2) of the generalized vc-KP equation (1.1). □

The formula (2.2) is a new bilinear form, which can also reduce to the ones obtained in Refs. [4, 7, 21, 24, 25, 49, 50] by choosing the appropriate coefficients h_i ($i = 1, \ldots, 7$).

(i). If $h_i = 0$ ($i = 3, 4, 5, 6, 7$), $h_1 = 1$ and $h_2 = 6$, Eq. (1.1) becomes the constant coefficient KdV equation. The corresponding bilinear form (2.2) reduces to

$$[D_xD_t + D_x^4]f \cdot f = 0, \tag{2.8}$$

which is also obtained in Refs. [4, 7, 49, 50], respectively.

(ii). In the case of $h_i = 0$ ($i = 3, 4, 6, 7$), $h_1 = 1$, $h_2 = 6$ and $h_5 = \pm 1$, Eq. (1.1) reduces to a general KP equation. The corresponding bilinear form (2.2) becomes

$$[D_xD_t + D_x^4 \pm D_y^2]f \cdot f = 0, \tag{2.10}$$

which is also researched in Refs. [4, 21, 49], respectively.

(iii). Assuming that $h_i = 0$ $(i = 3, 4, 7)$, $h_5 = 3\sigma_0^2/t^2$ and $h_6 = 1/2t$, Eq. (1.1) becomes the cylindrical KP model [24, 25]. The corresponding bilinear form (2.2) reduces to

$$[D_xD_t + h_1D_x^4 + 3\sigma_0^2/t^2D_y^2 + (h_6 + \partial_t \ln h_1h_2^{-1})\partial_x]f \cdot f = 0, \tag{2.12}$$

with σ_0 is an arbitrary constant, which is a new bilinear formulism for the cylindrical KP model.

3. Bilinear Bäcklund transformation and associated Lax pair

In this section, we construct the bilinear Bäcklund transformation and the Lax pair of the generalized vc-KP equation (1.1). Bilinear Bäcklund transformation is useful in constructing solutions and also serves as a characteristic of integrability for a given system. In the following, we derive a bilinear Bäcklund for the generalized vc-KP equation (1.1) by using the use of binary Bell polynomials.

Theorem 3.1. *Suppose that f is a solution of the bilinear equation* (2.2) *under the conditions* (1.12)*, i.e., the coefficients h_i $(i = 1, 2, 5, 6, 7)$ satisfy $h_2 = c_0h_1e^{\int h_6 dt}$, $h_5 = 3\alpha^2h_1$, $h_7 = 0$, then g satisfying*

$$\begin{aligned}
&(D_x^2 + \alpha D_y - \lambda)f \cdot g = 0,\\
&\left[D_t + h_1\left(D_x^3 - 3\alpha D_xD_y + 3\lambda D_x\right) + h_3D_x + h_4D_y + \gamma\right]f \cdot g = 0,
\end{aligned} \tag{3.1}$$

is another solution of the equation (2.2)*, where c_0, α are arbitrary parameters and $\gamma = \gamma(y, t)$ is an arbitrary function. So the system* (3.1) *is called a bilinear Bäcklund transformation for the generalized vc-KP equation* (1.1)*.*

Proof. Suppose the following expressions

$$q = 2\ln g, \quad q' = 2\ln f \tag{3.2}$$

are solutions of Eq. (2.5), respectively. The condition from the Eq. (2.5) can be changed into

$$\begin{aligned}
E(q') - E(q) =&(q' - q)_{xt} + h_1(q' - q)_{4x} + 3h_1(q' + q)_{2x}(q' - q)_{2x} + h_3(q' - q)_{2x} + h_4(q' - q)_{xy}\\
&+ h_5(q' - q)_{2y} + (h_6 + \partial_t \ln h_1h_2^{-1})(q' - q)_x + h_7(q' - q)_y = 0.
\end{aligned} \tag{3.3}$$

In order to obtain such conditions, the following new auxiliary variables are introduced

$$\upsilon = (q' - q)/2 = \ln(f/g), \quad \omega = (q' + q)/2 = \ln(fg), \tag{3.4}$$

then we can change Eq. (3.3) into the following form

$$\begin{aligned}
E(q') - E(q) =&E(\omega + \upsilon) - E(\omega - \upsilon) = \upsilon_{xt} + h_1(\upsilon_{4x} + 6\omega_{2x}\upsilon_{2x}) + h_3\upsilon_{2x} + h_4\upsilon_{xy}\\
&+ h_5\upsilon_{2y} + (h_6 + \partial_t \ln h_1h_2^{-1})\upsilon_x + h_7\upsilon_y\\
=&\partial_x[\mathscr{Y}_t(\upsilon) + h_1\mathscr{Y}_{3x}(\upsilon, \omega)] + \mathscr{R}(\upsilon, \omega) = 0,
\end{aligned} \tag{3.5}$$

where

$$\mathscr{R}(\upsilon,\omega) = 3h_1\text{Wronskian}[\mathscr{Y}_{2x}(\upsilon,\omega),\mathscr{Y}_x(\upsilon)] + h_3\upsilon_{2x} + h_4\upsilon_{xy} + h_5\upsilon_{2y} + (h_6 + \partial_t \ln h_1 h_2^{-1})\upsilon_x + h_7\upsilon_y.$$

To rewrite $\mathscr{R}(\upsilon,\omega)$ as $\mathscr{Y}$-polynomials in form of x-divergence form and to change Eq. (3.5) into some conditions, one can introduce a new constant

$$\mathscr{Y}_{2x}(\upsilon,\omega) + \alpha\mathscr{Y}_y(\upsilon,\omega) = \lambda, \tag{3.6}$$

where $\alpha = \alpha(t)$ is an function of t and λ is an arbitrary constant. By virtue of the Eq.(3.6), $\mathscr{R}(\upsilon,\omega)$ can be changed into

$$\mathscr{R}(\upsilon,\omega) = 3h_1\lambda\upsilon_{2x} - \alpha^{-1}\left[h_5\omega_{2x,y} + (2h_5 - 3\alpha^2 h_1)\upsilon_x\upsilon_{x,y} + 3\alpha^2 h_1\upsilon_{2x}\upsilon_y\right] + h_3\upsilon_{2x} + h_4\upsilon_{x,y} + (h_6 + \partial_t \ln h_1 h_2^{-1})\upsilon_x + h_7\upsilon_y, \tag{3.7}$$

which is equivalent to the following form

$$\mathscr{R}(\upsilon,\omega) = \partial_x\left[(3h_1\lambda + h_3)\mathscr{Y}_x(\upsilon) - 3\alpha h_1\mathscr{Y}_{x,y}(\upsilon,\omega) + h_4\mathscr{Y}_y(\upsilon)\right], \tag{3.8}$$

by taking

$$h_5 = 2h_5 - 3\alpha^2 h_1 = 3\alpha^2 h_1, \quad h_6 + \partial_t \ln h_1 h_2^{-1} = 0, \quad h_7 = 0,$$

namely,

$$h_2 = c_0 h_1 e^{\int h_6 dt}, \quad h_5 = 3\alpha^2 h_1, \quad h_7 = 0. \tag{3.9}$$

Then, using Eqs. (3.6)-(3.8), we obtain the following system

$$\begin{aligned} &\mathscr{Y}_{2x}(\upsilon,\omega) + \alpha\mathscr{Y}_y(\upsilon,\omega) - \lambda = 0, \\ &\partial_x\mathscr{Y}_t(\upsilon) + \partial_x\left\{h_1\left[\mathscr{Y}_{3x}(\upsilon,\omega) - 3\alpha\mathscr{Y}_{xy}(\upsilon,\omega) + 3\lambda\mathscr{Y}_x(\upsilon)\right] + h_3\mathscr{Y}_x(\upsilon) + h_4\mathscr{Y}_y(\upsilon)\right\} = 0. \end{aligned} \tag{3.10}$$

By virtue of property (A.6), Eq. (3.10) yields to the bilinear Bäcklund transformation (3.1) with $\gamma = \gamma(t)$ is an arbitrary function. □

Bäcklund transformation (3.1) can be used to construct exact solutions for the generalized vc-KP equation (1.1). Next, using the system (3.10), we will derive Lax pairs of the equation (1.1).

Theorem 3.2. *Under the conditions* (1.12) *and* $c_0 = 6$, *the generalized vc-KP equation* (1.1) *admits a Lax pair*

$$(\mathscr{L}_1 + \alpha\partial_y)\psi \equiv \psi_{2x} + \alpha\psi_y + (ue^{\int h_6 dt} - \lambda)\psi = 0, \tag{3.11a}$$

$$\begin{aligned} (\partial_t + \mathscr{L}_2)\psi \equiv\ &\psi_t + 4h_1\psi_{3x} - h_4\alpha^{-1}\psi_{2x} + \left(6h_1 ue^{\int h_6 dt} + 3h_1\lambda + h_3\right)\psi_x \\ &+ \left(3h_1 u_x e^{\int h_6 dt} - 3h_1\alpha\partial_x^{-1}u_y e^{\int h_6 dt} - h_4\alpha^{-1}ue^{\int h_6 dt} + h_4\alpha^{-1}\lambda\right)\psi = 0, \end{aligned} \tag{3.11b}$$

where u is a solution of the equation (1.1).

Proof. Linearizing the Eq. (3.10) into a Lax pair, we introduce a Hopf-Cole transformation $\upsilon = \ln\psi$. Using (A.8) and (A.9), one obtains

$$\mathscr{Y}_x(\upsilon) = \psi_x/\psi, \quad \mathscr{Y}_{2x}(\upsilon,\omega) = q_{2x} + \psi_{2x}/\psi, \quad \mathscr{Y}_{xy}(\upsilon,\omega) = q_{xy} + \psi_{xy}/\psi,$$

$$\mathscr{Y}_y(\upsilon) = \psi_y/\psi, \quad \mathscr{Y}_t(\upsilon) = \psi_t/\psi, \quad \mathscr{Y}_{3x}(\upsilon,\omega) = 3q_{2x}\psi_x/\psi + \psi_{3x}/\psi,$$

by means of which, Eq. (3.10) is then changed into the following form with λ and γ

$$(\mathscr{L}_1 + \alpha\partial_y)\psi \equiv \psi_{2x} + \alpha\psi_y + (q_{2x} - \lambda)\psi = 0, \tag{3.12a}$$

$$(\partial_t + \mathscr{L}_2)\psi \equiv \psi_t + 4h_1\psi_{3x} - h_4\alpha^{-1}\psi_{2x} + (6h_1q_{2x} + 3h_1\lambda + h_3)\psi_x + \left(3h_1q_{3x} - 3h_1\alpha q_{xy} - h_4\alpha^{-1}q_{2x} + h_4\alpha^{-1}\lambda\right)\psi = 0, \tag{3.12b}$$

which is equivalent to the Lax pair (3.11a) and (3.11b), respectively, by replacing q_{2x} with $ue^{\int h_6 dt}$. □

Corollary 3.3. *Using the conditions* (1.12) *and* $c_0 = 6$*, the Lax pair* (3.11a) *and* (3.11b) *of the generalized vc-KP equation* (1.1) *is equivalent to the following Lax pair*

$$(\mathscr{L}_1 + \alpha\partial_y)\psi \equiv \psi_{2x} + \alpha\psi_y + (ue^{\int h_6 dt} - \lambda)\psi = 0, \tag{3.13a}$$

$$(\partial_t + \mathscr{L}_2)\psi \equiv \psi_t - 4h_1\alpha\psi_{xy} - \left(h_1ue^{\int h_6 dt} - 7h_1\lambda - h_3\right)\psi_x + h_4\psi_y - \left(h_1u_xe^{\int h_6 dt} + 3h_1\alpha\partial_x^{-1}u_ye^{\int h_6 dt}\right)\psi = 0, \tag{3.13b}$$

where u is a solution of the equation (1.1).

The formulas (3.1), (3.11a) and (3.11b) are new bilinear Bäcklund transformation and Lax pair, respectively, which can also reduce to the ones obtained in Refs. [1],[4],[17]-[20], [24]-[27],[29],[50] by choosing the appropriate coefficients h_i $(i = 1, \ldots, 7)$. Without loss of generality, taking $c_0 = 6$, then $c(t) = e^{-\int h_6 dt}$.

(i). Assuming that $\alpha = h_i = 0$ $(i = 3, 4, 5, 6, 7)$, and $h_1 = 1$, $h_2 = 6$, Eq. (1.1) becomes the general KdV model. The corresponding Bäcklund transformation (3.1) reduces to

$$(D_x^2 - \lambda) f \cdot g = 0, \quad \left[D_t + D_x^3 + 3\lambda D_x\right] f \cdot g = 0, \tag{3.14}$$

which is studied in Refs. [4, 50]. The corresponding Lax pair (3.11a) and (3.11b) reduces to

$$(\mathscr{L}_1 + \alpha\partial_y)\psi \equiv \psi_{2x} + (u - \lambda)\psi = 0, \tag{3.15a}$$

$$(\partial_t + \mathscr{L}_2)\psi \equiv \psi_t + 4\psi_{3x} + 3(2u + \lambda)\psi_x + 3u_x\psi = 0, \tag{3.15b}$$

where u is a solution of the equation (1.1). The lax pair (3.15a) and (3.15b) is investigated by Lax, Ablowitz and co-workers in Refs. [1, 20], respectively.

(ii). For $h_i = 0$ $(i = 3, 4, 7)$, and $h_1 = 1/t^2$, $h_2 = 6/t^2$, $h_5 = 3\sigma_0^2/t^2$, $h_6 = 1/2t$, Eq. (1.1) becomes the cylindrical KP equation [24, 25]. The corresponding formula (3.1) reduces to

$$(D_x^2 + \sigma_0 D_y - \lambda) f \cdot g = 0, \quad \left[D_t + 1/t^2\left(D_x^3 - 3\sigma_0 D_x D_y + 3\lambda D_x\right) + \gamma\right] f \cdot g = 0, \tag{3.16}$$

which is a new one and not obtained in Refs. [24, 25]. The corresponding Lax pair (3.11a) and (3.11b) reduces to

$$(\mathscr{L}_1 + \alpha\partial_y)\psi \equiv \psi_{2x} + \sigma_0\psi_y + (u\sqrt{t} - \lambda)\psi = 0, \tag{3.17a}$$

$$(\partial_t + \mathscr{L}_2)\psi \equiv \psi_t + 4/t^2\psi_{3x} + \left(6u\sqrt{t}/t^2 + 3\lambda/t^2\right)\psi_x + \left(3u_x\sqrt{t}/t^2 - 3\sigma_0\partial_x^{-1}u_y\sqrt{t}/t^2\right)\psi = 0, \tag{3.17b}$$

where u is a solution of the equation (1.1). The lax pair (3.17a) and (3.17b) is a new one, which is not studied in Refs. [24, 25].

(iii). In the case of $h_1 = 1/t^2$, $h_2 = 6/t^2$, $h_3 = f(t) + yg(t)$, $h_4 = r(t)$, $h_5 = 3\sigma_0^2/t^2$, $h_6 = 1/2t$, $h_7 = 0$, Eq. (1.1) becomes a generalized cylindrical KP equation [17, 26]. The corresponding formula (3.1) reduces to

$$\begin{aligned}&(D_x^2 + \sigma_0 D_y - \lambda) f \cdot g = 0,\\ &\left[D_t + 1/t^2\left(D_x^3 - 3\sigma_0 D_x D_y + 3\lambda D_x\right) + (f + yg)D_x + rD_y + \gamma\right] f \cdot g = 0,\end{aligned} \tag{3.18}$$

which is also a new one and not obtained in Refs. [17, 26]. The corresponding Lax pair (3.11a) and (3.11b) reduces to

$$(\mathscr{L}_1 + \alpha\partial_y)\psi \equiv \psi_{2x} + \sigma_0\psi_y + (u\sqrt{t} - \lambda)\psi = 0, \tag{3.19a}$$

$$\begin{aligned}(\partial_t + \mathscr{L}_2)\psi \equiv{}& \psi_t + 4/t^2\psi_{3x} - \sigma_0^{-1} r(t)\psi_{2x} + \left[6u\sqrt{t}/t^2 + 3\lambda/t^2 + (f(t) + yg(t))\right]\psi_x\\ &+ \left[3u_x\sqrt{t}/t^2 - 3\sigma_0\partial_x^{-1}u_y\sqrt{t}/t^2 - \sigma_0^{-1}r(t)u\sqrt{t} + \sigma_0^{-1}r(t)\lambda\right]\psi = 0,\end{aligned} \tag{3.19b}$$

where u is a solution of the equation (1.1). The lax pair (3.19a) and (3.19b) is a new one, which is not obtained in Refs. [17, 26].

(iv). If $h_1 = f_2(t)$, $h_2 = f_1(t)$, $h_5 = g^2(t)$, $h_6 = 6f(t)$, $h_i = 0$ ($i = 3, 4, 7$), Eq. (1.1) becomes a variable-coefficient KP equation [27]. The corresponding formula (3.1) reduces to

$$\begin{aligned}&(D_x^2 + \sigma_0 D_y - \lambda) f \cdot g = 0,\\ &\left[D_t + 1/t^2\left(D_x^3 - 3\sigma_0 D_x D_y + 3\lambda D_x\right) + (f + yg)D_x + rD_y + \gamma\right] f \cdot g = 0,\end{aligned} \tag{3.20}$$

which is also a new one and not studied in Ref. [27]. The corresponding Lax pair (3.11a) and (3.11b) reduces to

$$(\mathscr{L}_1 + \alpha\partial_y)\psi \equiv \psi_{2x} + |g(t)|/\sqrt{3f_2(t)}\psi_y + (ue^{\int 6f(t)dt} - \lambda)\psi = 0, \tag{3.21a}$$

$$\begin{aligned}(\partial_t + \mathscr{L}_2)\psi \equiv{}& \psi_t + 4f_2(t)\psi_{3x} + \left(6f_2(t)ue^{\int 6f(t)dt} + 3f_2(t)\lambda\right)\psi_x\\ &+ \left(3f_2(t)u_x e^{\int 6f(t)dt} - 3f_2(t)|g(t)|/\sqrt{3f_2(t)}\partial_x^{-1}u_y e^{\int 6f(t)dt} - \right)\psi = 0,\end{aligned} \tag{3.21b}$$

where u is a solution of the equation (1.1). The lax pair (3.21a) and (3.21b) is a new one, which is not obtained in Refs. [27].

(v). Suppose $h_i = h_i(t)$ ($i = 1, 2, 3, 5$), $h_j = 0$ ($j = 4, 6, 7$), Eq. (1.1) becomes a generalized variable coefficient KP equation [18, 19, 29]. The corresponding formula (3.1) reduces to

$$\begin{aligned}&(D_x^2 + \alpha D_y - \lambda) f \cdot g = 0,\\ &\left[D_t + h_1\left(D_x^3 - 3\alpha D_x D_y + 3\lambda D_x\right) + h_3 D_x + \gamma\right] f \cdot g = 0,\end{aligned} \tag{3.22}$$

which is also a new one and not obtained in Refs. [18, 19, 29]. The corresponding Lax pair (3.11a) and (3.11b) reduces to

$$(\mathscr{L}_1 + \alpha\partial_y)\psi \equiv \psi_{2x} + \sqrt{h_5/3h_1}\psi_y + (u - \lambda)\psi = 0, \tag{3.23a}$$

$$(\partial_t + \mathscr{L}_2)\psi \equiv \psi_t + 4h_1\psi_{3x} + (6h_1u + 3h_1\lambda + h_3)\psi_x + \left(3h_1u_x - 3h_1\sqrt{h_5/3h_1}\partial_x^{-1}u_y\right)\psi = 0, \tag{3.23b}$$

where u is a solution of the equation (1.1). The lax pair (3.23a) and (3.23b) is a new one, which is not obtained in Refs. [18, 19, 29].

Starting from Lax pairs and Darboux transformation, the soliton-like solutions of the generalized vc-KP equation (1.1) can be established.

4. Darboux covariant Lax pair

Theorem 4.1. *Using the associated Lax pair* (3.12a)-(3.12b) *and assuming that the parameter λ is independent of variables x, y and t, the generalized vc-KP equation* (1.1) *admits a kind of Darboux covariant Lax pair as follows*

$$(\widehat{\mathscr{L}_1}+\alpha\partial_y)\phi=\lambda\phi,\quad \widehat{\mathscr{L}_1}=\partial_x^2+\widehat{q}_{2x}, \tag{4.1a}$$

$$(\partial_t+\widehat{\mathscr{L}_{2,cov}})\phi=0,\quad \widehat{\mathscr{L}_{2,cov}}=4h_1\partial_x^3-h_4\alpha^{-1}\partial_x^2+(6h_1\widehat{q}_{2x}+h_3)\,\partial_x+3h_1\widehat{q}_{3x}-3h_1\alpha\widehat{q}_{xy}-h_4\alpha^{-1}\widehat{q}_{2x}, \tag{4.1b}$$

whose form is Darboux covariant, namely,

$$T(\mathscr{L}_1+\alpha\partial_y)(q)T^{-1}=(\widehat{\mathscr{L}_1}+\alpha\partial_y)(\widehat{q}), \tag{4.2a}$$

$$T(\partial_t+\mathscr{L}_{2,cov})(q)T^{-1}=(\partial_t+\widehat{\mathscr{L}_{2,cov}})(\widehat{q}), \tag{4.2b}$$

with $\widehat{q}=q+2\ln\phi$, under a certain gauge transformation

$$T=\phi\partial_x\phi^{-1}=\partial_x-\sigma,\quad \sigma=\partial_x\ln\phi. \tag{4.3}$$

The integrability condition of the Darboux covariant Lax pair (4.1a) *and* (4.1b) *precisely gives rise to Eq.* (1.1) *in Lax representation*

$$[\partial_t+\widehat{\mathscr{L}_{2,cov}},\widehat{\mathscr{L}_1}+\alpha\partial_y]=[\widehat{q}_{x,t}+h_1(\widehat{q}_{4x}+3\widehat{q}_{2x}^2)+h_3\widehat{q}_{2x}+h_4\widehat{q}_{xy}+h_5\widehat{q}_{2y}]_x=0, \tag{4.4}$$

if one chooses $\partial_y h_4=h_6+\partial_t\ln h_1h_2^{-1}$, $\partial_y h_1=h_7=0$. The equation (4.4) *is equivalent to equation* (2.5)*, which implies that Lax equations* (4.1a) *and* (4.1b) *is also a Lax pair for the generalized vc-KP equation* (1.1).

Proof. Let ϕ be a solution of the Lax pair (3.12a). The following transformation (4.3) change the operator $\mathscr{L}_1(q)+\alpha\partial_y-\lambda$ into a new one as follows

$$T(\mathscr{L}_1(q)+\alpha\partial_y-\lambda)T^{-1}=\widehat{\mathscr{L}_1}(\widehat{q})+\alpha\partial_y-\lambda, \tag{4.5}$$

which admitting the following form

$$\widehat{\mathscr{L}_1}(\widehat{q})=\mathscr{L}_1(\widehat{q}=q+\triangle q),\quad \text{with}\ \ \triangle q=2\ln\phi. \tag{4.6}$$

Using transformation (4.3), one should look for another one $\mathscr{L}_{2,\mathrm{cov}}(q)$, which satisfies the following form

$$\widehat{\mathscr{L}_{2,\mathrm{cov}}}(\widehat{q})=\mathscr{L}_{2,\mathrm{cov}}(\widehat{q}=q+\triangle q). \tag{4.7}$$

Let ϕ be a solution of the following system

$$(\mathscr{L}_1 + \alpha\partial_y)\phi = \lambda\phi, \quad \mathscr{L}_1 = \partial_x^2 + q_{2x}, \tag{4.8a}$$

$$(\partial_t + \mathscr{L}_{2,\mathrm{cov}})\phi = 0, \quad \mathscr{L}_{2,\mathrm{cov}} = 4h_1\partial_x^3 + b_1\partial_x^2 + b_2\partial_x + b_3, \tag{4.8b}$$

with b_i $(i = 1, 2, 3)$ are undetermined functions. To determine b_i $(i = 1, 2, 3)$, one can show that (4.3) change $\partial_t + \mathscr{L}_{2,\mathrm{cov}}$ into the following form

$$T(\partial_t + \mathscr{L}_{2,\mathrm{cov}})T^{-1} = \partial_t + \widehat{\mathscr{L}_{2,\mathrm{cov}}}, \quad \widehat{\mathscr{L}_{2,\mathrm{cov}}} = 4h_1\partial_x^3 + \widehat{b}_1\partial_x^2 + \widehat{b}_2\partial_x + \widehat{b}_3, \tag{4.9}$$

with $\widehat{b}_j$ $(j = 1, 2, 3)$ and $\widehat{\mathscr{L}_{2,\mathrm{cov}}}$ are determined by

$$\widehat{b}_j = b_j(q) + \triangle b_j = b_j(q + \triangle q), \quad j = 1, 2, 3. \tag{4.10}$$

Using (4.3) and (4.9), one has

$$\begin{aligned} &\triangle b_1 = 0, \quad \triangle b_2 = 12h_1\sigma_x + b_{1,x} + \sigma b_{1,x}, \\ &\triangle b_3 = 12h_1\sigma_{2x} + 12h_1\sigma\sigma_x + \sigma b_{1,x} + b_{2,x} + 2\sigma_x\widehat{b}_1. \end{aligned} \tag{4.11}$$

By virtue of (4.10), one should just express $\widehat{b}_i$ $i = 1, 2, 3$ in the following form

$$\widehat{b}_j = \mathscr{H}_j(q, q_x, q_y, q_{2x}, q_{xy}, q_{2y}, \cdots), \quad j = 1, 2, 3, \tag{4.12}$$

and satisfies

$$\triangle\mathscr{H}_j = \mathscr{H}_j(q + \triangle q, q_x + \triangle q_x, q_y + \triangle q_y, \cdots) - \mathscr{H}_j(q, q_x, q_y, \cdots) = \triangle b_j, \tag{4.13}$$

where $\triangle q_{n_1 x, n_2 y} = 2\partial_x^{n_1}\partial_y^{n_2} \ln q$, $n_1, n_2 = 1, 2, \ldots$, and $\triangle b_j$ can be solved by Eq. (4.11).

Direct calculation shows that

$$\widehat{b}_1 = c_1(y, t), \tag{4.14}$$

by using Eqs.(4.11)-(4.13), where $c_1(y, t)$ being an arbitrary function about y and t.

Using Eq.(4.13), one has

$$\triangle b_2 = \triangle\mathscr{H}_2 = \mathscr{H}_{2,q}\triangle q + \mathscr{H}_{2,q_x}\triangle q_x + \mathscr{H}_{2,q_y}\triangle q_y + \cdots = 12h_1\sigma_x = 6h_1\triangle q_{2x}. \tag{4.15}$$

It implies that we can determine $\widehat{b}_2$ up to an arbitrary constant $c_2(y, t)$, namely,

$$\widehat{b}_2 = \mathscr{H}_2(q_{2x}) = 6h_1 q_{2x} + c_2(y, t), \tag{4.16}$$

where $c_2(y, t)$ being an arbitrary function about y and t.

By means of Eq. (4.8a), one obtains

$$q_{3x} = -\alpha\sigma_{xy} - (\sigma_x + \sigma^2)_x. \tag{4.17}$$

Using Eqs.(4.14), (4.16) and (4.17) into Eq.(4.11), one has

$$\triangle b_3 = 6h_1\sigma_{2x} - 6h_1\alpha\sigma_{xy} + 2c_1\sigma_x = 3h_1\triangle q_{3x} - 3h_1\alpha\triangle q_{xy} + c_1\triangle q_{2x}, \tag{4.18}$$

which can be verified that the third condition

$$\triangle\mathscr{H}_3 = \mathscr{H}_{3,q}\triangle q + \mathscr{H}_{3,q_x}\triangle q_x + \mathscr{H}_{3,q_y}\triangle q_y + \cdots = \triangle b_3, \tag{4.19}$$

can be satisfied by choosing

$$\widehat{b_3} = \mathscr{H}_3(q, q_x, q_y, q_{2x}, q_{xy}, q_{2y}, q_{3x}, \cdots) = 3h_1q_{3x} - 3h_1\alpha q_{xy} + c_1(y,t)q_{2x} + c_3(y,t), \tag{4.20}$$

where $c_3(y,t)$ is an arbitrary function of y and t.

Taking $c_1(y,t) = -\alpha^{-1}h_4$, $c_2(y,t) = h_3$, $c_3(y,t) = 0$ in Eqs.(4.14), (4.16) and (4.20), we obtain the Darboux covariant evolution equation (4.1b) by using (4.8a), (4.8b).

Through a tedious calculations of the Lie bracket $[\partial_t + \widehat{\mathscr{L}_{2,\text{cov}}}, \widehat{\mathscr{L}_1} + \alpha\partial_y]$, one obtains the Eq.(4.4) by choosing $\partial_y h_4 = h_6 + \partial_t \ln h_1 h_2^{-1}$, $\partial_y h_1 = h_7 = 0$. □

From above, we can investigate the higher ones by using the same method

$$\widehat{\mathscr{L}_{n_0,\text{cov}}(q)} = 4h_1\partial_x^{n_0} + \widehat{b}_1\partial_x^{n_0-1} + \cdots + \widehat{b}_s, \quad s = 5, 6, 7, \cdots, \tag{4.21}$$

which can obtain other new ones of the Eq. (1.1).

5. Infinite conservation laws

In this section, we derive the infinite conservation laws for the generalized vc-KP equation (1.1) by using the binary Bell polynomials.

Theorem 5.1. *Under the conditions* (1.12)*, the generalized vc-KP equation* (1.1) *admits an infinite conservation laws*

$$\mathscr{I}_{n,t} + \mathscr{J}_{n,x} + \mathscr{G}_{n,y} = 0, \quad n = 1, 2, \ldots. \tag{5.1}$$

The conversed densities $\mathscr{I}_n's$ are obtained as follows

$$\begin{aligned}
\mathscr{I}_1 &= -\frac{1}{2}q_{2x} = -\frac{1}{2}e^{\int h_6 dt}u,\\
\mathscr{I}_2 &= \frac{1}{4}q_{3x} + \frac{1}{4}\alpha q_{xy} = \frac{1}{4}e^{\int h_6 dt}\left(\alpha\partial_x^{-1}u_y + u_{2x}\right),\\
\mathscr{I}_{n+1} &= -\frac{1}{2}\left(\mathscr{I}_{n,x} + \alpha\partial_x^{-1}\mathscr{I}_{n,y} + \sum_{i=1}^{n}\mathscr{I}_i\mathscr{I}_{n-i}\right), \quad n = 2, 3, \ldots,
\end{aligned} \tag{5.2}$$

and the first fluxes $\mathscr{J}_n's$ are obtained as follows

$$\begin{aligned}
\mathscr{J}_1 &= h_1\mathscr{I}_{1,2x} - 6h_1\alpha\partial_x^{-1}\mathscr{I}_{2,y} + h_3\mathscr{I}_1 - 6h_1\mathscr{I}_1^2,\\
\mathscr{J}_2 &= h_1\mathscr{I}_{2,2x} - 6h_1\alpha\mathscr{I}_1\partial_x^{-1}\mathscr{I}_{1,y} - 6h_1\alpha\partial_x^{-1}\mathscr{I}_{3,y} - 12h_1\mathscr{I}_1\mathscr{I}_2 + h_3\mathscr{I}_2,\\
\mathscr{J}_n &= h_1\left(\mathscr{I}_{n,2x} - 6\sum_{k=1}^{n}\mathscr{I}_k\mathscr{I}_{n+1-k} - 2\sum_{k_1+k_2+k_3=n}\mathscr{I}_{k_1}\mathscr{I}_{k_2}\mathscr{I}_{k_3}\right) - 6h_1\alpha\left(\partial_x^{-1}\mathscr{I}_{n+1,y} + \sum_{k=1}^{n}\mathscr{I}_k\partial_x^{-1}\mathscr{I}_{n-k,y}\right)\\
&\quad + h_3\mathscr{I}_n, \quad n = 3, 4, \ldots.
\end{aligned} \tag{5.3}$$

and the second fluxes $\mathscr{G}_n's$ are obtained as follows

$$\mathscr{G}_1 = 6h_1\alpha\mathscr{I}_2 + h_4\mathscr{I}_1 + h_5\partial_x^{-1}\mathscr{I}_{1,y},$$
$$\mathscr{G}_2 = 3h_1\alpha\mathscr{I}_1^2 + 6h_1\alpha\mathscr{I}_3 + h_4\mathscr{I}_2 + h_5\partial_x^{-1}\mathscr{I}_{2,y},$$
$$\mathscr{G}_n = 3h_1\alpha\sum_{k=1}^{n}\mathscr{I}_k\mathscr{I}_{n-k} + 6h_1\alpha\mathscr{I}_{n+1} + h_4\mathscr{I}_n + h_5\partial_x^{-1}\mathscr{I}_{n,y}, \ \ n = 2, 3, \ldots. \tag{5.4}$$

Proof. Changing (3.3) into the divergence form and using (3.5), one can rewrite $\mathscr{R}(\upsilon, \omega)$ into a new form

$$\mathscr{R}(\upsilon, \omega) = [(3h_1\lambda + h_3)\upsilon_x - 3h_1\alpha\upsilon_x\upsilon_y]_x + [-3h_1\alpha\omega_{2x} + h_4\upsilon_x]_y. \tag{5.5}$$

which is equivalent to the following form

$$\omega_{2x} + \upsilon_x^2 + \alpha\upsilon_y - \lambda = 0,$$
$$\partial_t[\upsilon_x] + \partial_x\left[h_1\upsilon_{3x} + 3h_1\upsilon_x\omega_{2x} + h_1\upsilon_x^3 + (3h_1\lambda + h_3)\,\upsilon_x - 3h_1\alpha\upsilon_x\upsilon_y\right]$$
$$+ \partial_y\left[3h_1\alpha\upsilon_x^2 + h_4\upsilon_x + h_5\upsilon_y - 3h_1\alpha\lambda\right] = 0, \tag{5.6}$$

by using the fact $\partial_x(\upsilon_t) = \partial_t(\upsilon_x) = \upsilon_{xt}$.

Using the relationship (3.4) and the following new function

$$\eta = (q_x' - q_x)/2, \tag{5.7}$$

one obtains

$$\upsilon_x = \eta, \ \ \omega_x = q_x + \eta. \tag{5.8}$$

By using (5.8) into (5.6), Eq. (3.5) can be changed into a Riccati-type equation

$$q_{2x} + \eta_x + \eta^2 + \alpha\partial_x^{-1}\eta_y - \varepsilon^2 = 0, \tag{5.9}$$

which is a new potential function about q, and a divergence-type equation

$$\eta_t + \partial_x\left[h_1\left(\eta_{2x} - 2\eta^3 - 6\alpha\eta\partial_x^{-1}\eta_y + 6\varepsilon^2\eta\right) + h_3\eta\right] + \partial_y\left[3h_1\alpha\eta^2 + h_4\eta + h_5\partial_x^{-1}\eta_y - 3h_1\alpha\varepsilon^2\right] = 0, \tag{5.10}$$

in which one can obtain Eq. (5.10) by virtue of the equation (5.9) and take $\lambda = \varepsilon^2$.

Introducing the following series

$$\eta = \varepsilon + \sum_{n=1}^{\infty}\mathscr{I}_n(q, q_x, q_{2x}, \cdots)\varepsilon^{-n}, \tag{5.11}$$

into Eq. (5.9) and collecting the coefficients of ε, one can get the formulas (5.2) for $\mathscr{I}_n$.

In addition, substituting the expression (5.11) into Eq. (5.10), one obtains

$$\sum_{n=1}^{\infty}\mathscr{I}_{n,t}\varepsilon^{-n} + \partial_x\Bigg\{h_1\left[\sum_{n=1}^{\infty}\mathscr{I}_{n,2x}\varepsilon^{-n} - 2\left(\sum_{n=1}^{\infty}\mathscr{I}_n\varepsilon^{-n}\right)^3 - 6\varepsilon\left(\sum_{n=1}^{\infty}\mathscr{I}_n\varepsilon^{-n}\right)^2 + 4\varepsilon^3\right] + h_3\left(\sum_{n=1}^{\infty}\mathscr{I}_n\varepsilon^{-n} + \varepsilon\right)$$
$$-6h_1\alpha\left[\left(\sum_{n=1}^{\infty}\mathscr{I}_n\varepsilon^{-n}\right)\left(\partial_x^{-1}\sum_{n=1}^{\infty}\mathscr{I}_{n,y}\varepsilon^{-n}\right)\right] - 6h_1\alpha\varepsilon\partial_x^{-1}\sum_{n=1}^{\infty}\mathscr{I}_{n,y}\varepsilon^{-n}\Bigg\}$$
$$+ \partial_y\left\{3h_1\alpha\left[\left(\sum_{n=1}^{\infty}\mathscr{I}_n\varepsilon^{-n}\right)^2 + 2\varepsilon\sum_{n=1}^{\infty}\mathscr{I}_n\varepsilon^{-n}\right] + h_4\left(\sum_{n=1}^{\infty}\mathscr{I}_n\varepsilon^{-n} + \varepsilon\right) + h_5\left(\partial_x^{-1}\sum_{n=1}^{\infty}\mathscr{I}_{n,y}\varepsilon^{-n} + \varepsilon x\right)\right\}$$
$$= 0, \tag{5.12}$$

from which one can obtain the infinite conservation laws (5.1)

$$\mathscr{I}_{n,t} + \mathscr{J}_{n,x} + \mathscr{G}_{n,y} = 0, \;\; n = 1, 2, \ldots .$$

In Eq. (5.1), the conversed densities $\mathscr{I}_n' s$ are obtained by recursion formulas (5.2), and the first fluxes $\mathscr{J}_n' s$ and the second fluxes $\mathscr{G}_n' s$, respectively, are obtained by (5.3) and (5.4) through a cumbersome calculation. □

From above, one concludes that the first fluxes $\mathscr{J}_n' s$ (5.3) and the second fluxes $\mathscr{G}_n' s$ (5.4) can be introduced from u, and the formula $\mathscr{I}_{n,t} + \mathscr{J}_{n,x} + \mathscr{G}_{n,y} = 0, (n = 1, 2, \ldots)$ implies that infinite conserved densities of the generalized vc-KP equation (1.1) can be obtained by using $\{\mathscr{I}_n, n = 1, 2, \ldots, \}$. Using Eqs. (5.2), (5.3) and (5.4), one can easily obtain $\mathscr{I}_n$, $\mathscr{J}_n$ and $\mathscr{G}_n$. And the generalized vc-KP equation (1.1) can be expressed in the form of the first equation for conservation law (5.1).

6. Soliton solution and Riemann theta function periodic wave solution

Under the conditions (1.12) and $c_0 = 6$, we can discuss the solutions of the generalized vc-KP equation (1.1) by using the bilinear form (2.2). The following subsections are independent to each other, and the parameters are also independent.

6.1 Soliton solution

Theorem 6.1. *Assuming δ=0, under the conditions* (1.12) *and $c_0 = 6$, the generalized vc-KP equation* (1.1) *admits a N-soliton solution as follows*

$$\begin{aligned} &u = 12h_1h_2^{-1}(\ln f)_{xx}, \\ &f = \sum_{\rho=0,1} \exp\left(\sum_{j=1}^{N} \rho_j \eta_j + \sum_{1\le j<i\le N}^{N} \rho_i\rho_j A_{ij} \right), \end{aligned} \tag{6.1}$$

where $\eta_j = \mu_j x + \nu_j y - (h_1\mu_j^3 + h_3\mu_j + h_4\nu_j + h_5\mu_j^{-1}\nu_j^2)t + c_j$ and $\exp(A_{ij}) = \frac{3h_1\mu_i^2\mu_j^2(\mu_i-\mu_j)^2 - h_5(\mu_i\nu_j-\mu_j\nu_i)^2}{3h_1\mu_i^2\mu_j^2(\mu_i+\mu_j)^2 - h_5(\mu_i\nu_j-\mu_j\nu_i)^2}$ $(1 \le j < i \le N)$, while μ_j, ν_j are the parameters characterizing the j-th soliton, $\sum_{1\le j<i\le N}^{N}$ is the summation over all possible pairs chosen from N elements under the condition $1 \le j < i \le N$, and $\sum_{\rho=0,1}$ denotes the summation over all possible combinations of ρ_i, $\rho_j = 0, 1$ $(i, j = 1, 2, \ldots, N)$.

Proof. Substituting (6.1) into the bilinear form (2.2) yields

$$\begin{aligned} \sum_{\rho=0,1}\sum_{\rho'=0,1} \mathscr{D}\Bigg(&-\sum_{j=1}^{N}(\rho_j - \rho_j')(h_1\mu_j^3 + h_3\mu_j + h_4\nu_j + h_5\mu_j^{-1}\nu_j), \sum_{j=1}^{N}(\rho_j - \rho_j')\mu_j, \sum_{j=1}^{N}(\rho_j - \rho_j')\nu_j \Bigg) \\ &\times \exp\left(\sum_{j=1}^{N}(\rho_j + \rho_j')\eta_j + \sum_{1\le j<i\le N}^{N}(\rho_i\rho_j + \rho_i'\rho_j')A_{ij} \right) = 0, \end{aligned} \tag{6.2}$$

in which the bilinear operator $\mathscr{D}$ is given by Eq.(2.2) with $\delta = 0$. Let the coefficient of the factor

$$\exp\left(\sum_{j=1}^{m} \eta_j + 2\sum_{j=m+1}^{n} \eta_j \right), \tag{6.3}$$

on the left hand of (6.2) be $\mathscr{F}$, it follows that

$$\mathscr{F} = \sum_{\rho=0,1}\sum_{\rho'=0,1}\mathscr{C}(\rho,\rho')\mathscr{D}\left(-\sum_{j=1}^{N}(\rho_j-\rho'_j)(h_1\mu_j^3+h_3\mu_j+h_4\nu_j+h_5\mu_j^{-1}\nu_j),\sum_{j=1}^{N}(\rho_j-\rho'_j)\mu_j,\sum_{j=1}^{N}(\rho_j-\rho'_j)\nu_j\right)$$
$$\times\exp\left(\sum_{1\le j<i\le N}^{N}(\rho_i\rho_j+\rho'_i\rho'_j)A_{ij}\right)=0, \tag{6.4}$$

where the coefficient $\mathscr{C}(\rho,\rho')$ denotes that the summations over ρ and ρ' performed under the following conditions

$$\rho_j=\begin{cases}1-\rho'_j, & \text{if } 1\le j\le m,\\ \rho'_j=1, & \text{if } m+1\le j\le n,\\ \rho'_j=0, & \text{if } n+1\le j\le N.\end{cases} \tag{6.5}$$

By introducing a new variable

$$\varpi_j=\rho_j-\rho'_j, \tag{6.6}$$

one obtains the following equality

$$\exp\left(\sum_{1\le j<i\le N}^{N}(\rho_i\rho_j+\rho'_i\rho'_j)A_{ij}\right)=\sum_{1\le j<i\le N}^{m}\frac{1}{2}(1+\varpi_i\varpi_j)A_{ij}+\sum_{i=1}^{m}\sum_{j=m+1}^{n}A_{ij}+\sum_{1\le j<i\le N}^{n}\sum_{j=m+1}^{n}A_{ij}. \tag{6.7}$$

On account of ϖ_i, $\varpi_j=\pm1$ and the relations

$$\mathscr{D}\left(h_1\mu_j^3+h_3\mu_j+h_4\nu_j+h_5\mu_j^{-1}\nu_j,\mu_j,\nu_j\right)=\mathscr{D}\left(-h_1\mu_j^3-h_3\mu_j-h_4\nu_j-h_5\mu_j^{-1}\nu_j,-\mu_j,-\nu_j\right),$$
$$\exp(A_{ij})=-\frac{\mathscr{D}\left(h_1(\mu_i^3-\mu_j^3)+h_3(\mu_i-\mu_j)+h_4(\nu_i-\nu_j)+h_5(\mu_i^{-1}\nu_i-\mu_j^{-1}\nu_j),\mu_j-\mu_i,\nu_j-\nu_i\right)}{\mathscr{D}\left(-h_1(\mu_i^3+\mu_j^3)-h_3(\mu_i+\mu_j)-h_4(\nu_i+\nu_j)-h_5(\mu_i^{-1}\nu_i+\mu_j^{-1}\nu_j),\mu_i+\mu_j,\nu_i+\nu_j\right)}, \tag{6.8}$$

one obtains

$$\sum_{1\le j<i\le N}^{m}\frac{1}{2}(1+\varpi_i\varpi_j)A_{ij}=-\frac{\mathscr{D}\left(h_1(\mu_i^3-\mu_j^3)+h_3(\mu_i-\mu_j)+h_4(\nu_i-\nu_j)+h_5(\mu_i^{-1}\nu_i-\mu_j^{-1}\nu_j),\mu_j-\mu_i,\nu_j-\nu_i\right)}{\mathscr{D}\left(-h_1(\mu_i^3+\mu_j^3)-h_3(\mu_i+\mu_j)-h_4(\nu_i+\nu_j)-h_5(\mu_i^{-1}\nu_i+\mu_j^{-1}\nu_j),\mu_i+\mu_j,\nu_i+\nu_j\right)}\varpi_i\varpi_j. \tag{6.9}$$

Substituting Eqs.(6.6)-(6.9) into Eq.(6.4) yields

$$\mathscr{F}=\mathscr{A}\sum_{\nu=\pm1}\mathscr{D}\left(-\sum_{j=1}^{N}\varpi_j(h_1\mu_j^3+h_3\mu_j+h_4\nu_j+h_5\mu_j^{-1}\nu_j),\sum_{j=1}^{N}\varpi_j\mu_j,\sum_{j=1}^{N}\varpi_j\nu_j\right)$$
$$\times\prod_{j<i}^{N}\mathscr{D}\left(h_1(\mu_i^3-\mu_j^3)+h_3(\mu_i-\mu_j)+h_4(\nu_i-\nu_j)+h_5(\mu_i^{-1}\nu_i-\mu_j^{-1}\nu_j),\mu_j-\mu_i,\nu_j-\nu_i\right)\varpi_i\varpi_j=0, \tag{6.10}$$

where $\mathscr{A}=\mathscr{A}(\exp(A_{ij}))$ is independent of the summation indices ϖ_i $(i=1,2,\ldots,N)$. If we can verify the identity (6.10) for $\mathscr{A}\equiv1$, $N=1,2,\ldots$, then (6.1) is the solution of Eq. (1.1). Using the bilinear form (2.2), one can rewrite (6.10) as follows

$$\widehat{\mathscr{F}}_N(\mu_1,\nu_1,\mu_2,\nu_2,\ldots,\mu_N,\nu_N)$$
$$\equiv\mathscr{A}\sum_{\varpi=\pm1}\left\{-\sum_{i,j=1}^{N}\varpi_i\varpi_j(h_1\mu_i^3+h_3\mu_i+h_4\nu_i+h_5\mu_i^{-1}\nu_i)\mu_j+h_1\left(\sum_{j=1}^{N}\varpi_j\nu_j\right)^4+h_3\left(\sum_{j=1}^{N}\varpi_j\mu_j\right)^2\right.$$
$$\left.+h_4\sum_{j=1}^{N}\varpi_i\varpi_j\mu_i\nu_j+h_5\left(\sum_{j=1}^{N}\varpi_j\nu_j\right)^2\right\}\prod_{j<i}^{N}\left[3h_1\mu_i^2\mu_j^2(\varpi_i\mu_i-\varpi_j\mu_j)^2-h_5(\mu_i\nu_j-\mu_j\nu_i)^2\right]=0. \tag{6.11}$$

$\widehat{\mathscr{F}}_N(\mu_1, \nu_1, \mu_2, \nu_2, \ldots, \mu_N, \nu_N)$ is a symmetric and homogeneous polynomial, and is also an even function of μ_j, ν_j $(j = 1, 2, \ldots, N)$. Suppose $(\mu_1, \nu_1) = (\pm\mu_2, \pm\nu_2)$, then we have the following relationship

$$\widehat{\mathscr{F}}_N(\mu_1, \nu_1, \ldots, \mu_N, \nu_N) = 8(3h_1\mu_1^6 - h_5\mu_1^2\nu_1^2)\prod_{j=3}^{N}\left[3h_1\mu_1^4\mu_j^4(\mu_1^2-\mu_j^2)^4 + h_5(\mu_1^2\nu_j^2 - \mu_j^2\nu_1^2)^4\right]^2 \widehat{\mathscr{F}}_{N-2}(\mu_3, \nu_3, \ldots, \mu_N, \nu_N). \tag{6.12}$$

For $\mathscr{A} \equiv 1$, $n = 1, 2$, the identity (6.11) is easily verified. Let's assume that the identity hold for $N-2$, utilizing the relationship (6.12), it is seen that $\widehat{\mathscr{F}}_N(\mu_1, \mu_2, \ldots, \mu_N)$ can be the factor by a symmetric homogeneous polynomial as follows

$$\widehat{\mathscr{F}}_N(\mu_1, \nu_1, \ldots, \mu_N, \nu_N) = \prod_{i=1}^{N}(3h_1\mu_i^6 - h_5\mu_i^2\nu_i^2)\prod_{j<i}^{N}\left[3h_1\mu_i^4\mu_j^4(\mu_i^2-\mu_j^2)^4 + h_5(\mu_i^2\nu_j^2 - \mu_j^2\nu_i^2)^4\right]^2 \widetilde{\mathscr{F}}_N(\mu_1, \nu_1, \ldots, \mu_N, \nu_N). \tag{6.13}$$

According to the degrees of Eqs.(6.11) and (6.13), $\widehat{\mathscr{F}}_N(\mu_1, \nu_1, \ldots, \mu_N, \nu_N)$ must be zero for $\mathscr{A} \equiv 1$, $n \geq 2$, and the identity is proved. Hence, the expression (6.1) is the N-soliton solution of the generalized vc-KP equation (1.1). □

Based on the Theorem 6.1, one can easily obtain the following corollary.

Corollary 6.2. *For the case $N = 1$, the one-soliton solution of the generalized vc-KP equation* (1.1) *can be written as follows:*

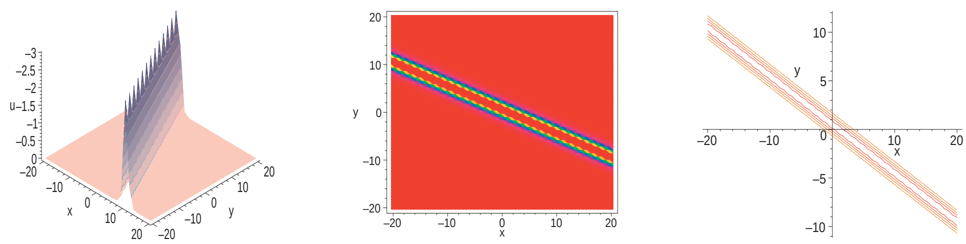
$$u = 12h_1h_2^{-1}\left[\ln(1 + e^{\eta})\right]_{xx}, \tag{6.14}$$

where $\eta = \mu x + \nu y - (h_1\mu^3 + h_3\mu + h_4\nu + h_5\mu^{-1}\nu^2)t + c$. For the case $N = 2$, the following expression

$$u = 12h_1h_2^{-1}\left[\ln(1 + e^{\eta_1} + e^{\eta_2} + e^{\eta_1+\eta_2+A_{12}})\right]_{xx}, \tag{6.15}$$

with $\eta_i = \mu_i x + \nu_i y - (h_1\mu_i^3 + h_3\mu_i + h_4\nu_i + h_5\mu_i^{-1}\nu_i^2)t + c_i$ $i = 1, 2$, $e^{A_{12}} = \frac{3h_1\mu_1^2\mu_2^2(\mu_1-\mu_2)^2 - h_5(\mu_1\nu_2-\mu_2\nu_1)^2}{3h_1\mu_1^2\mu_2^2(\mu_1+\mu_2)^2 - h_5(\mu_1\nu_2-\mu_2\nu_1)^2}$, describes the two-soliton solution for equation (1.1).

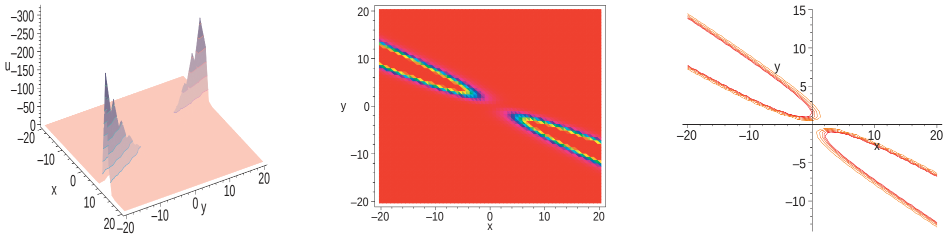
Based on the soliton solutions obtained by the Hirota's method, we present some figures to describe the propagation situations of the solitary waves. Figures 1 and 2 show the pulse propagation of the fundamental soliton along the distance (x, y)-surface with suitable choice of the parameters in Eq.(6.14). In Figures 3 and 4, we choose the same value of μ_1 and μ_2 but different ν_1 and ν_2. In this case, the phases of the two solitons are the same and two sets of parallel solitons are obtained via Eq.(6.15).



(a) (b) (c)

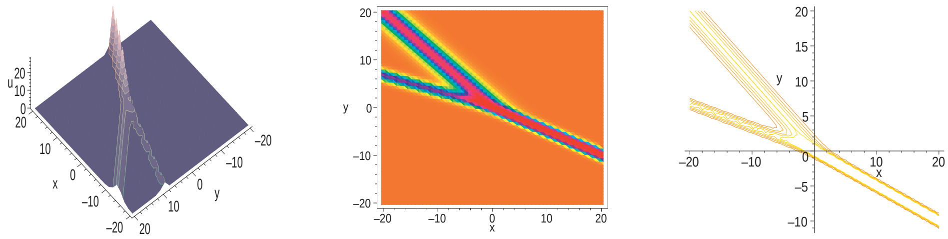
Fig. 1. (Color online) Propagation of the solitary wave for the generalized vc-KP equation (1.1) via expression (6.14) with parameters: h_1=1, h_2=-sech$^2(t)$, $h_3 = -1$, $h_4 = 1$, $h_5 = 2$, $\mu = 1$, $\nu = 2$ and $c = -1$. (a) Perspective view of the wave.

(*b*) Overhead view of the wave. (*c*) The corresponding contour plot.



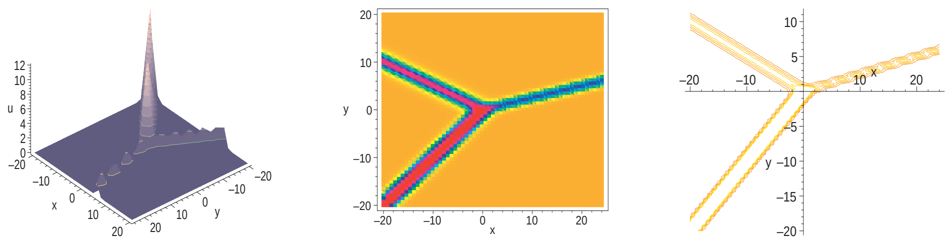
(*a*) (*b*) (*c*)

Fig. 2. (Color online) Propagation of the solitary wave for the generalized vc-KP equation (1.1) via expression (6.14) with parameters: $h_1 = y^2$, h_2=-sech$^2(t)$, $h_3 = t$, $h_4 = y$, $h_5 = 2$, $\mu = 1$, $\nu = 2$ and $c = -1$. (*a*) Perspective view of the wave. (*b*) Overhead view of the wave. (*c*) The corresponding contour plot.



(*a*) (*b*) (*c*)

Fig. 3. (Color online) Evolution plots of the two solitary waves for the generalized vc-KP equation (1.1) via expression (6.15) with parameters: $h_1 = 1$, h_2=sech$^2(t)$, $h_3 = 1$, $h_4 = -t$, $h_5 = t$, $\mu_1 = 1$, $\nu_1 = 3$, $\mu_2 = 2$, $\nu_2 = 4$ and $c_1 = c_2 = 0$. (*a*) Perspective view of the wave. (*b*) Overhead view of the wave. (*c*) The corresponding contour plot.



(*a*) (*b*) (*c*)

Fig. 4. (Color online) Evolution plots of the two solitary waves for the generalized vc-KP equation (1.1) via expression (6.15) with parameters: $h_1 = 1$, h_2=sech$^2(t)$, $h_3 = 1$, $h_4 = -1$, $h_5 = t$, $\mu_1 = 1$, $\nu_1 = 2$, $\mu_2 = 2$, $\nu_2 = -2$ and $c_1 = c_2 = 0$. (*a*) Perspective view of the wave. (*b*) Overhead view of the wave. (*c*) The corresponding contour plot.

6.2 Riemann theta function periodic wave solution

Using a multidimensional Riemann theta function, in Refs.[51, 52] we proposed two key theorems to systematically construct Riemann theta function periodic wave solutions for nonlinear equations and discrete soliton equations, respectively. Using the results in Ref.[51], we can directly obtain some periodic wave solutions for the generalized vc-KP equation (1.1) (see details in Appendix: B).

Considering the conditions (1.12), we consider the following bilinear form when δ is nonzero constant in Eq.(2.2)

$$\mathscr{L}(D_x, D_y, D_t)f\cdot f \equiv \left(D_xD_t + h_1D_x^4 + h_3D_x^2 + h_4D_xD_y + h_5D_y^2 - \delta\right)f\cdot f = 0. \tag{6.16}$$

Let now consider the Riemann theta function

$$\vartheta(\boldsymbol{\xi}) = \vartheta(\boldsymbol{\xi},\boldsymbol{\tau}) = \sum_{\boldsymbol{n}\in\mathbb{Z}^N} e^{\pi i\langle \boldsymbol{n}\boldsymbol{\tau},\boldsymbol{n}\rangle + 2\pi i\langle\boldsymbol{\xi},\boldsymbol{n}\rangle}, \tag{6.17}$$

where the integer value vector $\boldsymbol{n} = (n_1, n_2, \ldots, n_N)^T \in \mathbb{Z}^N$, complex phase variables $\boldsymbol{\xi} = (\xi_1, \xi_2, \ldots, \xi_N)^T \in \mathbb{Z}^N$, and $-i\boldsymbol{\tau}$ is a positive definite and real-valued symmetric $N\times N$ matrix.

Theorem 6.3. *Assuming that $\vartheta(\xi,\tau)$ is a Riemann theta function for $N = 1$ with $\xi = kx + ly + \omega t + \varepsilon$, the generalized vc-KP equation* (1.1) *admits a one-periodic wave solution as follows*

$$u = 12h_1h_2^{-1}\partial_x^2 \ln\vartheta(\xi,\tau), \tag{6.18}$$

where

$$\omega = \frac{b_1a_{22} - b_2a_{12}}{a_{11}a_{22} - a_{12}a_{21}},\quad \delta = \frac{b_2a_{11} - b_1a_{21}}{a_{11}a_{22} - a_{12}a_{21}}, \tag{6.19}$$

with

$$\begin{aligned}
&\wp = e^{\pi i\tau},\quad a_{11} = \sum_{n=-\infty}^{+\infty} 16n^2\pi^2k\wp^{2n^2},\quad a_{12} = \sum_{n=-\infty}^{+\infty}\wp^{2n^2},\quad a_{21} = \sum_{n=-\infty}^{+\infty} 4\pi^2(2n-1)^2k\wp^{2n^2-2n+1},\\
&a_{22} = \sum_{n=-\infty}^{+\infty}\wp^{2n^2-2n+1},\quad b_1 = \sum_{n=-\infty}^{+\infty}\left(256h_1n^4\pi^4k^4 - 16h_3n^2\pi^2k^2 - 16h_4n^2\pi^2kl - 16h_5n^2\pi^2l^2\right)\wp^{2n^2},\\
&b_2 = \sum_{n=-\infty}^{+\infty}\left(16h_1\pi^4(2n-1)^4k^4 - 4h_3\pi^2(2n-1)^2k^2 - 4h_4\pi^2(2n-1)^2kl - 4h_5\pi^2(2n-1)^2l^2\right)\wp^{2n^2-2n+1},
\end{aligned} \tag{6.20}$$

and the other parameters k, l, τ and ε are free.

Proof. In order to obtain one-periodic wave solutions of Eq. (1.1), we consider one-Riemann theta function $\vartheta(\xi,\tau)$ as $N = 1$

$$\vartheta(\xi,\tau) = \sum_{n=-\infty}^{+\infty} e^{\pi in^2\tau + 2\pi in\xi}, \tag{6.21}$$

where the phase variable $\xi = kx + ly + \omega t + \varepsilon$ and the parameter $\mathrm{Im}\tau > 0$. According to the Theorem A in Appendix (see details in Ref.[51]), k, l, ω and ε satisfy the following system

$$\sum_{n=-\infty}^{+\infty}\mathscr{L}(4n\pi ik, 4n\pi il, 4n\pi i\omega)e^{2n^2\pi i\tau} = 0, \tag{6.22a}$$

$$\sum_{n=-\infty}^{+\infty}\mathscr{L}(2\pi i(2n-1)k, 2\pi i(2n-1)l, 2\pi i(2n-1)\omega)e^{(2n^2-2n+1)\pi i\tau} = 0. \tag{6.22b}$$

Substituting the bilinear form $\mathscr{L}$ (6.16) into system (6.22a), (6.22b) yields

$$\sum_{n=-\infty}^{+\infty}\left(16n^2\pi^2k\omega - 256h_1n^4\pi^4k^4 + 16h_3n^2\pi^2k^2 + 16h_4n^2\pi^2kl + 16h_5n^2\pi^2l^2 + \delta\right)e^{2n^2\pi i\tau} = 0, \tag{6.23a}$$

$$\sum_{n=-\infty}^{+\infty}\left(4\pi^2(2n-1)^2k\omega - 16h_1\pi^4(2n-1)^4k^4 + 4h_3\pi^2(2n-1)^2k^2 + 4h_4\pi^2(2n-1)^2kl + 4h_5\pi^2(2n-1)^2l^2 + \delta\right)e^{(2n^2-2n+1)\pi i\tau} = 0. \tag{6.23b}$$

The notations are the same as the system (6.20), the system (6.23a), (6.23b) is simplified into a linear system for the frequency ω and the integration constant δ, namely,

$$\begin{pmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}\begin{pmatrix}\omega\\ \delta\end{pmatrix} = \begin{pmatrix} b_1\\ b_2\end{pmatrix}. \tag{6.24}$$

Now solving this system, we get a one-periodic wave solution of Eq. (1.1)

$$u = 12h_1h_2^{-1}\partial_x^2 \ln\vartheta(\xi,\tau),$$

which provided the vector $(\omega,\delta)^T$. It solves the system (6.24) with the theta function $\vartheta(\xi,\tau)$ given by Eq.(6.21). The other parameters k, l, τ and ε are free. □

Theorem 6.4. *Assuming that $\vartheta(\xi_1,\xi_2,\boldsymbol{\tau})$ is a Riemann theta function for $N=2$ with $\xi_i = k_ix + l_iy + \omega_it + \varepsilon_i$ $(i=1,2)$, the generalized vc-KP equation* (1.1) *admits a two-periodic wave solution as follows*

$$u = u_0 + 12h_1h_2^{-1}\partial_x^2 \ln\vartheta(\xi_1,\xi_2,\boldsymbol{\tau}), \tag{6.25}$$

where the parameters ω_1, ω_2, u_0 and δ satisfy the linear system

$$\boldsymbol{H}(\omega_1,\omega_2,u_0,\delta)^T = \boldsymbol{b}, \tag{6.26}$$

with

$$\begin{aligned}
&\boldsymbol{H} = (h_{ij})_{4\times4},\ \ \boldsymbol{b} = (b_1,b_2,b_3,b_4)^T,\ \ h_{i1} = \sum_{(n_1,n_2)\in\mathbb{Z}^2} 4\pi^2\langle 2\boldsymbol{n}-\boldsymbol{\theta_i},\boldsymbol{k}\rangle(2n_1-\theta_i^1)\mathfrak{I}_i(\boldsymbol{n}),\\
&h_{i2} = \sum_{(n_1,n_2)\in\mathbb{Z}^2} 4\pi^2\langle 2\boldsymbol{n}-\boldsymbol{\theta_i},\boldsymbol{k}\rangle(2n_2-\theta_i^2)\mathfrak{I}_i(\boldsymbol{n}),\ \ h_{i3} = -\sum_{(n_1,n_2)\in\mathbb{Z}^2} 16h_1\pi^4\langle 2\boldsymbol{n}-\boldsymbol{\theta_i},\boldsymbol{k}\rangle^4\mathfrak{I}_i(\boldsymbol{n}),\\
&h_{i4} = \sum_{(n_1,n_2)\in\mathbb{Z}^2} \mathfrak{I}_i(\boldsymbol{n}),\ \ b_i = \sum_{(n_1,n_2)\in\mathbb{Z}^2}\Big[16h_1\pi^4\langle 2\boldsymbol{n}-\boldsymbol{\theta_i},\boldsymbol{k}\rangle^4 - 4h_3\pi^2\langle 2\boldsymbol{n}-\boldsymbol{\theta_i},\boldsymbol{k}\rangle^2 - 4h_4\pi^2\langle 2\boldsymbol{n}-\boldsymbol{\theta_i},\boldsymbol{k}\rangle\langle 2\boldsymbol{n}-\boldsymbol{\theta_i},\boldsymbol{l}\rangle\\
&\qquad - 4h_5\pi^2\langle 2\boldsymbol{n}-\boldsymbol{\theta_i},\boldsymbol{l}\rangle^2\Big]\mathfrak{I}_i(\boldsymbol{n}),\\
&\mathfrak{I}_i(\boldsymbol{n}) = \wp_1^{n_1^2+(n_1-\theta_i^1)^2}\wp_2^{n_2^2+(n_2-\theta_i^2)^2}\wp_3^{n_1n_2+(n_1-\theta_i^1)(n_2-\theta_i^2)},\ \ \wp_1 = e^{\pi i\tau_{11}},\ \ \wp_2 = e^{\pi i\tau_{22}},\ \ \wp_1 = e^{2\pi i\tau_{12}},\ \ i=1,2,3,4.
\end{aligned} \tag{6.27}$$

and $\boldsymbol{\theta_i} = (\theta_i^1,\theta_i^2)^T$, $\boldsymbol{\theta_1} = (0,0)^T$, $\boldsymbol{\theta_2} = (1,0)^T$, $\boldsymbol{\theta_3} = (0,1)^T$, $\boldsymbol{\theta_4} = (1,1)^T$, $i=1,2,3,4$, the other parameters k_i, l_i, τ_{ij} and ε_i $(i,j=1,2)$ are free.

Proof. To obtain two-periodic wave solutions of Eq. (1.1), we consider two-Riemann theta function $\vartheta(\xi_1,\xi_2,\tau)$ as $N=2$

$$\vartheta(\xi_1,\xi_2,\tau) = \sum_{\boldsymbol{n}\in\mathbb{Z}^2} e^{\pi i\langle\boldsymbol{\tau n},\boldsymbol{n}\rangle + 2\pi i\langle\boldsymbol{\xi},\boldsymbol{n}\rangle}, \tag{6.28}$$

where the phase variable $\boldsymbol{\xi} = (\xi_1,\xi_2)^T \in \mathbb{C}^2$, $\xi_i = k_ix + l_iy + \omega_it + \varepsilon_i$, $i=1,2$, $\boldsymbol{n} = (n_1,n_2)^T \in \mathbb{Z}^2$, and $-i\boldsymbol{\tau}$ is a positive definite and real-valued symmetric 2×2 matrix which can take the form

$$\begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{pmatrix},\ \ \mathrm{Im}(\tau_{11}) > 0,\ \ \mathrm{Im}(\tau_{22}) > 0,\ \ \tau_{11}\tau_{22} - \tau_{12} < 0. \tag{6.29}$$

By considering a variable transformation

$$u = u_0 + 12h_1h_2^{-1}\partial_x^2 \ln\vartheta(\xi_1,\xi_2,\tau), \tag{6.30}$$

and integrating with respect to x, the $\mathscr{L}$ becomes the following bilinear form

$$\widehat{\mathscr{L}}(D_x,D_y,D_t)f\cdot f \equiv \left(D_xD_t + h_1D_x^4 + h_1u_0D_x^4 + h_3D_x^2 + h_4D_xD_y + h_5D_y^2 - \delta\right)f\cdot f = 0. \tag{6.31}$$

According to the Theorem B in Appendix (see details in Ref.[51]), k_i, ω_i and ε_i $(i=1,2)$ satisfy the following system

$$\sum_{\boldsymbol{n}\in\mathbb{Z}^2}\widehat{\mathscr{L}}(2\pi i\langle 2\boldsymbol{n}-\boldsymbol{\theta_i},\boldsymbol{k}\rangle, 2\pi i\langle 2\boldsymbol{n}-\boldsymbol{\theta_i},\boldsymbol{l}\rangle, 2\pi i\langle 2\boldsymbol{n}-\boldsymbol{\theta_i},\boldsymbol{\omega}\rangle)\, e^{\pi i[\langle\boldsymbol{\tau}(\boldsymbol{n}-\boldsymbol{\theta_i}),\boldsymbol{n}-\boldsymbol{\theta_i}\rangle + \langle\boldsymbol{\tau n},\boldsymbol{n}\rangle]} = 0, \tag{6.32}$$

where $\boldsymbol{\theta_i} = (\theta_i^1, \theta_i^2)^T$, $\boldsymbol{\theta_1} = (0,0)^T$, $\boldsymbol{\theta_2} = (1,0)^T$, $\boldsymbol{\theta_3} = (0,1)^T$, $\boldsymbol{\theta_4} = (1,1)^T$, $i = 1,2,3,4$.

Substituting the bilinear form $\widehat{\mathscr{L}}$ (6.31) into system (6.32) yields

$$\sum_{\boldsymbol{n}\in\mathbb{Z}^2} \Big[4\pi^2\langle 2\boldsymbol{n}-\boldsymbol{\theta_i},\boldsymbol{k}\rangle\langle 2\boldsymbol{n}-\boldsymbol{\theta_i},\boldsymbol{\omega}\rangle - 16h_1\pi^4\langle 2\boldsymbol{n}-\boldsymbol{\theta_i},\boldsymbol{k}\rangle^4 - 16h_1u_0\pi^4\langle 2\boldsymbol{n}-\boldsymbol{\theta_i},\boldsymbol{k}\rangle^4 + 4h_3\pi^2\langle 2\boldsymbol{n}-\boldsymbol{\theta_i},\boldsymbol{k}\rangle^2 + 4h_4\pi^2\langle 2\boldsymbol{n}-\boldsymbol{\theta_i},\boldsymbol{k}\rangle\langle 2\boldsymbol{n}-\boldsymbol{\theta_i},\boldsymbol{l}\rangle + 4h_5\pi^2\langle 2\boldsymbol{n}-\boldsymbol{\theta_i},\boldsymbol{l}\rangle^2 + \delta\Big] e^{\pi i[\langle\boldsymbol{\tau}(\boldsymbol{n}-\boldsymbol{\theta_i}),\boldsymbol{n}-\boldsymbol{\theta_i}\rangle+\langle\boldsymbol{\tau}\boldsymbol{n},\boldsymbol{n}\rangle]} = 0, \quad i = 1,2,3,4. \tag{6.33}$$

The notations are the same as the system (6.27), Eqs.(6.33) can be written as a linear system about the frequency ω_1, ω_2, u_0 and the integration constant δ, namely,

$$\begin{pmatrix} h_{11} & h_{12} & h_{13} & h_{14} \\ h_{21} & h_{22} & h_{23} & h_{24} \\ h_{31} & h_{32} & h_{33} & h_{34} \\ h_{41} & h_{42} & h_{43} & h_{44} \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ u_0 \\ \delta \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}. \tag{6.34}$$

Now solving this system, we get a two-periodic wave solution of Eq. (1.1)

$$u = u_0 + 12h_1h_2^{-1}\partial_x^2 \ln\vartheta(\xi_1,\xi_2,\boldsymbol{\tau}),$$

which provided the vector $(\omega_1,\omega_2,u_0,\delta)^T$. It solves the system (6.34) with the theta function $\vartheta(\xi_1,\xi_2,\boldsymbol{\tau})$ given by Eq.(6.28). The other parameters k_i, l_i, τ_{ij} and ε_i $(i,j = 1,2)$ are free. □

We now present some figures to describe the propagation situations of the periodic waves. Figure 5 shows the propagation of the one periodic wave via solution (6.18). Figure 6 shows the propagation of the degenerate two-periodic wave via solution (6.25). And Figures 7 and 8 show the propagation of the asymmetric and symmetric two-periodic waves via solution (6.25).

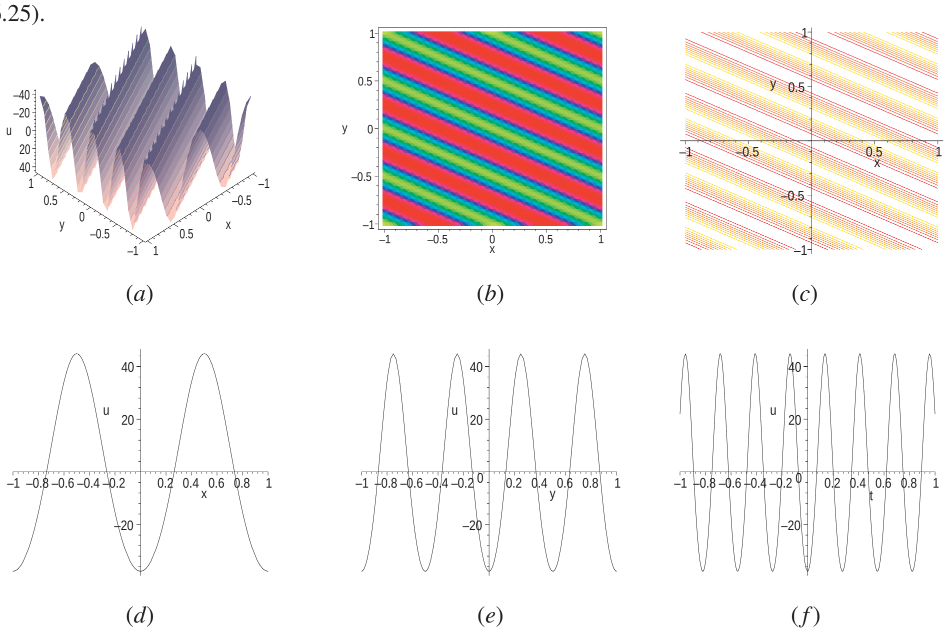


Fig. 5. (Color online) A one-periodic wave of the generalized vc-KP equation (1.1) via expression (6.18) with parameters: $h_1 = 1$, $h_2 = 1$, $h_3 = 2$, $h_4 = 4$, $h_5 = 6$, $k = 1$, $l = 2$, $\tau = i$ and $\varepsilon = 0$. This figure shows that every one-periodic wave is one-dimensional, and it can be viewed as a superposition of overlapping solitary waves, placed one period apart. (a) Perspective view of the real part of the periodic wave $\mathrm{Re}(u)$. (b) Overhead view of the wave, the green lines are crests

and the red lines are troughs. (c) The corresponding contour plot. (d) Wave propagation pattern of the wave along the x axis. (e) Wave propagation pattern of wave along the y axis. (f) Wave propagation pattern of wave along the t axis.

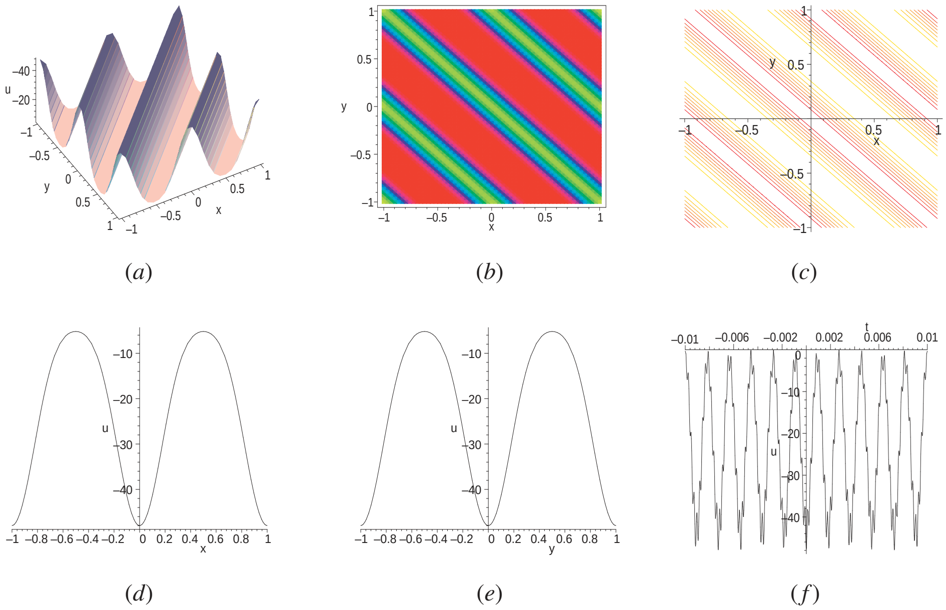


Fig. 6. (Color online) A degenerate two-periodic wave of the generalized vc-KP equation (1.1) via expression (6.25) with parameters: $h_1 = 1$, $h_2 = 2$, $h_3 = 4$, $h_4 = 6$, $h_5 = 8$, $k_1 = l_1 = 1$, $k_2 = l_2 = -1$, $\tau_{11} = i$, $\tau_{12} = 0.5i$, $\tau_{22} = 2i$ and $\varepsilon_1 = \varepsilon_2 = 0$. This figure shows that degenerate two-periodic wave is almost one-dimensional. (a) Perspective view of the real part of the periodic wave Re(u). (b) Overhead view of the wave, the green points are crests and the red points are troughs. (c) The corresponding contour plot. (d) Wave propagation pattern of the wave along the x axis. (e) Wave propagation pattern of wave along the y axis. (f) Wave propagation pattern of wave along the t axis.

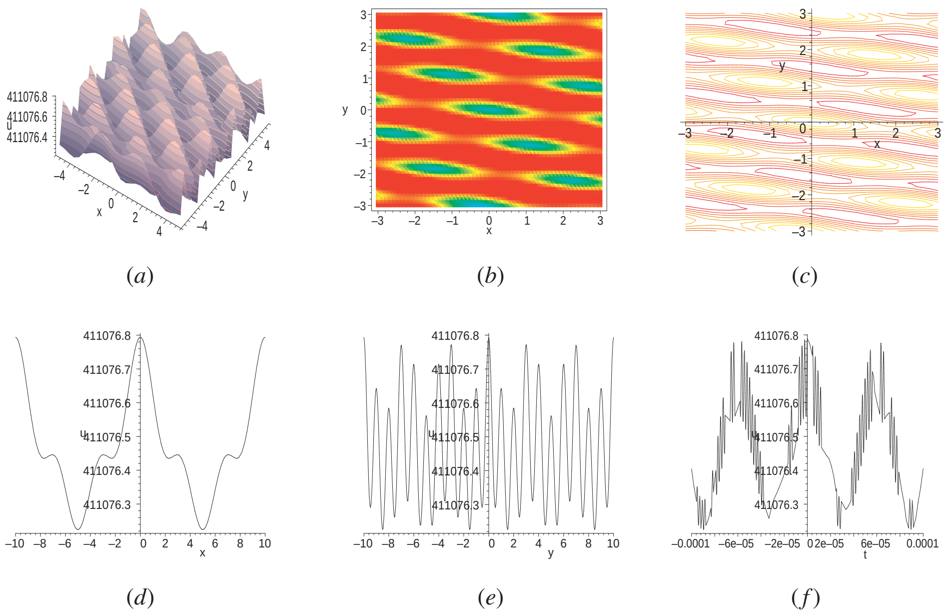


Fig. 7. (Color online) An asymmetric two-periodic wave of the generalized vc-KP equation (1.1) via expression (6.25) with parameters: $h_1 = -1$, $h_2 = 2$, $h_3 = 4$, $h_4 = 6$, $h_5 = 8$, $k_1 = 0.1$, $l_1 = 1$, $k_2 = l_2 = 0.3$, $\tau_{11} = i$, $\tau_{12} = 0.5i$, $\tau_{22} = 2i$

and $\varepsilon_1 = \varepsilon_2 = 0$. This figure shows that the asymmetric two-periodic wave is spatially periodic in three directions, but it need not to be periodic in either the x, y or t directions. (a) Perspective view of the real part of the periodic wave $\mathrm{Re}(u)$. (b) Overhead view of the wave, the green points are crests and the red points are troughs. (c) The corresponding contour plot. (d) Wave propagation pattern of the wave along the x axis. (e) Wave propagation pattern of wave along the y axis. (f) Wave propagation pattern of wave along the t axis.

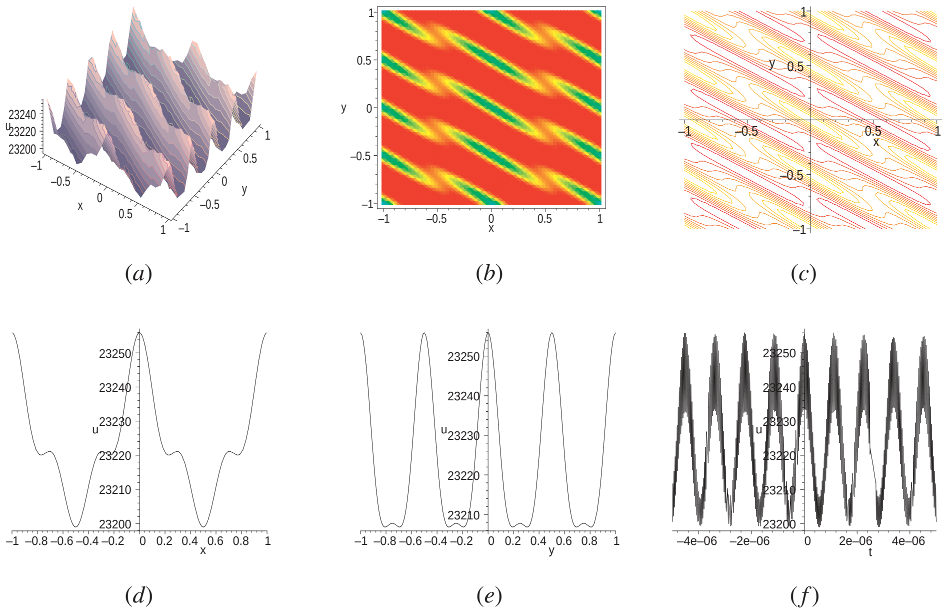


Fig. 8. (Color online) An symmetric two-periodic wave of the generalized vc-KP equation (1.1) via expression (6.25) with parameters: $h_1 = -1, h_2 = 2, h_3 = 4, h_4 = 6, h_5 = 8, k_1 = 1, l_1 = 2, k_2 = 3, l_2 = 4, \tau_{11} = i, \tau_{12} = 0.5i, \tau_{22} = 2i$ and $\varepsilon_1 = \varepsilon_2 = 0$. This figure shows that the symmetric two-periodic wave is periodic in three directions. (a) Perspective view of the real part of the periodic wave $\mathrm{Re}(u)$. (b) Overhead view of the wave, the green points are crests and the red points are troughs. (c) The corresponding contour plot. (d) Wave propagation pattern of the wave along the x axis. (e) Wave propagation pattern of wave along the y axis. (f) Wave propagation pattern of wave along the t axis.

6.3 Asymptotic property of Riemann theta function periodic waves

Based on the results of Ref. [51], the relation between the one- and two- periodic wave solutions (6.18), (6.25) and the one- and two- soliton solutions (6.14), (6.15) can be directly established as follows.

Theorem 6.5. *If the vector* $(\omega, \delta)^T$ *is a solution of the system* (6.24) *for the one-periodic wave solution* (6.18)*, we let*

$$k = \frac{\mu}{2\pi i}, \quad l = \frac{\nu}{2\pi i}, \quad \varepsilon = \frac{c + \pi\tau}{2\pi i}, \tag{6.35}$$

where μ*,* ν *and* c *are given in Eq.*(6.14)*. Then we have the following asymptotic properties*

$$\delta \to 0, \quad 2\pi i\xi \to \eta + \pi\tau, \quad \vartheta(\xi, \tau) \to 1 + e^{\eta}, \quad \text{when } \wp \to 0. \tag{6.36}$$

It implies that the one-periodic solution (6.18) *converges to the one-soliton solution* (6.14) *under a small amplitude limit, that is* $(u, \wp) \to (u_1, 0)$.

Proof. By using the system (6.20), a_{ij}, b_i, $i, j = 1, 2$, can be rewritten as the series about $\wp$

$$
\begin{aligned}
a_{11} &= 32\pi^2 k\left(\wp^2 + 4\wp^8 + 9\wp^{18} + \cdots + n^2\wp^{2n^2} + \cdots\right),\ \ a_{12} = 1 + 2\left(\wp^2 + \wp^8 + \wp^{18} + \cdots + \wp^{2n^2} + \cdots\right),\\
a_{21} &= 8\pi^2 k\left(\wp + 9\wp^5 + 25\wp^{13} + \cdots + (2n-1)^2\wp^{2n^2-2n+1} + \cdots\right),\ \ a_{22} = 2\left(\wp + \wp^5 + \wp^{13} + \cdots + \wp^{2n^2-2n+1} + \cdots\right),\\
b_1 &= 32\pi^2\Big[\left(16h_1\pi^2k^4 - h_3k^2 - h_4kl - h_5l^2\right)\wp^2 + \left(256h_1\pi^2k^4 - 4h_3k^2 - 4h_4kl - 4h_5l^2\right)\wp^8 + \cdots\\
&\quad + \left(16h_1n^4\pi^2k^4 - h_3n^2k^2 - h_4n^2kl - h_5n^2l^2\right)\wp^{2n^2} + \cdots\Big],\\
b_2 &= 8\pi^2\Big[\left(4h_1\pi^2k^4 - h_3k^2 - h_4kl - h_5l^2\right)\wp + \left(324h_1\pi^2k^4 - 9h_3k^2 - 9h_4kl - 9h_5l^2\right)\wp^5 + \cdots\\
&\quad + \left(4h_1(2n-1)^4\pi^2k^4 - h_3(2n-1)^2k^2 - h_4(2n-1)^2kl - h_5(2n-1)^2l^2\right)\wp^{2n^2-2n+1} + \cdots\Big].
\end{aligned}
\tag{6.37}
$$

With the aid of Proposition C in Appendix, we have

$$
\begin{aligned}
&A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ A_1 = \begin{pmatrix} 0 & 0 \\ 8\pi^2 k & 2 \end{pmatrix},\ A_2 = \begin{pmatrix} 32\pi^2 k & 2 \\ 0 & 0 \end{pmatrix},\ A_5 = \begin{pmatrix} 0 & 0 \\ 72\pi^2 k & 2 \end{pmatrix},\ A_3 = A_4 = 0,\ \ \ldots,\\
&B_1 = \begin{pmatrix} 0 \\ 8\pi^2\triangle_1 \end{pmatrix},\ B_2 = \begin{pmatrix} 32\pi^2\triangle_2 \\ 0 \end{pmatrix},\ B_5 = \begin{pmatrix} 0 \\ 72\pi^2\triangle_3 \end{pmatrix},\ B_0 = B_3 = B_4 = 0,\ \ \ldots,
\end{aligned}
\tag{6.38}
$$

where $\triangle_1 = 4h_1\pi^2k^4 - h_3k^2 - h_4kl - h_5l^2$, $\triangle_2 = 16h_1\pi^2k^4 - h_3k^2 - h_4kl - h_5l^2$ and $\triangle_3 = 36h_1\pi^2k^4 - h_3k^2 - h_4kl - h_5l^2$.

Substituting the system (6.38) into formulas (D.7), one can obtain

$$
X_0 = \begin{pmatrix} -k^{-1}\triangle_1 \\ 0 \end{pmatrix},\ X_2 = \begin{pmatrix} 8k^{-1}\triangle_1 \\ 32\pi^2\triangle_1 \end{pmatrix},\ X_4 = -\begin{pmatrix} 89k^{-1}\triangle_1 + 9k^{-1}\triangle_3 \\ 320\pi^2\triangle_1 \end{pmatrix},\ X_1 = X_3 = 0,\ \ \ldots. \tag{6.39}
$$

From (D.2), one then has

$$
\begin{aligned}
\omega &= -k^{-1}\triangle_1 + 8k^{-1}\triangle_1\wp^2 - (89k^{-1}\triangle_1 + 9k^{-1}\triangle_3)\wp^4 + o(\wp^4),\\
\delta &= 32\pi^2\triangle_1\wp^2 - 320\pi^2\triangle_1\wp^4 + 0(\wp^4),
\end{aligned}
\tag{6.40}
$$

which implies by using relation (6.35) that

$$
\delta \to 0,\ \ 2\pi i\omega \to -(h_1\mu^3 + h_3\mu + h_4\nu + h_5\mu^{-1}\nu^2),\ \ \text{when}\ \ \wp \to 0. \tag{6.41}
$$

In order to show that one-periodic wave (6.18) degenerates to the one-soliton solution (6.14) under the limit $\wp \to 0$, we first expand the periodic function $\vartheta(\xi, \tau)$ in the form of

$$
\vartheta(\xi, \tau) = 1 + \left(e^{2\pi i\xi} + e^{-2\pi i\xi}\right)\wp + \left(e^{4\pi i\xi} + e^{-4\pi i\xi}\right)\wp^4 + \cdots. \tag{6.42}
$$

Using the transformation (6.35), one has

$$
\begin{aligned}
&\vartheta(\xi, \tau) = 1 + e^{\widehat{\xi}} + \left(e^{-\widehat{\xi}} + e^{2\widehat{\xi}}\right)\wp^2 + \left(e^{-2\widehat{\xi}} + e^{3\widehat{\xi}}\right)\wp^6 + \cdots \to 1 + e^{\widehat{\xi}},\ \ \text{when}\ \ \wp \to 0,\\
&\widehat{\xi} = 2\pi i\xi - \pi\tau = \mu x + \nu y + 2\pi i\omega t + c.
\end{aligned}
\tag{6.43}
$$

Combining Eqs.(6.41) and (6.43), one deduces that

$$
\begin{aligned}
&\widehat{\xi} \to \mu x + \nu y - (h_1\mu^3 + h_3\mu + h_4\nu + h_5\mu^{-1}\nu^2)t + c,\ \ \text{when}\ \ \wp \to 0,\\
&2\pi i\xi \to \eta + \pi\tau,\qquad \text{when}\ \ \wp \to 0.
\end{aligned}
\tag{6.44}
$$

With the aid of Eqs.(6.43) and (6.44), one can obtain

$$\vartheta(\xi) \to 1 + e^{\eta}, \quad \text{when} \;\; \wp \to 0. \tag{6.45}$$

From above, we conclude that the one-periodic solution (6.18) just converges to the one-soliton solution (6.14) as the amplitude $\wp \to 0$. □

Theorem 6.6. *If* $(\omega_1, \omega_2, u_0, \delta)^T$ *is a solution of the system* (6.26) *for the two-periodic wave solution* (6.25)*, we take*

$$k_i = \frac{\mu_i}{2\pi i}, \;\; l_i = \frac{\nu_i}{2\pi i}, \;\; \varepsilon_i = \frac{c_i + \pi\tau_{ij}}{2\pi i}, \;\; \tau_{12} = \frac{A_{12}}{2\pi i}, \;\; i = 1, 2, \tag{6.46}$$

where μ_i*,* ν_i*,* c_i*,* $i = 1, 2$*, and* A_{12} *are given in Eq.*(6.15)*. Then we have the following asymptotic relations*

$$\begin{aligned} &u_0 \to 0, \;\; \delta \to 0, \;\; 2\pi i\xi_i \to \eta_i + \pi\tau_{ij}, \;\; i = 1, 2, \\ &\vartheta(\xi_1, \xi_2, \boldsymbol{\tau}) \to 1 + e^{\eta_1} + e^{\eta_2} + e^{\eta_1+\eta_2+A_{12}}, \;\; \textit{when} \;\; \wp_1, \wp_2 \to 0. \end{aligned} \tag{6.47}$$

It implies that the two-periodic solution (6.25) *converges to the two-soliton solution* (6.15) *under a small amplitude limit, that is* $(u, \wp_1, \wp_2) \to (u_1, 0, 0)$*.*

Proof. The proof is similar to the one of Theorem 6.5. □

7. Conclusions and discussions

In this paper, under the conditions (1.12), we have systematically researched integrability features of the generalized vc-KP equation (1.1), which is an important model of various nonlinear real situations in hydrodynamics, plasma physics and some other nonlinear science when the inhomogeneities of media and nonuniformities of boundaries are taken into consideration. Using the properties of the binary Bell polynomials, we systematically construct the bilinear representation, Bäcklund transformation, Lax pair and Darboux covariant Lax pair, respectively, which can be reduced to the ones of several integrable equations such as KdV (1.2), KP (1.3), cylindrical KdV (1.4), cylindrical KP and generalized cylindrical KP (1.5) equations etc. Based on its Lax equation, the infinite conservation laws of the equation also can be constructed. Using the bilinear formula and the recent results in Ref. [51, 52], we have present the soliton solutions and Riemann theta function periodic wave solutions of the vc-KP equation (1.1). And we are also able to choose different parameters and functions to obtain some solutions, and also analyze their graphics in Figures 1-4 and 5-8, respectively. Finally, a limiting procedure is presented to analyze in detail, the relations between the periodic wave solutions and soliton solutions. In conclusion, the generalized vc-KP equation (1.1) is completely integrable under the conditions (1.12) in the sense that it admits bilinear Bäcklund transformation, Lax pair and infinite conservation laws. And the integrable constraint conditions (1.12) on the variable coefficients can be naturally found in the procedure of applying binary Bell polynomials. The results presented in this paper may provide further evidence of structures and complete integrability of these equations.

Acknowledgments

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Appendix A: Multidimensional Bell polynomials

In the following, we simply recall some necessary notations on multidimensional binary Bell polynomials, for details refer, for instance, to Lembert and Gilson's work [8-10].

Suppose $f{=}f(x_1, x_2, \ldots, x_n)$ be a multi-variables function in $\mathbb{C}^\infty$, the expression as follows

$$Y_{n_1x_1,\ldots,n_rx_r}(f) \equiv Y_{n_1,\ldots,n_r}(f_{l_1x_1}, \ldots, f_{l_rx_r}) = e^{-f}\partial_{x_1}^{n_1}\cdots\partial_{x_r}^{n_r}e^f, \tag{A.1}$$

is called muliti-dimensional Bell polynomials, where $f_{l_1x_1,\ldots,l_rx_r} = \partial_{x_1}^{l_1}\cdots\partial_{x_r}^{l_r}$ $(0 \le l_i \le n_i, i = 1, 2, \ldots, r)$. Taking $n = 1$, Bell polynomials are presented as follows

$$Y_{nx}(f) \equiv Y_n(f_1, \ldots, f_n) = \sum \frac{n!}{s_1!\cdots s_n!(1!)^{s_1}\cdots(n!)^{s_n}} f_1^{s_1}\cdots f_n^{s_n}, \quad n = \sum_{k=1}^{n} ks_k,$$
$$Y_x(f) = f_x, \quad Y_{2x}(f) = f_{2x} + f_x^2, \quad Y_{3x}(f) = f_{3x} + 3f_xf_{2x} + f_x^3, \cdots . \tag{A.2}$$

To make the link between the Bell polynomials and the Hirota D-operator, the multi-dimensional binary Bell polynomials can be defined as follows [9]

$$\mathscr{Y}_{n_1x_1,\ldots,n_rx_r}(\upsilon, \omega) = Y_{n_1,\ldots,n_r}(f)\Big|_{f_{l_1x_1,\ldots,l_rx_r}=\begin{cases}\upsilon_{l_1x_1,\ldots,l_rx_r}, & l_1+\cdots+l_r \text{ is odd},\\ \omega_{l_1x_1,\ldots,l_rx_r}, & l_1+\cdots+l_r \text{ is even},\end{cases}} \tag{A.3}$$

$$\mathscr{Y}_x(\upsilon, \omega) = \upsilon_x, \quad \mathscr{Y}_{2x}(\upsilon, \omega) = \upsilon_x^2 + \omega_{2x}, \quad \mathscr{Y}_{x,t}(\upsilon, \omega) = \upsilon_x\upsilon_t + \omega_{xt}, \quad \mathscr{Y}_{3x}(\upsilon, \omega) = \upsilon_{3x} + 3\upsilon_x\omega_{2x} + \upsilon_x^3, \cdots , \tag{A.4}$$

which inherit the easily recognizable partial structure of the Bell polynomials.

To find the relationship of $\mathscr{Y}$-polynomials and the Hirota bilinear equation $D_{x_1}^{n_1}\cdots D_{x_1}^{n_r}F\cdot G$ [4], one should investigate the following identity[9]

$$\mathscr{Y}_{n_1x_1,\ldots,n_rx_r}(\upsilon = \ln F/G,\ \omega = \ln FG) = (FG)^{-1}D_{x_1}^{n_1}\cdots D_{x_r}^{n_r}F\cdot G, \tag{A.5}$$

where F and G are both the functions of x and t. In case of $F = G$, Eq. (A.5) can be changed into

$$F^{-2}D_{x_1}^{n_1}\cdots D_{x_r}^{n_r}F\cdot F = \mathscr{Y}(0, q = 2\ln F) = \begin{cases}0, & n_1+\cdots+n_r \text{ is odd},\\ P_{n_1x_1,\ldots,n_rx_r}(q), & n_1+\cdots+n_r \text{ is even}.\end{cases} \tag{A.6}$$

By using (A.6) and the following structure

$$P_{2x}(q) = q_{2x}, \quad P_{x,t}(q) = q_{xt}, \quad P_{4x}(q) = q_{4x} + 3q_{2x}^2, \quad P_{6x}(q) = q_{6x} + 15q_{2x}q_{4x} + 15q_{2x}^3, \ldots . \tag{A.7}$$

one can characterize P-polynomials. The binary Bell polynomials $\mathscr{Y}_{n_1x_1,\ldots,n_rx_r}(\upsilon, \omega)$ can be rewritten as P- and Y-polynomials

$$\begin{aligned}(FG)^{-1}D_{x_1}^{n_1}\cdots D_{x_r}^{n_r}F\cdot G &= \mathscr{Y}_{n_1x_1,\ldots,n_rx_r}(\upsilon, \omega)|_{\upsilon=\ln F/G,\omega=\ln FG}\\ &= \mathscr{Y}_{n_1x_1,\ldots,n_rx_r}(\upsilon, \upsilon + q)|_{\upsilon=\ln F/G,\omega=\ln FG}\\ &= \sum_{n_1+\cdots+n_r=even}\sum_{l_1=0}^{n_1}\cdots\sum_{l_r=0}^{n_r}\prod_{i=0}^{r}\binom{n_i}{l_i}P_{l_1x_1,\ldots,l_rx_r}(q)Y_{(n_1-l_1)x_1,\ldots,(n_r-l_r)x_r}(\upsilon).\end{aligned} \tag{A.8}$$

Multidimensional Bell polynomials admits the following key property

$$Y_{n_1x_1,\dots,n_rx_r}(\upsilon)|_{\upsilon=\ln\psi} = \psi_{n_1x_1,\dots,n_rx_r}/\psi. \tag{A.9}$$

It implies that the Hopf-Cole transformation $\upsilon = \ln\psi$, that is, $\psi = F/G$ is a linear transformation of $\mathscr{Y}_{n_1x_1,\dots,n_rx_r}(\upsilon,\omega)$. By using (A.8) and (A.9), one can then construct the Lax system of the nonlinear equations.

Appendix B: Riemann theta function periodic wave

Based on the results in Ref. [51], we consider one-periodic wave solutions of nonlinear evolution equation (NLEE). Then Riemann theta function reduces the following Fourier series in n

$$\vartheta(\xi,\tau) = \sum_{n=-\infty}^{+\infty} e^{\pi i n^2\tau+2\pi i n\xi}, \tag{B.1}$$

where the phase variable $\xi = kx_1 + lx_2 + \cdots + \rho x_N + \omega t + \varepsilon$ and the parameter $\mathrm{Im}(\tau) > 0$.

Theorem A.(Ref.[51]) *Assuming that $\vartheta(\xi,\tau)$ is a Riemann theta function for $N = 1$ with $\xi = kx_1 + lx_2 + \cdots + \rho x_N + \omega t + \varepsilon$ and k, l, $\cdots$, ρ, ω, ε satisfy the following system*

$$\sum_{n=-\infty}^{\infty} \mathscr{L}(4n\pi ik, 4n\pi il, \cdots, 4n\pi i\rho, 4n\pi i\omega)\, e^{2n^2\pi i\tau} = 0, \tag{B.2a}$$

$$\sum_{n=-\infty}^{\infty} \mathscr{L}(2\pi i(2n-1)k, 2\pi i(2n-1)l, \cdots, 2\pi i(2n-1)\rho, 2\pi i(2n-1)\omega)\, e^{(2n^2-2n+1)\pi i\tau} = 0. \tag{B.2b}$$

Then the following expression

$$u = u_0 + a\partial_\Lambda^n \ln\vartheta(\xi), \tag{B.3}$$

is the one-periodic wave solution of the NLEE.

Let us now consider the case when N=2, the Riemann theta function takes the form of

$$\vartheta(\xi,\tau) = \vartheta(\xi_1,\xi_2,\tau) = \sum_{n\in\mathbb{Z}^2} e^{\pi i\langle\tau n,n\rangle+2\pi i\langle\xi,n\rangle}, \tag{B.4}$$

where $n = (n_1,n_2)^T \in \mathbb{Z}^2$, $\xi = (\xi_1,\xi_2) \in \mathbb{C}^2$, $\xi_i = k_ix_1 + l_ix_2 + \cdots + \rho_ix_N + \omega_it + \varepsilon_i$, $i = 1,2$, and $-i\tau$ is a positive definite whose real-valued symmetric 2×2 matrix is

$$\tau = \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{12} & \tau_{22} \end{pmatrix}, \quad \mathrm{Im}(\tau_{11}) > 0, \quad \mathrm{Im}(\tau_{22}) > 0, \quad \tau_{11}\tau_{22} - \tau_{12}^2 < 0. \tag{B.5}$$

Theorem B.([51]) *Assuming that $\vartheta(\xi_1,\xi_2,\tau)$ is one Riemann theta function with $N = 2$, $\xi_i = k_ix_1 + l_ix_2 + \cdots + \rho_ix_N + \omega_it + \varepsilon_i$, $i = 1,2$ and k_i, l_i, $\cdots$, ρ_i, ω_i, ε_i $(i = 1,2)$ satisfy the following system*

$$\sum_{\boldsymbol{n}\in\mathbb{Z}^2} \mathscr{L}(2\pi i\langle 2\boldsymbol{n}-\boldsymbol{\theta}_i,k\rangle, 2\pi i\langle 2\boldsymbol{n}-\boldsymbol{\theta}_i,l\rangle, \cdots, 2\pi i\langle 2\boldsymbol{n}-\boldsymbol{\theta}_i,\rho\rangle, 2\pi i\langle 2\boldsymbol{n}-\boldsymbol{\theta}_i,\omega\rangle)\, e^{\pi i[\langle\boldsymbol{\tau}(\boldsymbol{n}-\boldsymbol{\theta}_i),\boldsymbol{n}-\boldsymbol{\theta}_i\rangle+\langle\boldsymbol{\tau}\boldsymbol{n},\boldsymbol{n}\rangle]} = 0, \tag{C.1}$$

where $\boldsymbol{\theta}_i = (\theta_i^1,\theta_i^2)^T$, $\boldsymbol{\theta}_1 = (0,0)^T$, $\boldsymbol{\theta}_2 = (1,0)^T$, $\boldsymbol{\theta}_3 = (0,1)^T$, $\boldsymbol{\theta}_4 = (1,1)^T$, $i = 1,2,3,4$. *Then the following expression*

$$u = u_0 + a\partial_\Lambda^n \ln\vartheta(\xi_1,\xi_2), \tag{C.2}$$

is the two-periodic wave solution of the NLEE.

Finally, we present a key proposition to investigate the asymptotic property of periodic waves. We write the system (6.24) into power series of

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = A_0 + A_1\wp + A_2\wp^2 + \cdots, \tag{D.1}$$

$$\begin{pmatrix} \omega \\ c \end{pmatrix} = X_0 + X_1\wp + X_2\wp^2 + \cdots, \tag{D.2}$$

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = B_0 + B_1\wp + B_2\wp^2 + \cdots. \tag{D.3}$$

Substituting Eqs.(D.1)-(D.3) into Eq.(6.24) leads to the following recursion relations

$$A_0X_0 = B_0, \quad A_0X_n + A_1X_{n-1} + \cdots + A_nX_0 = B_n, \quad n \geq 1, \ n \in \mathbb{N}, \tag{D.4}$$

form which we then recursively get each vector $X_i, i = 0, 1, \cdots$.

Proposition C. ([51]) *Assuming that the matrix A_0 is reversible, we can obtain*

$$X_0 = A_0^{-1}B_0, \quad X_n = A_0^{-1}\left(B_n - \sum_{i=1}^{n} A_iB_{n-1}\right), \quad n \geq 1, \ n \in \mathbb{N}. \tag{D.5}$$

If the matrix A_0 and A_1 are not inverse,

$$A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0 \\ -8\pi^2k & 2 \end{pmatrix}, \tag{D.6}$$

we can obtain

$$\begin{aligned} X_0 &= \begin{pmatrix} \frac{2B_0^{(1)}-B_1^{(2)}}{8\pi^2k} & B_0^{(1)} \end{pmatrix}^T, \quad X_1 = \begin{pmatrix} \frac{2B_1^{(1)}-(B_2-A_2X_0)^{(2)}}{8\pi^2k} & B_1^{(1)} \end{pmatrix}^T, \cdots, \\ X_n &= \begin{pmatrix} \frac{2\left(B_{n+1}-\sum_{i=2}^{n}A_iX_{n-i}\right)^{(1)}-\left(B_{n+1}-\sum_{i=2}^{n+1}A_iX_{n+1-i}\right)^{(2)}}{8\pi^2k}, & \left(B_{n+1}-\sum_{i=2}^{n}A_iX_{n-i}\right)^{(1)} \end{pmatrix}^T, \quad n \geq 2, \quad n \in \mathbb{N}, \end{aligned} \tag{D.7}$$

where $\alpha^{(1)}$ and $\alpha^{(2)}$ denote the first and second component of a two-dimensional vector α, respectively.

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