# ON THE INTEGRAL EQUATION OF RENEWAL THEORY

BY WILLY FELLER

*Brown University*

1. **Introduction.** In this paper we consider the behavior of the solutions of the integral equation

$$(1.1) \qquad u(t) = g(t) + \int_0^t u(t-x)f(x)\,dx,$$

where $f(t)$ and $g(t)$ are given non-negative functions.[1] This equation appears, under different forms, in population theory, the theory of industrial replacement and in the general theory of self-renewing aggregates, and a great number of papers have been written on the subject.[2] Unfortunately most of this literature is of a heuristic nature so that the precise conditions for the validity of different methods or statements are seldom known. This literature is, moreover, abundant in controversies and different conjectures which are sometimes supported or disproved by unnecessarily complicated examples. All this renders an orientation exceedingly difficult, and it may therefore be of interest to give a rigorous presentation of the theory. It will be seen that some of the previously announced results need modifications to become correct.

The existence of a solution $u(t)$ of (1.1) could be deduced directly from a well-known result of Paley and Wiener [21] on general integral equations of form (1.1).[3] However, the case of non-negative functions $f(t)$ and $g(t)$, with which we are here concerned, is much too simple to justify the deep methods used by Paley and Wiener in the general case. Under the present conditions, the existence of a solution can be proved in a simple way using properties of completely monotone functions, and this method has also the distinct advantage of showing some properties of the solutions, which otherwise would have to be proved separately. It will be seen in section 3 that the existence proof becomes most natural if equation (1.1) is slightly generalized. Introducing the summatory functions

$$(1.2) \qquad U(t) = \int_0^t u(x)\,dx, \qquad F(t) = \int_0^t f(x)\,dx, \qquad G(t) = \int_0^t g(x)\,dx,$$

[1] For the interpretation of the equation cf. section 2.

[2] Lotka's paper [8] contains a bibliography of 74 papers on our subject published before 1939. Yet it is stated that even this list "is not the result of an exhaustive search." At the end of the present paper the reader will find a list of 16 papers on (1.1) which have appeared during the two years since the publication of Lotka's paper.

[3] This has been remarked also by Hadwiger [3].

equation (1.1) can be rewritten in the form

$$(1.3) \qquad U(t) = G(t) + \int_0^t U(t - x)\, dF(x).$$

However, (1.3) has a meaning even if $F(t)$ and $G(t)$ are not integrals, provided $F(t)$ is of bounded total variation and the integral is interpreted as a Stieltjes integral. Now for many practical applications (and even for numerical calculations) this generalized form of the integral equation seems to be the most appropriate one and, as a matter of fact, it has sometimes been used in a more or less hidden form (e.g., if all individuals of the parent population are of the same age). Our existence theorem refers to this generalized equation.

We then turn to one of the main problems of the theory, namely the asymptotic behavior of $u(t)$ as $t \to \infty$. It is generally supposed that the solution $u(t)$ "in general" either behaves like an exponential function, or that it approaches in an oscillating manner a finite limit $q$; the latter case should arise if $\int_0^\infty f(t)\, dt = 1$, thus in particular in the cases of a stable population and of industrial replacement. However, special examples have been constructed to show that this is not always so.[4] In order to simplify the problem and to get more general conditions, we shall first (section 4) consider only the question of convergence in mean, that is to say, we shall study the asymptotic behavior not of $u(t)$ itself but of the mean value $u^*(t) = \frac{1}{t} \int_0^t u(x)\, dx$. The question can be solved completely using only the simplest Tauberian theorems for Laplace integrals. Of course, if $u(t) \to q$ then also $u^*(t) \to q$, but not conversely. The investigation of the precise asymptotic behavior of $u(t)$ is more delicate and requires more refined tools (section 5).

Most of section 6 is devoted to a study of Lotka's well-known method of expanding $u(t)$ into a series of oscillatory components, and it is hoped that this study will help clarify the true nature of this expansion. It will be seen that Lotka's method can be justified (with some necessary modifications) even in some cases for which it was not intended, e.g., if the characteristic equation has multiple or negative real roots, or if it has only a finite number of roots. On the other hand limitations of the method will also become apparent: thus it can occur in special cases that a formal application of the method will lead to a function $u(t)$ which apparently solves the given equation, whereas in reality it is the solution of quite a different equation.

Of course, most of the difficulties mentioned above arise only when the function $f(t)$ has an infinite tail. However, it is known that even computational considerations sometimes require the use of such curves, and, as matter of fact,

---

[4] Cf. Hadwiger [2] and also Hadwiger, "Zur Berechnung der Erneuerungsfunktion nach einer Formel von V. A. Kostitzin," *Mitt. Verein. schweizerischer Versich.-Math.*, Vol. 34 (1937), pp. 37–43.

exponential and Pearsonian curves have been used most frequently in connection with (1.1). It will be seen that even in these special cases customary methods may lead to incorrect results. Besides, our considerations show how much the solution $u(t)$ is influenced by the values of $f(t)$ for $t \to \infty$, and, accordingly, that extreme caution is needed in practice. The last section contains some simple remarks on the practical computation of the solution.

**2. Generalities on equations (1.1) and (1.3).** This section contains a few remarks on the meaning of our integral equation and on an alternative form under which it is encountered in the literature. A reader interested only in the abstract theory may pass immediately to section 3.

Equation (1.1) can be interpreted in various ways; the most important among them are the following two:

(i) In the theory of industrial replacement (as outlined in particular by Lotka), it is assumed that each individual dropping out is immediately replaced by a new member of zero age. $f(t)$ denotes the density of the probability at the moment of installment that an individual will drop out at age $t$. The function $g(t)$ is defined by

$$g(t) = \int_0^t \eta(x) f(t - x)\, dx, \tag{2.1}$$

where $\eta(x)$ represents the age distribution of the population at the moment $t = 0$ (so that the number of individuals of an age between $x$ and $x + \delta x$ is $\eta(x)\delta x + o(\delta x)$). Obviously $g(t)$ then represents the rate of dropping out at time $t$ of individuals belonging to the parent population. Finally, $u(t)$ denotes the rate of dropping out at time $t$ of individuals of the total population. Now each individual dropping out at time $t$ belongs either to the parent population, or it came to the population by the process of replacement at some moment $t - x$ $(0 < x < t)$, and hence $u(t)$ satisfies (1.1). It is worthwhile to note that in this case

$$\int_0^\infty f(t)\, dt = 1, \tag{2.2}$$

since $f(t)$ represents a density of probability.

(ii) In population theory $u(t)$ measures the rate of female births at time $t > 0$. The function $f(t)$ now represents the reproduction rate of females at age $t$ (that is to say, the average number of female descendants born during $(t, t + \delta t)$ from a female of age $t$ is $f(t)\delta t + o(\delta t)$). If $\eta(x)$ again stands for the age distribution of the parent population at $t = 0$, the function $g(t)$ of (2.1) will obviously measure the rate of production of females at time $t$ by members of the parent population. Thus we are again led to (1.1), with the difference, however, that this time either of the inequalities

$$\int_0^\infty f(t)\, dt \lesseqgtr 1 \tag{2.3}$$

may occur; the value of this integral shows the tendency of increase or decrease in the total population.

Theoretically speaking, $f(t)$ and $g(t)$ are two arbitrary non-negative functions. It is true that $g(t)$ is connected with $f(t)$ by (2.1); but, since the age distribution $\eta(x)$ is arbitrary, $g(t)$ can also be considered as an arbitrarily prescribed function.

It is hardly necessary to interpret the more general equation (1.3) in detail: it is the straightforward generalization of (1.1) to the case where the increase or decrease of the population is not necessarily a continuous process. This form of the equation is frequently better adapted to practical needs. Indeed, the functions $f(t)$ and $g(t)$ are usually determined from observations, so that only their mean values over some time units (years) are known. In such cases it is sometimes simpler to treat $f(t)$ and $g(t)$ as discontinuous functions, using equation (1.3) instead of (1.1). For some advantages of such a procedure see section 7. It may also be mentioned that the most frequently (if not the only) special case of (1.1) studied is that where $g(t) = f(t)$. Now it is apparent from (2.1) that this means that all members of the parent population are of zero age: in this case, however, there is no continuous age-distribution $\eta(x)$. Instead we have to use a discontinuous function $\eta(x)$ and write (2.1) in the form of a Stieltjes integral. Thus discontinuous functions and Stieltjes integrals present themselves automatically, though in a somewhat disguised form, even in the simplest cases.

At this point a remark may be inserted which will prove useful for a better understanding later on (section 6). In the current literature we are frequently confronted not with (1.1) but with

$$u(t) = \int_0^\infty u(t - x)f(x)\,dx, \tag{2.4}$$

together with the explanation that it is asked to find a solution of (2.4) which reduces, for $t < 0$, to a prescribed function $h(t)$. Now such a function, as is known, exists only under very exceptional conditions, and (2.4) is by no means equivalent to (1.1). The current argument can be boiled down to the following. Suppose first that the function $g(t)$ of (1.1) is given in the special form

$$g(t) = \int_t^\infty h(t - x)f(x)\,dx, \tag{2.5}$$

where $h(x)$ is a non-negative function defined for $x < 0$. Since the solution $u(t)$ of (1.1) has a meaning only for $t > 0$, we are free to *define* that $u(-t) = h(-t)$ for $t > 0$. This arbitrary definition, then, formally reduces (1.1) to (2.4). It should be noted, however, that this function $u(t)$ does not, in general, satisfy (2.4) for $t < 0$, for $h(t)$ was prescribed arbitrarily. Thus we are not, after all, concerned with (2.4) but with (1.1), which form of the equation is, by the way, the more general one for our purposes. If there really existed a solution of (2.4) which reduced to $h(t)$ for $t < 0$, we could of course define $g(t)$ by (2.5) and transform (2.4) into (1.1) by splitting the interval $(0, \infty)$ into the subintervals

(0, $t$) and ($t$, $\infty$). However, as was already mentioned, a solution of the required kind does not exist in general. It will also be seen (section 6) that the true nature of the different methods and the limits of their applicability can be understood only when the considerations are based on the proper equation (1.1) and not on (2.4).

### 3. Existence of solutions.

THEOREM 1. *Let $F(t)$ and $G(t)$ be two finite non-decreasing functions which are continuous to the right*[5]. *Suppose that*

$$F(0) = G(0) = 0, \tag{3.1}$$

*and that the Laplace integrals*[6]

$$\varphi(s) = \int_0^\infty e^{-st}\, dF(t), \qquad \gamma(s) = \int_0^\infty e^{-st}\, dG(t) \tag{3.2}$$

*converge at least for $s > \sigma \geq 0$*[7]. *In case that $\lim_{s \to \sigma + 0} \varphi(s) > 1$, let $\sigma' > \sigma$ be the root*[8] *of the characteristic equation $\varphi(s) = 1$; in case $\lim_{s \to \sigma+0} \varphi(s) \leq 1$, put $\sigma' = \sigma$.*

*Under these conditions there exists for $t > 0$ one and only one finite non-decreasing function $U(t)$ satisfying* (1.3). *With this function the Laplace integral*

$$\omega(s) = \int_0^\infty e^{-st}\, dU(t) \tag{3.3}$$

---

[5] It is needless to emphasize that this restriction is imposed only to avoid trivial ambiguities.

[6] The integrals (3.2) should be interpreted as Lebesgue-Stieltjes integrals over *open* intervals; thus

$$\varphi(s) = \lim_{\epsilon \to +0} \int_\epsilon^\infty e^{-st}\, dF(t),$$

which implies that $\varphi(s) \to 0$ as $s \to \infty$. Alternatively it can be supposed that $F(t)$ and $G(t)$ have no discontinuities at $t = 0$. Continuity of $F(t)$ at $t = 0$ means that there is no reproduction at zero age. This assumption is most natural for our problem, but is by no means necessary. In order to investigate the case where $F(t)$ has a saltus $c > 0$ at $t = 0$, one should take the integrals (3.2) over the closed set $[0, \infty]$, so that

$$\varphi(s) = c + \lim_{\epsilon \to +0} \int_\epsilon^\infty e^{-st}\, dF(t).$$

It is readily seen that Theorem 1 and its proof remain valid if $0 < c < 1$. However, if $c > 1$, then (1.3) plainly has no solution $U(t)$. The continuity of $G(t)$ at $t = 0$ is of no importance and is not used in the sequel.

[7] The condition is formulated in this general way in view of later applications (cf., e.g., the lemma of section 4). In all cases of practical interest $\sigma = 0$.

[8] $\varphi(s)$ is, of course, monotonic for $s > \sigma$ and tends to zero as $s \to \infty$. In order to ensure the existence of a root of $\varphi(s) = 1$, it is sufficient to suppose that the saltus $c$ of $F(t)$ at $t = 0$ is less than 1 (cf. footnote 6).

*converges for* $s > \sigma'$, *and*

$$\omega(s) = \frac{\gamma(s)}{1 - \varphi(s)}. \tag{3.4}$$

PROOF: A trivial computation shows that for any finite non-decreasing solution $U(t)$ of (1.3) and any $T > 0$ we have

$$\int_0^T e^{-st}\,dU(t) = \int_0^T e^{-st}\,dG(t) + \int_0^T e^{-sx}\,dF(x) \int_0^{T-x} e^{-st}\,dU(t);$$

herein all terms are non-negative and hence by (3.2)

$$\int_0^T e^{-st}\,dU(t) \leq \gamma(s) + \varphi(s) \int_0^T e^{-st}\,dU(t).$$

Now $\varphi(s) < 1$ for $s > \sigma'$, and hence it is seen that the integral (3.3) exists for $s > \sigma'$ and satisfies (3.4). On the other hand it is well-known that the values of $\omega(s)$ for $s > \sigma'$ determine the corresponding function $U(t)$ uniquely, except for an additive constant, at all points of continuity. However, from (1.3) and (3.1) it follows that $U(0) = 0$ and, since by (1.3) $U(t)$ is continuous to the right, the monotone solution $U(t)$ of (1.3), if it exists, is determined uniquely.

To prove the existence of $U(t)$ consider a function $\omega(s)$ defined for $s > \sigma'$ by (3.4). It is clear from (3.2) that $\varphi(s)$ and $\gamma(s)$ are completely monotone functions, that is to say that $\varphi(s)$ and $\gamma(s)$ have, for $s > \sigma$, derivatives of all orders and that $(-1)^n\varphi^{(n)}(s) \geq 0$ and $(-1)^n\gamma^{(n)}(s) \geq 0$. We can therefore differentiate (3.4) any number of times, and it is seen that $\omega^{(n)}(s)$ is continuous for $s > \sigma'$. Now a simple inductive argument shows that $(-1)^n\omega^{(n)}(s)$ is a product of $\{1 - \varphi(s)\}^{-(n+1)}$ by a finite number of completely monotone functions. It follows that $(-1)^n\omega^{(n)}(s) \geq 0$, so that $\omega(s)$ is a completely monotone function, at least for $s > \sigma'$. Hence it follows from a well-known theorem of S. Bernstein and D. V. Widder[9] that there exists a non-decreasing function $U(t)$ such that (3.3) holds for $s > \sigma'$. Moreover, this function can obviously be so defined that $U(0) = 0$ and that it is continuous to the right. Using $U(t)$ let us form a new function

$$V(t) = \int_0^t U(t - x)\,dF(x). \tag{3.5}$$

$V(t)$ is clearly non-negative and non-decreasing. It is readily verified (and, of course, well-known) that

$$\psi(s) \equiv \int_0^\infty e^{-st}\,dV(t) = \omega(s)\varphi(s).$$

It follows, therefore, from (3.4) that $\psi(s) = \omega(s) - \gamma(s)$, and this implies, by the

[9] This theorem has been repeatedly proved by several authors; for a recent proof cf. Feller [19].

uniqueness theorem for Laplace transforms, that $V(t) = U(t) - G(t)$. Combining this result with (3.5) it is seen that $U(t)$ is a solution of (1.3).

THEOREM 2. *Suppose that $f(t)$ and $g(t)$ are measurable, non-negative and bounded in every finite interval $0 \le t \le T$. Let the integrals*

$$\varphi(s) = \int_0^\infty e^{-st} f(t)\,dt, \qquad \gamma(s) = \int_0^\infty e^{-st} g(t)\,dt \tag{3.6}$$

*converge for $s > \sigma$. Then there exists one and only one non-negative solution $u(t)$ of (1.1) which is bounded in every finite interval*[10]. *With this function the integral*

$$\omega(s) = \int_0^\infty e^{-st} u(t)\,dt \tag{3.7}$$

*converges at least for $s > \sigma'$, where $\sigma' = \sigma$ if $\lim_{s \to \sigma+0} \varphi(s) \le 1$, and otherwise $\sigma' > \sigma$ is defined as the root of the characteristic equation $\varphi(s) = 1$. For $s > \sigma'$ equation (3.4) holds.*

*If $f(t)$ is continuous except, perhaps, at a finite number of points then $u(t) - g(t)$ is continuous.*

PROOF: Define $F(t)$ and $G(t)$ by (1.2). Under the present conditions these functions satisfy the conditions of Theorem 1, and hence (1.3) has a non-decreasing solution $U(t)$. Consider, then, an arbitrary interval $0 \le t \le T$ and suppose that in this interval $f(t) < M$ and $g(t) < M$. If $0 \le t < t + h \le T$ we have by (1.3)

$$\begin{aligned}
0 &\le \frac{1}{h}\{U(t+h) - U(t)\} \\
&= \frac{1}{h}\{G(t+h) - G(t)\} + \frac{1}{h}\int_t^{t+h} U(t+h-x) f(x)\,dx \\
&\qquad + \frac{1}{h}\int_0^t \{U(t+h-x) - U(t-x)\} f(x)\,dx \\
&\le M + MU(T) + \frac{M}{h}\int_0^t \{U(t+h-x) - U(t-x)\}\,dx \\
&= M + MU(T) + \frac{M}{h}\int_t^{t+h} U(y)\,dy - \frac{M}{h}\int_0^h U(y)\,dy \\
&< M + 2MU(T).
\end{aligned}$$

Thus $U(t)$ has bounded difference ratios and is therefore an integral. The derivative $U'(t)$ exists for almost all $t$ and $0 \le U'(t) \le M$. Accordingly we can differentiate (1.3) formally, and since $U(0) = 0$ it follows that $u(t) = U'(t)$ satisfies (1.1) for almost all $t$. However, changing $u(t)$ on a set of measure zero does not affect the integral in (1.1), and since $g(t)$ is defined for all $t$ it is seen that

[10] Without the assumptions of positiveness and boundedness this theorem reduces to a special case of a theorem by Paley and Wiener [21]; cf. section 1, p. 243.

$u(t)$ can be defined, in a unique way, so as to satisfy (1.1) and obtain (1.3). Since the solution of (1.3) was uniquely determined it follows that the solution $u(t)$ is also unique. Obviously equations (3.7) and (3.3) define the same function $\omega(s)$, so that (3.4) holds, and (3.7) converges for $s > \sigma'$.

Finally, if $f(t)$ has only a finite number of jumps, the continuity of $u(t) - g(t)$ becomes evident upon writing (1.1) in the form

$$u(t) - g(t) = \int_0^t u(x) f(t - x)\, dx.$$

4. **Asymptotic properties.** In this section we shall be concerned with the asymptotic behavior as $t \to \infty$ not of $u(t)$ itself but of the mean value $u^*(t) = \frac{1}{t}\int_0^t u(\tau)\, d\tau$. If $u(t)$ tends to the (not necessarily finite) limit $C$, then obviously also $u^*(t) \to C$, whereas the converse is not necessarily true. For the proof of the theorem we shall need the following obvious but useful

LEMMA: *If $u(t) \geq 0$ is a solution of (1.1) and if*

$$(4.1) \qquad u_1(t) = e^{kt} u(t), \qquad f_1(t) = e^{kt} f(t), \qquad g_1(t) = e^{kt} g(t),$$

*then $u_1(t)$ is a solution of*

$$u_1(t) = g_1(t) + \int_0^t u_1(t - x) f_1(x)\, dx.$$

THEOREM 3: *Suppose that using the functions defined in Theorem 2 the integrals*

$$(4.2) \qquad \int_0^\infty f(t)\, dt = a, \qquad \int_0^\infty g(t)\, dt = b,$$

*are finite.*

(i) *In order that*

$$(4.3) \qquad u^*(t) = \frac{1}{t}\int_0^t u(\tau)\, d\tau \to C$$

*as $t \to \infty$, where $C$ is a positive constant, it is necessary and sufficient that $a = 1$, and that the moment,*

$$(4.4) \qquad \int_0^\infty t\, f(t)\, dt = m$$

*be finite. In this case*

$$(4.5) \qquad C = \frac{b}{m}.$$

(ii) *If $a < 1$ we have*

$$(4.6) \qquad \int_0^\infty u(t)\, dt = \frac{b}{1 - a}.$$

(iii) *If* $a > 1$ *let* $\sigma'$ *be the positive root of the characteristic equation* $\varphi(s) = 1$ (cf. (3.2)) *and put*[11]

$$\int_0^\infty e^{-\sigma' t}\, t\, f(t)\, dt = m_1. \tag{4.7}$$

*Then*

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t e^{-\sigma'\tau} u(\tau)\, d\tau = \frac{b}{m_1}. \tag{4.8}$$

REMARK: The case $a = 1$ corresponds in demography to a population of stationary size. In the theory of industrial replacement only the case $a = 1$ occurs; the moment $m$ is the average lifetime of an individual. The case $a > 1$ corresponds in demography to a population in which the fertility is greater than the mortality. As is seen from (4.8), in this case the mean value of $u(t)$ increases exponentially. It is of special interest to note that in a population with $a < 1$ the integral (4.6) always converges.

PROOF: By (4.2) and (3.7)

$$\lim_{s\to+0} \phi(s) = a, \qquad \lim_{s\to+0} \gamma(s) = b. \tag{4.9}$$

If $a < 1$, it follows from (3.4) that $\lim \omega(s) = b/(1 - a)$ is finite. Since $u(t) \geq 0$ this obviously implies that (4.6) holds. This proves (ii).

If $a = 1$ and $m$ is finite, it is readily seen that

$$\lim_{s\to+0} \frac{1 - \varphi(s)}{s} = m,$$

and hence by (3.4)

$$\lim_{s\to+0} s\omega(s) = \lim_{s\to+0} \gamma(s) \lim_{s\to+0} \frac{s}{1 - \varphi(s)} = \frac{b}{m}.$$

By a well-known Tauberian theorem for Laplace integrals of non-negative functions[12] it follows that $u^*(t) \to \frac{b}{m}$. Conversely, if (4.3) holds it is readily seen that[13]

[11] (4.2) implies the finiteness of $m_1$.

[12] Cf. e.g. Doetsch [18], p. 208 or 210.

[13] Indeed, if (4.3) holds and if $U(t)$ is defined by (1.2), then there is a $M = M(\epsilon)$ such that $| U(t) - Ct | < M + \epsilon t$. Now

$$\varphi(s) = s \int_0^\infty e^{-st} U(t)\, dt,$$

and hence

$$s\varphi(s) - C = s^2 \int_0^\infty e^{-st}(U(t) - Ct)\, dt,$$

or

$$| s\varphi(s) - C | \leq s^2 \int_0^\infty e^{-st}(M + \epsilon t)\, dt = sM + \epsilon.$$

$$\lim_{s \to +0} s\omega(s) = C,$$

which in turn implies by (3.4) and (4.9) that

$$\lim_{s \to +0} \frac{1 - \varphi(s)}{s} = \frac{b}{C}.$$

This obviously means that the moment (4.4) exists and equals $b/C$. This proves (i).

Finally case (iii) reduces immediately to (ii) using the above lemma with $k = -\sigma'$. This finishes the proof.

It may be remarked that the finiteness of the integrals (4.2) is by no means necessary for (4.3). This is shown by the following

EXAMPLE: Let

$$f(t) = \frac{1}{2\sqrt{\pi}\, t^{3/2}} e^{-1/4t}, \qquad g(t) = \frac{1}{\sqrt{\pi t}} e^{-1/4t}.$$

It is readily seen that with these functions $a = 1$, but $b = \infty$. Now[14] $\varphi(s) = e^{-\sqrt{s}}$ and $\gamma(s) = e^{-\sqrt{s}}/\sqrt{s}$, so that

$$\omega(s) = \frac{e^{-\sqrt{s}}}{\sqrt{s}\,(1 - e^{-\sqrt{s}})}.$$

Thus $s\omega(s) \to 1$ as $s \to +0$, and hence $u^*(t) \to 1$. In this particular case it can even be shown that the solution $u(t)$ itself tends to 1 as $t \to \infty$.

In practice, however, the integrals (4.2) will always exist, and accordingly we restrict the consideration to this case.

**5. Closer study of asymptotic properties.** In this section we shall deal almost exclusively with the most important special case, namely where

$$\int_0^\infty f(t)\, dt = 1. \tag{5.1}$$

The question has been much discussed whether in this case necessarily $u(t) \to C$ as $t \to \infty$, which statement, if true, would be a refinement of (4.3). Hadwiger [2] has constructed a rather complicated example to show that $u(t)$ does not necessarily approach a limit. Now this can also be seen directly and without any computations. Indeed, if $u(t) \to C$ and if (5.1) holds, then obviously

$$\lim_{t \to \infty} \int_0^t u(t - x) f(x)\, dx = C,$$

and hence it follows from (1.1) that $g(t) \to 0$. In order that $u(t) \to C$ it is therefore

[14] The integrals can be evaluated by elementary methods, and are known; cf. e.g. Doetsch [18], p. 25.

necessary that $g(t) \to 0$, and this proves the assertion. In Hadwiger's example $\limsup g(t) = \infty$, which makes his computations unnecessary.

It can be shown in a similar manner that not even the condition $g(t) \to 0$ is sufficient to ensure that $u(t) \to C$. Some restriction as to the total variation of $f(t)$ seems both necessary and natural (conditions on the existence of derivatives are not sufficient). In the following theorem we shall prove the convergence of $u(t)$ under a condition which is, though not strictly necessary, sufficiently wide to cover all cases of any possible practical interest.

THEOREM 4: *Suppose that with the functions $f(t)$ and $g(t)$ of Theorem 2*

$$\int_0^\infty f(t)\,dt = 1, \qquad \int_0^\infty g(t)\,dt = b < \infty. \tag{5.2}$$

*Suppose moreover that there exists an integer $n \geqq 2$ such that the moments*

$$m_k = \int_0^\infty t^k f(t)\,dt, \qquad k = 1, 2, \cdots, n, \tag{5.3}$$

*are finite, and that the functions $f(t)$, $tf(t)$, $t^2 f(t)$, $\cdots$, $t^{n-2}f(t)$ are of bounded total variation over $(0, \infty)$. Suppose finally that*

$$\lim_{t\to\infty} t^{n-2} g(t) = 0 \quad \textit{and} \quad \lim_{t\to\infty} t^{n-2}\int_t^\infty g(x)\,dx = 0. \tag{5.4}$$

*Then*

$$\lim_{t\to\infty} u(t) = \frac{b}{m_1} \tag{5.5}$$

*and*

$$\lim_{t\to\infty} t^{n-2}\left\{u(t) - \frac{b}{m_1}\right\} = 0. \tag{5.6}$$

REMARK: As it was shown in section 4, the case where $\int_0^\infty f(t)\,dt > 1$ can readily be reduced to the above theorem by applying the lemma of section 4 with $k = \sigma'$, where $\sigma'$ is the positive root of $\varphi(s) = 1$: it is only necessary to suppose that $e^{-\sigma' t} f(t)$ is of bounded total variation and that $e^{-\sigma' t} g(t) \to 0$. Obviously all moments of $e^{-\sigma' t} f(t)$ exist, so that the above theorem shows that $u_1(t) = e^{-\sigma' t} u(t)$ tends to the finite limit $b'/m_1'$, where

$$b' = \int_0^\infty e^{-\sigma' t} g(t)\,dt, \qquad m_1' = \int_0^\infty e^{-\sigma' t}\, t f(t)\,dt.$$

Thus in this case and under the above assumptions $u(t) \sim \dfrac{b'}{m_1'} e^{\sigma' t}$, so that the renewal function increases exponentially as could be expected. If however

$$\int_0^\infty f(t)\,dt < 1,$$

$u(t)$ will in general *not* show an exponential character. If $f(t)$ is of bounded variation and has a finite moment of second order, and if $g(t) \to 0$, then it can be shown that $u(t) \to 0$. However, the lemma of section 4 can be applied only if the integral defining $\varphi(s)$ converges in some negative $s$-interval containing a value $s'$ such that $\varphi(s') = 1$, and this is in general not the case.

PROOF: The proof of Theorem 4 will be based on a Tauberian theorem due to Haar[15]. With some specializations and obvious changes this theorem can be formulated as follows.

Suppose that $l(t)$ is, for $t \geq 0$, non-negative and continuous, and that the Laplace integral

$$\lambda(s) = \int_0^\infty e^{-st} l(t)\, dt \tag{5.7}$$

converges for $s > 0$. Consider $\lambda(s)$ as a function of the complex variable $s = x + iy$ and suppose that the following conditions are fulfilled:

(i) For $y \neq 0$ the function $\lambda(s)$ (which is always regular for $x > 0$) has continuous boundary values $\lambda(iy)$ as $x \to +0$, for $x \geq 0$ and $y \neq 0$

$$\lambda(s) = \frac{C}{s} + \psi(s), \tag{5.8}$$

where $\psi(iy)$ has finite derivatives $\psi'(iy), \cdots, \psi^{(r)}(iy)$ and $\psi^{(r)}(iy)$ is bounded in every finite interval;

$$\int_{-\infty}^{+\infty} e^{ity} \lambda(x + iy)\, dy \tag{ii}$$

converges for some fixed $x > 0$ uniformly with respect to $t \geq T > 0$;

(iii) $\lambda(x + iy) \to 0$ as $y \to \pm\infty$, uniformly with respect to $x \geq 0$;

(iv) $\lambda'(iy), \lambda''(iy), \cdots, \lambda^{(r)}(iy)$ tend to zero as $y \to \pm\infty$;

(v) The integrals

$$\int_{-\infty}^{y_1} e^{ity} \lambda^{(r)}(iy)\, dy \quad \text{and} \quad \int_{y_2}^{\infty} e^{ity} \lambda^{(r)}(iy)\, dy$$

(where $y_1 < 0$ and $y_2 > 0$ are fixed) converge uniformly with respect to $t \geq T > 0$.

Under these conditions

$$\lim t^r \{l(t) - C\} = 0. \tag{5.9}$$

Now the hypotheses of this theorem are too restrictive to be applied to the solution $u(t)$ of (1.1). We shall therefore replace (1.1) by the more special equation

$$v(t) = h(t) + \int_0^t v(t - x) f(x)\, dx, \tag{5.10}$$

[15] Haar [20] or Doetsch [18], p. 269.

where

$$h(t) = \int_0^t f(t - x) f(x)\, dx. \tag{5.11}$$

Plainly Theorem 2 can be applied to (5.10). It is also plain that $h(t)$ is bounded and non-negative and that (by (5.1))

$$\int_0^\infty h(t)\, dt = 1, \tag{5.12}$$

$$\chi(s) \equiv \int_0^\infty e^{-st} h(t)\, dt = \varphi^2(s). \tag{5.13}$$

Accordingly we have by Theorem 2

$$\zeta(s) \equiv \int_0^\infty e^{-st} v(t)\, dt = \frac{\varphi^2(s)}{1 - \varphi(s)}. \tag{5.14}$$

We shall first verify that $\zeta(s)$ satisfies the conditions of Haar's theorem with $r = n - 2$. For this purpose we write

$$f(t) = f_1(t) - f_2(t), \tag{5.15}$$

where $f_1(t)$ and $f_2(t)$ are non-decreasing and non-negative functions which are, by assumption, bounded:

$$0 \le f_1(t) < M, \qquad 0 \le f_2(t) < M. \tag{5.16}$$

(a) We show that $v(t)$ is continuous. Now by Theorem 2 the solution $v(t)$ of (5.10) is certainly continuous if $h(t)$ is continuous; however, that $h(t)$ is continuous follows directly from (5.11) and the fact that the functions

$$\int_0^t f_1(t - x) f(x)\, dx \quad \text{and} \quad \int_0^t f_2(t - x) f(x)\, dx$$

are continuous.

(b) In view of (5.1) the function $\varphi(s)$ exists for $x = \Re(s) \ge 0$. Obviously $|\varphi(x + iy)| < 1$ for $x > 0$. Now

$$\begin{aligned} 1 - \varphi(iy) &= \int_0^\infty (1 - e^{-iyt}) f(t)\, dt \\ &= \int_0^\infty (1 - \cos yt) f(t)\, dt + i \int_0^\infty \sin yt \cdot f(t)\, dt, \end{aligned}$$

and, since $1 - \cos yt \ge 0$ and $f(t) \ge 0$, the equality $\varphi(iy) = 1$ for $y \ne 0$ would imply that $f(t) = 0$ except on a set of measure zero. It is therefore seen that $\varphi(x + iy) \ne 1$ for all $x > 0$ and for $x = 0, y \ne 0$.

It follows furthermore from (5.3) that for $k = 1, \cdots, n$ and $x \ge 0$ the derivatives

$$\varphi^{(k)}(s) = \int_0^\infty (-t)^k e^{-st} f(t)\, dt$$

exist and that

$$\lim_{x\to+0} \varphi^{(k)}(x+iy) = \varphi^{(k)}(iy).$$

Finally, it is readily seen that in the neighborhood of $y = 0$ we have

$$\varphi(iy) = \int_0^\infty e^{-iyt} f(t)\, dt$$

$$(5.17) \qquad = 1 - m_1 iy + \frac{m_2}{2}(iy)^2 - + \cdots$$

$$+ (-1)^{n-1} \frac{m_{n-1}}{(n-1)!}(iy)^{n-1} + O(|y|^n).$$

(c) From what was said under (b) it follows by (5.14) that $\zeta(s)$ is regular for $x > 0$, and that $\zeta(s), \zeta'(s), \cdots, \zeta^{(n)}(s)$ approach continuous boundary values as $s = x + iy$ approaches a point of the imaginary axis other than the origin. Now put

$$(5.18) \qquad \psi(s) = \frac{\varphi^2(s)}{1-\varphi(s)} - \frac{1}{m_1 s},$$

so that by (5.14)

$$(5.19) \qquad \zeta(s) = \frac{1}{m_1 s} + \psi(s).$$

For $x > 0$ and $x = 0$, $y \neq 0$ the function $\psi(x + iy)$ is obviously continuous; the derivatives $\psi'(iy), \cdots, \psi^{(n)}(iy)$ exist. To investigate the behavior of $\psi(iy)$ in the neighborhood of $y = 0$ put

$$(5.20) \qquad P(y) = m_1 - \frac{m_2}{2}(iy) + - \cdots - (-1)^{n-1}\frac{m_{n-1}}{(n-1)!}(iy)^{n-2}.$$

By (5.17), (5.18) and (5.20)

$$(5.21) \qquad \psi(iy) = \left[\frac{\{1 - iyP(y)\}^2}{P(y)} - \frac{1}{m_1}\right]\frac{1}{iy} + O(|y|^{n-2}).$$

Now the expression in brackets represents an analytic function of $y$ which vanishes at $y = 0$. Hence $\psi(iy) = \mathfrak{P}(y) + O(|y|^{n-2})$, where $\mathfrak{P}(y)$ denotes a power series. It follows that the derivatives $\psi'(iy), \cdots, \psi^{(n-2)}(iy)$ exist for all real $y$ (including $y = 0$) and are bounded for sufficiently small $|y|$: since they are continuous functions they are bounded in every finite interval.

(d). Next we show that there exists a constant $A > 0$ such that for sufficiently large $|y|$

$$(5.22) \qquad |\varphi(x + iy)| < \frac{A}{|y|}$$

uniformly in $x \geq 0$. By (5.15)

$$(5.23) \qquad \varphi(s) = \int_0^\infty \{\cos yt - i \sin yt\} e^{-xt}\{f_1(t) - f_2(t)\}\, dt.$$

Now $f_1(t)$ is non-decreasing and accordingly by the second mean-value theorem we have for any $T > 0$ and $y$

$$\int_0^T \cos yt \cdot f_1(t)\, dt = f_1(T) \int_\tau^T \cos yt\, dt = f_1(T) \frac{\sin Ty - \sin \tau y}{y},$$

where $\tau$ is some value between 0 and $T$ (depending, of course, on $y$; at points of discontinuity, $f_1(T)$ should be replaced by $\lim_{t \to T-0} f_1(t)$). Hence by (5.16)

$$\left| \int_0^\infty \cos yt \cdot e^{-xt} \cdot f_1(t)\, dt \right| < \frac{2M}{|y|}.$$

Treating the other terms in (5.23) in a like manner, (5.22) follows.

Combining (5.22) with (5.14) it is seen that for sufficiently large $|y|$

$$|\zeta(s)| < \frac{2A^2}{y^2}$$

uniformly in $x \geq 0$. This shows that the assumptions (ii) and (iii) of Haar's theorem are satisfied for $\lambda(s) = \zeta(s)$. In order to prove that also conditions (iv) and (v) are satisfied it suffices to notice that the proof of (5.22) used only the fact that $f(t)$ is of bounded total variation. Now $\varphi^{(k)}(s)$ is the Laplace transform of $(-t)^k f(t)$, and, since $t^k f(t)$ is of bounded total variation for $k \leq n - 2$, it follows that

$$|\varphi^{(k)}(s)| = O(|y|^{-1}), \qquad k = 1, 2, \cdots, n - 2,$$

for sufficiently large $|y|$, uniformly in $x \geq 0$. Differentiating (5.14) $k$ times it is also seen that

$$|\zeta^{(k)}(s)| = O(|y|^{-2}), \qquad k = 1, 2, \cdots, n - 2,$$

as $y \to +\infty$, uniformly with respect to $x \geq 0$.

This enumeration shows that $v(s) = l(t)$ and $\lambda(s) = \zeta(s)$ satisfy all hypotheses of Haar's theorem with $r = n - 2$ and $C = 1/m_1$. Hence

$$\lim_{t \to \infty} t^{k-2} \left\{ v(t) - \frac{1}{m_1} \right\} = 0. \tag{5.24}$$

Returning now to (5.14) we get

$$\omega(s) = \gamma(s) + \gamma(s)\varphi(s) + \gamma(s)\zeta(s),$$

or, by the uniqueness property of Laplace integrals,

$$\begin{aligned} u(t) &= g(t) + \int_0^t g(x) f(t - x)\, dx + \int_0^t g(x) v(t - x)\, dx \\ &= g(t) + u_1(t) + u_2(t) \end{aligned} \tag{5.25}$$

(which relation can also be checked directly using (5.10)). Let us begin with the last term. We have by (5.2)

$$u_2(t) - \frac{b}{m_1} \equiv \int_0^t g(t - x)\left\{v(x) - \frac{1}{m_1}\right\} dx,$$

and hence

$$t^{n-2}\left|u_2(t) - \frac{b}{m_1}\right| \leq 2^{n-2} \int_{t/2}^t g(t - x)x^{n-2}\left|v(x) - \frac{1}{m_1}\right| dx$$
$$+ t^{n-2} \int_{t/2}^t g(y)\left|v(t - y) - \frac{1}{m_1}\right| dy.$$

If $t$ is sufficiently large we have by (5.24) in the first integral $x^{n-2}\left|v(x) - \frac{1}{m_1}\right| < \epsilon$. In the second integral $v(t - y) - \frac{1}{m_1}$ is bounded, and hence by (5.4)

$$\lim_{t\to\infty} t^{n-2}\left|u_2(t) - \frac{b}{m_1}\right| = 0.$$

The same argument applies (even with some simplifications) also to the second term in (5.24); it follows that

$$\lim t^{n-2}u_1(t) = 0,$$

whilst $t^{n-2}g(t) \to 0$ by assumption (5.4). Now the assertion (5.6) of our theorem follows in view of (5.25) if the last three relationships are added. This finishes the proof of Theorem 4.

It seems that the solution $u(t)$ is generally supposed to oscillate around its limit $b/m_1$ as $t \to \infty$. It goes without saying that such a behavior is a priori more likely than a monotone character. It should, however, be noticed that there is no reason whatsoever to suppose that $u(t)$ *always* oscillates around its limit. Again no computation is necessary to see this, as shown by the following

EXAMPLE: Differentiating (1.1) formally we get

$$u'(t) = g'(t) + g(0)f(t) + \int_0^t u'(t - x)f(x)\, dx,$$

which shows that, if $g(t)$ and $f(t)$ are sufficiently regular, $u'(t)$ satisfies an integral equation of the same type as $u(t)$. Thus if

$$g'(t) + g(0)f(t) \geq 0$$

for all $t$, we shall have $u'(t) \geq 0$, and $u(t)$ is a monotone function. In particular, if $g'(t) + g(0)f(t) = 0$, then $u'(t) = 0$ and $u(t) = \text{const}$. For example, let $f(t) = g(t) = e^{-t}$. Then $\varphi(s) = \gamma(s) = 1/(s + 1)$ and hence $\omega(s) = 1/s$, which is the Laplace transform of $u(t) = 1$. It is also seen directly that $u(t) \equiv 1$ is the solution. We have however the following

THEOREM 5[16]: *If the functions $f(t)$ and $g(t)$ of Theorem 4 vanish identically for $t \geq T > 0$, then the solution $u(t)$ of* (1.1) *oscillates around its limit $b/m$ as $t \to \infty$.*

[16] Under some slight additional hypotheses and with quite different methods this theorem was proved by Richter [16].

PROOF: For $t \geq T$ equation (1.1) reduces to

$$u(t) = \int_{t-T}^{t} u(t-x)f(x)\,dx,$$

and since $\int_{t-T}^{t} f(x)\,dx = 1$ it follows that the maxima of $u(t)$ in the intervals $nT < t < (n+1)T$ form, for sufficiently large integers $n$, a non-increasing sequence. Similarly the corresponding minima do not decrease. Since $u(t) \to b/m_1$, by Theorem 4, it follows that the minima do not exceed $b/m_1$ and the maxima are not smaller than $b/m_1$.

6. **On Lotka's method.** Probably the most widely used method for treating equation (1.1) in connection with problems of the renewal theory is Lotka's method. As a matter of fact this method consists of two independent parts. The first step aims at obtaining the exact solution of (1.1) in the form of a series of exponential terms (this is achieved by an adaptation of a method which was used by P. Herz and Herglotz for other purposes. The second part of Lotka's theory consists of devices for a convenient approximative computation of the first few terms of the series. While restricting ourselves formally to Lotka's theory, it will be seen that some of the following remarks apply equally to other methods.

Lotka's method rests essentially on the fundamental assumption that the characteristic equation

$$\varphi(s) = 1 \tag{6.1}$$

has infinitely many distinct simple[17] roots $s_0, s_1, \cdots$, and that the solution $u(t)$ of (1.1) can be expanded into a series

$$u(t) = \sum_k A_k e^{s_k t} \tag{6.2}$$

where the $A_k$ are complex constants. The argument usually rests on an assumed completeness-property of the roots. Thus, starting from (2.4) it is required that (6.2) reduces to $h(t)$ for $t < 0$; in other words, that an arbitrarily prescribed function $h(x)$ be, for $x < 0$, respresentable in the form

$$h(x) = \sum_k A_k e^{s_k x} \qquad (x < 0). \tag{6.3}$$

In practice we are, of course, usually not concerned with $h(t)$ but with $g(t)$ (cf. (2.5)), and according to Lotka's theory the coefficients $A_k$ of the solution (6.2) of (1.1) can be computed directly from $g(t)$ in a way similar to the computation of the Fourier coefficients.

Lotka's method is known to lead to correct results in many cases and also to

[17] Hadwiger [3] objected to the assumption that all roots of (6.1) be simple. The modifications which are necessary to cover the case of multiple roots also will be indicated below.

have distinct computational merits. On the other hand it seems to require a safer justification, since its fundamental assumptions are rarely realized. Thus clearly an arbitrary function $h(x)$ cannot be represented in the form (6.3): to see this it suffices to note that (6.1) frequently has only a finite number of roots (cf. also below). It should also be noted that, the series (6.3) having regularity properties as are assumed in Lotka's theory, any function representable in the form (6.3) is necessarily a solution of the integral equation (2.4), whereas the theory requires us to construct a solution $u(t)$ which reduces to an *arbitrarily* prescribed function $h(t)$ for $t < 0$, (which frequently is an empirical function, determined by observations). Nevertheless, it is possible to give sound foundations to Lotka's method so that it can be used (with some essential limitations and modifications) sometimes even in cases for which it originally was not intended. For this purpose it turns out to be necessary that all considerations be based on the more general equation (1.1), instead of (2.4) (cf. also section 2).

Before proceeding it is necessary to make clear *what is really meant by a root of* (6.1). The function $\varphi(s)$ is defined by (3.2), and the integral will in general converge only for $s$-values situated in the half-plane $\Re(s) > \sigma$. Usually only roots situated in this half-plane are considered[18]. It is also argued that $\varphi(s)$ is, for real $s$, a monotone function, so that (6.1) has at most one real root: accordingly the terms of (6.2) are called "oscillatory components." However, the function $\varphi(s)$ can usually be defined by analytic continuation even outside the half-plane $\Re(s) > \sigma$, and, if this is done, (6.1) will in general also have roots in the half-plane $\Re(s) < \sigma$. It will be seen in the sequel that these roots play exactly the same role for the solution $u(t)$ as the other ones, and that the applicability of Lotka's method depends on the behavior of $\varphi(s)$ in the entire complex $s$-plane. It may be of interest to quote an example where (6.1) has infinitely many real and no other roots.

Example[19]: Let

$$f(t) = \frac{1}{2\sqrt{\pi}\, t^{3/2}} e^{-1/4t}, \qquad t > 0; \tag{6.4}$$

---

[18] This was stated in particular by Hadwiger [3] and Hadwiger and Ruchti [6]; accordingly the results of the latter paper (obtained by methods quite different from Lotka's) need some modifications.

[19] Cf. the example at the end of section 4. A function closely related to (6.4) plays an important role in two recent papers by Hadwiger [4] and [5]. Hadwiger's conclusion, if it could be justified, would fundamentally change the aspect of the whole theory. The conclusion reached by Hadwiger seems to be that for any biological population the reproduction function should be of the form $u(t) = \Sigma u_n(t)$, where $u_n(t)$ represents the contribution of the $n$th generation and

$$u_n(t) = \frac{an}{\sqrt{\pi}\, t^{3/2}} e^{-At+Can-n^2a^2/t}. \tag{*}$$

Here $a$, $A$ and $C$ are constants. Clearly (*) is a generalization of (6.4). Now his conclusion is based on the arbitrary assumption that $u_n(t)$ should be of the form $u_n(t) = \psi(x, na)$

It is easily seen that $\varphi(s) = e^{-\sqrt{s}}$. The integral (3.2) converges only for $\Re(s) \geq 0$, but $\varphi(s)$ is defined as a two-valued function in the entire $s$-plane. The roots of (6.1) are obviously $s_k = -4\, k^2\pi^2$, so that all of them are real and simple. If $g(t) = f(t)$, we get by (3.4)

$$\omega(s) = \frac{e^{-\sqrt{s}}}{1 - e^{-\sqrt{s}}} = \sum_1^\infty e^{-n\sqrt{s}}, \qquad s \text{ real}, > 0.$$

Now $e^{-n\sqrt{s}}$ is the Laplace transform of $\frac{n}{2\sqrt{\pi}\, t^{3/2}} e^{-n^2/4t}$, and hence it is readily seen that the solution $u(t)$ can be written in the form

$$u(t) = \frac{1}{2\sqrt{\pi}\, t^{3/2}} \sum_1^\infty n e^{-n^2/4t}; \tag{6.5}$$

of course, this expansion is not of form (6.2) and shows no oscillatory character.

From now on we shall consistently denote by $\varphi(s)$ the function defined by the integral (3.4) and by the usual process of analytic continuation; accordingly we shall take into consideration *all* roots of (6.1). The main limitation of Lotka's theory can then be formulated in the following way: Lotka's method depends only on the function $g(t)$ and on the roots of (6.1). Now two different functions $f(t)$ can lead to characteristic equations having the same roots. Lotka's method would be applicable to both only if the corresponding two integral equations (1.1) had the same solution $u(t)$. This, however, is not necessarily the case. Thus, if Lotka's method is applied, and if all computations are correctly performed, and if the resulting series for $u(t)$ converges uniformly, there is no possibility of telling which equation is really satisfied by the resulting $u(t)$: it can happen that one has unwittingly solved some unknown equation of type (1.1) which, by chance, leads to a characteristic equation having the same roots as the characteristic equation of the integral equation with which one was really concerned. Indeed this happens in the following example which is familiar in connection with our problem. It is illustrative also for other purposes: thus it shows not only limitations of Lotka's method, but also that this method can be modified so as to become applicable in some cases where the characteristic equation has only a finite number of roots.

---

where $\psi(x, a)$ is independent of $n$. To my mind Hadwiger's result shows only the impractibility of this axiom. However, Hadwiger's result is not correct even under his assumption. Indeed, he derives for $\psi(x, a)$ the functional equation

$$\psi(x, a + b) = \int_0^t \psi(x - \xi, a)\psi(\xi, b)\, d\xi, \tag{**}$$

which is well-known from the theory of stochastic processes. Now Hadwiger merely verifies the known result that (*) leads to a solution of (**). However, (**) has infinitely many other solutions (it is possible to write down expressions for their Laplace transforms, although it is difficult to express the solutions themselves explicitly). This, of course, renders Hadwiger's result illusory.

EXAMPLE: *Pearson type III-curves.*[20] Consider the integral equation (1.1) in the following two cases:

$$\text{(I)} \qquad f(t) = g(t) = f_{\mathrm{I}}(t) = \frac{1}{\Gamma(\frac{3}{2})} t^{1/2} e^{-t}$$

and

$$\text{(II)} \qquad f(t) = g(t) = f_{\mathrm{II}}(t) = \tfrac{1}{2} t^2 e^{-t}.$$

It is readily seen (and well known) that the corresponding Laplace transforms are

$$\text{(I)} \qquad \varphi_{\mathrm{I}}(s) = \frac{1}{(s+1)^{3/2}}$$

and

$$\text{(II)} \qquad \varphi_{\mathrm{II}}(s) = \frac{1}{(s+1)^3},$$

respectively. Thus in both cases the characteristic equation has the same roots, namely

$$s_1 = 0, \qquad s_{2,3} = -\frac{3}{2} \pm \frac{i}{2}\sqrt{3},$$

of which only the first one lies in the half-plane of convergence of the integral (3.4). Lotka's method is not applicable since there are only three roots. However, in the second case, an expansion of type (6.2) is possible. Indeed, we have by (3.4)

$$\begin{aligned}\omega_{\mathrm{II}}(s) = \frac{\varphi_{\mathrm{II}}(s)}{1 - \varphi_{\mathrm{II}}(s)} &= \frac{1}{s^3 + 3s^2 + 3s} \\ &= \frac{1}{3s} - \frac{\frac{1}{6} - \frac{i}{2\sqrt{3}}}{s + \frac{3}{2} - \frac{i}{2}\sqrt{3}} - \frac{\frac{1}{6} + \frac{i}{2\sqrt{3}}}{s + \frac{3}{2} + \frac{i}{2}\sqrt{3}};\end{aligned}$$

now $1/(s + a)$ is the Laplace transform of $e^{-at}$, and hence we obtain the solution $u(t)$ in the form

$$\begin{aligned}u_{\mathrm{II}}(t) &= \frac{1}{3} - \left(\frac{1}{6} - \frac{i}{2\sqrt{3}}\right) e^{\frac{1}{2}(-3+i\sqrt{3})t} - \left(\frac{1}{6} + \frac{i}{2\sqrt{3}}\right) e^{\frac{1}{2}(-3-i\sqrt{3})t} \\ &= \frac{1}{3} - \frac{1}{3} e^{-3t/2} \cos \frac{\sqrt{3}}{2} t - \frac{1}{\sqrt{3}} e^{-3t/2} \sin \frac{\sqrt{3}}{2} t,\end{aligned}$$

[20] General Pearson curves have been investigated recently in connection with (1.1) by Brown [1], Hadwiger and Ruchti [6] and Rhodes [15]. Hadwiger and Ruchti use a method of their own, but they are also led to the study of the characteristic equation (6.1) in a slightly disguised form: their result needs a modification since they arbitrarily drop the roots lying in the halfplane of divergence of the integral $\varphi(s)$.

which is an expansion of type (6.2). In the first of the above examples we get for real positive $s$

$$\omega_{\mathrm{I}}(s) = \frac{\varphi_{\mathrm{I}}(s)}{1 - \varphi_{\mathrm{I}}(s)} = \sum_{n=1}^{\infty} \frac{1}{(s+1)^{3n/2}},$$

and it is readily seen that this is the Laplace transform of the solution

$$u_{\mathrm{I}}(t) = e^{-t} \sum_{n=1}^{\infty} \frac{1}{\Gamma(3n/2)} t^{3(n-2)/2}.$$

The series is convergent for $t > 0$, but obviously this solution cannot be represented in a form similar to (6.2).

A similar remark applies to the general Pearson-type III curve

$$f(t) = At^{\beta}e^{-\alpha t},$$

where $A, \alpha, \beta$ are positive constants; the corresponding Laplace transform is

$$\varphi(s) = A\Gamma(\beta + 1) \frac{1}{(s + \alpha)^{\beta+1}}.$$

These preparatory remarks enable us to formulate rigorous conditions for the existence of an expansion of type (6.2). The following theorem shows the limits of Lotka's method, but at the same time it also represents an extension of it. In the formulation of the theorem we have considered only the case of absolute convergence of (6.2). This was done to avoid complications lacking any practical significance whatsoever. The conditions can, of course, be relaxed along customary lines.

THEOREM 6: *In order that the solution $u(t)$ of Theorem 2 be representable in form (6.2), where the series converges absolutely for $t \geq 0$ and where the $s_k$ denote the roots of the characteristic equation*[21] *(6.1), it is necessary and sufficient that the Laplace transform $\omega(s)$ admit an expansion*

$$\omega(s) \equiv \frac{\gamma(s)}{1 - \varphi(s)} = \sum \frac{A_k}{s - s_k} \tag{6.6}$$

*and that $\Sigma| A_k |$ converges absolutely. The coefficients $A_k$ are determined by*

$$A_k = -\frac{\gamma'(s_k)}{\varphi(s_k)}. \tag{6.7}$$

*In particular, it is necessary that $\omega(s)$ be a one-valued function.*[22]

PROOF: All roots $s_k$ of (6.1) satisfy the inequality $\Re(s_k) \leq \sigma'$, where $\sigma'$ was defined in Theorem 2. It is therefore readily seen that in case $\Sigma | A_k |$ converges, the Laplace transform of (6.2) can be computed for sufficiently large

[21] The number of roots may be finite or infinite. It should also be noted that it is not required that $s_k \to \infty$. If the $s_k$ have a point of accumulation, $\omega(s)$ will have an essential singularity. That this actually can happen can be shown by examples.

[22] This was not so in our example I.

positive $s$-values by termwise integration so that (6.6) certainly holds for sufficiently large positive $s$. Now with $\Sigma \mid A_k \mid$ converging, (6.6) defines $\omega(s)$ uniquely for all complex $s$ (with singularities at the points $s_k$ and the points of accumulation of $s_k$, if any). Since the analytic continuation is unique, it follows that (6.6) holds for all $s$. The series $\Sigma \mid A_k \mid$ must, of course, converge if (6.2) is to converge absolutely for $t = 0$, and this proves the necessity of our condition. Conversely, if $\omega(s) = \dfrac{\gamma(s)}{1 - \varphi(s)}$ is given by (6.6), and if $\Sigma \mid A_k \mid$ converges, then $\omega(s)$ is the Laplace transform of a function $u(t)$ defined by (6.2). Since the Laplace transform is unique, $u(t)$ is the solution of (1.1) by Theorem 2. The series (6.2) converges absolutely for $t \geq 0$ since $\mid A_k e^{s_k t} \mid \leq \mid A_k \mid e^{\sigma' t}$. Finally (6.7) follows directly from (6.6).

It is interesting to compare (6.7) with formulas (50) and (56) of Lotka's paper [8]. Lotka considers the special case $g(t) = f(t)$; in this case $\gamma(s_k) = \varphi(s_k) = 1$, and (6.7) reduces to $A_k = -\dfrac{1}{\varphi'(s_k)}$. If $s_k$ lies in the domain of convergence of the integral $\varphi(s) = \int_0^\infty e^{-st} f(t)\, dt$, that is, if $\Re(s_k) \geq \sigma$ then

$$\frac{1}{A_k} = \int_0^\infty e^{-st}\, t f(t)\, dt, \tag{6.8}$$

in accordance with Lotka's result. However, (6.8) becomes meaningless for the roots with $\Re(s_k) < \sigma$, whereas (6.7) is applicable in all cases.

Theorem 6 can easily be generalized to the case where the *characteristic equation has multiple roots.* The expansion (6.6) (which reduces to the customary expansion into partial fractions whenever $\omega(s)$ is meromorphic) is to be replaced by

$$\omega(s) = \sum_k \left\{ \frac{A_k^{(1)}}{s - s_k} + \frac{A_k^{(2)}}{(s - s_k)^2} + \cdots + \frac{A_k^{(m_k)}}{(s - s_k)^{m_k}} \right\}, \tag{6.9}$$

where $m_k$ is the multiplicity of the root $s_k$. This leads us formally to an expansion

$$u(t) = \sum_k e^{s_k t} \left\{ A_k^{(1)} + A_k^{(2)} \frac{t}{1!} + \cdots + A_k^{(m_k)} \frac{t^{m_k - 1}}{(m_k - 1)!} \right\}, \tag{6.10}$$

which now replaces (6.2). Generalizing Theorem 6 it is easy to formulate some simple conditions under which (6.11) will really represent a solution of (1.1). Other conditions which ensure that (6.9) is the transform of (6.10) are known from the general theory of Laplace transforms; such conditions usually use only function-theoretical properties of (6.9) and are applicable in particular when $\omega(s)$ is meromorphic. We mention in particular a theorem of Churchill [17] which can be used for our purposes.

7. **On the practical computation of the solution.** There are at hand two main methods for the practical computation of the solution of (1.1). One of them

has been developed by Lotka and consists of an approximate computation of a few coefficients in the series (6.2). The other method uses an expansion

$$(7.1) \qquad u(t) = \sum_{n=0}^{\infty} u_n(t),$$

where $u_n(t)$ represents the contribution of the $n$th "generation" and is defined by $x$

$$(7.2) \qquad u_0(t) = g(t), \qquad u_{n+1}(t) = \int_0^t u_n(t-x)f(x)\,dx.$$

Now the Laplace transform of $u_{n+1}(t)$ is $\gamma(s)\varphi^n(s)$, and hence (7.2) corresponds to the expansion

$$(7.3) \qquad \omega(s) = \frac{\gamma(s)}{1-\varphi(s)} = \gamma(s)\sum_{n=0}^{\infty}\varphi^n(s).$$

In practice the functions $g(t)$ and $f(t)$ are usually not known exactly. Frequently their values are obtained from some statistical material, so that only their integrals over some time units, e.g. years, are actually known or, in other words, only the values

$$(7.4) \qquad f_n = \frac{1}{\delta}\int_{n\delta}^{(n+1)\delta} f(t)\,dt, \qquad g_n = \frac{1}{\delta}\int_{n\delta}^{(n+1)\delta} g(t)\,dt,$$

are given, where $\delta > 0$ is a given constant. Ordinarily in such cases some theoretical forms (e.g. Pearson curves) are fitted to the empirical data and equation (1.1) is solved with these theoretical functions. Now such a procedure is sometimes not only very troublesome, but also somewhat arbitrary. Consider for example the limit of $u(t)$ as $t \to \infty$; this asymptotic value is the main point of interest of the theory and all practical computations. However, as has been shown above, this limit depends only on the moments of the first two orders of $f(t)$ and $g(t)$, and, unless the fitting is done by the method of moments, the resulting value will depend on the special procedure of fitting. Accordingly it will sometimes happen that it is of advantage to use the empirical material as it is, and this can, at least in principle, always be done.

If only the values (7.4) are used it is natural to consider $f(t)$ and $g(t)$ as step-functions defined by

$$(7.5) \qquad \left.\begin{aligned} f(t) &= f_n, \\ g(t) &= g_n, \end{aligned}\right\} \quad \text{for} \quad n\delta \le t < (n+1)\delta.$$

In practice only a finite number among the $f_n$ and $g_n$ will be different from zero: accordingly the Laplace transforms $\gamma(s)$ and $\varphi(s)$ reduce to trigonometrical polynomials, so that the analytic study of $\omega(s) = \dfrac{\gamma(s)}{1-\varphi(s)}$ becomes particularly simple. Lotka's method can be applied directly in this case.

For a convenient computation of (7.1) it is better to return to the more general equation (1.3), instead of (1.1). The summatory functions $F(t)$ and $G(t)$ should not be defined by (1.2) in this case, but simply by

$$F(t) = \sum_{n=0}^{\left[\frac{t}{\delta}\right]} f_n, \qquad G(t) = \sum_{n=0}^{\left[\frac{t}{\delta}\right]} g_n. \tag{7.6}$$

It is readily seen that the solution $U(t)$ of (1.3) can be written in the form $U(t) = \sum_{n=0}^{\infty} U_n(t)$, where

$$U_0(t) = G(t), \qquad U_{n+1}(t) = \int_0^t U_n(t-x)\,dF(x);$$

in our case $U_n(t)$ will again be a step-function with jumps at the points $k\delta$, the corresponding saltus being

$$u_0^{(k)} = g_k, \qquad u_{n+1}^{(k)} = \sum_{r=0}^{k} u_n^{(k-r)} f_r.$$

Thus we arrive at exactly the same result as would have been obtained if the integrals (7.2) had been computed, starting from (7.4), by the ordinary methods for numerical integration of tabulated functions. It is of interest to note that this method of approximate evaluation of the integrals (7.2) leads to the *exact values of the renewal function* of a population where all changes occur in a discontinuous way at the end of time intervals of length $\delta$ in such a way that each change equals the mean value of the changes of the given population over the corresponding time interval.

## REFERENCES

### *I. Papers on the integral equation of renewal theory*

*Note*: Lotka's paper [8] contains a list of 74 papers on the subject published before 1939. The following list is to bring Lotka's list up to June 1941; however no claims to completeness are made.


[1] A. W. Brown, A note on the use of a Pearson type III function in renewal theory," *Annals of Math. Stat.* Vol. 11 (1940), pp. 448–453.

[2] H. Hadwiger, "Zur Frage des Beharrungszustandes bei kontinuierlich sich erneuernden Gesamtheiten," *Archiv f. mathem. Wirtschafts- und Sozialforschung*, Vol. 5 (1939), pp. 32–34.

[3] H. Hadwiger, "Über die Integralgleichung der Bevölkerungstheorie," *Mitteilungen Verein. schweizer Versicherungsmathematiker* (*Bull. Assoc. Actuaires suisses*), Vol. 38 (1939), pp. 1–14.

[4] H. Hadwiger, "Eine analytische Reproduktionsfunktion für biologische Gesamtheiten," *Skand. Aktuarietidskrift* (1940), pp. 101–113.

[5] H. Hadwiger, "Natürliche Ausscheidefunktionen für Gesamtheiten und die Lösung der Erneuerungsgleichung," *Mitteilungen Verein. Schweiz. Versich.-Math.*, Vol. 40 (1940), pp. 31–39.

[6] H. HADWIGER and W. RUCHTI, "Über eine spezielle Klasse analytischer Geburtenfunktionen," *Metron*, Vol. 13 (1939), No. 4, pp. 17–26.

[7] A. LINDER, "Die Vermehrungsrate der stabilen Bevölkerung," *Archiv f. mathem. Wirtschafts- und Sozialforschung*, Vol. 4 (1938), pp. 136–156.

[8] A. LOTKA, "A contribution to the theory of self-renewing aggregates, with special reference to industrial replacement," *Annals of Math. Stat.*, Vol. 10 (1939), pp. 1–25.

[9] A. LOTKA, "On an integral equation in population analysis," *Annals of Math. Stat.*, Vol. 10 (1939), pp. 144–161.

[10] A. LOTKA, "Théorie analytique des associations biologiques II," *Actualités Scientifiques No. 780*, Paris, 1939.

[11] A. LOTKA, "The theory of industrial replacement," *Skand. Aktuarietidskrift* (1940), pp. 1–14.

[12] A. LOTKA, "Sur une équation intégrale de l'analyse démographique et industrielle," *Mitt. Verein. Schweiz. Versich.-Math.*, Vol. 40 (1940), pp. 1–16.

[13] H. MÜNZNER, "Die Erneuerung von Gesamtheiten," *Archiv f. math. Wirtschafts- u. Sozialforschung*, Vol. 4 (1938).

[14] G. A. D. PREINREICH, "The theory of industrial replacement," *Skand. Aktuarietidskrift* (1939), pp. 1–19.

[15] E. C. RHODES, "Population mathematics, I, II, III," *Roy. Stat. Soc. Jour.*, Vol. 103 (1940), pp. 61–89, 218–245, 362–387.

[16] H. RICHTER, "Die Konvergenz der Erneuerungsfunktion," *Blätter f. Versicherungsmathematik*, Vol. 5 (1940), pp. 21–35.

[16a] H. HADWIGER, "Über eine Funktionalgleichung der Bevölkerungstheorie und eine spezielle Klasse analytischer Lösungen," *Bl. f. Versicherungsmathematik*, Vol. 5 (1941), pp. 181–188.

[16b] G. A. D. PRIENREICH, "The present status of renewal theory," Waverly Press, Baltimore (1940).

*II. Other papers quoted*

[17] R. V. CHURCHILL, "The inversion of the Laplace transformation by a direct expansion in series and its application to boundary-value problems," *Math. Zeits.*, Vol. 42 (1937), pp. 567–579.

[18] G. DOETSCH, *Theorie und Anwendung der Laplace Transformation.* J. Springer, Berlin, 1937.

[19] W. FELLER, "Completely monotone functions and sequences," *Duke Math. Jour.*, Vol. 5 (1939), pp. 661–674.

[20] A. HAAR, "Über asymptotische Entwicklungen von Funktionen," *Math. Ann.*, Vol. 96 (1927), pp. 69–107.

[21] R. E. A. C. PALEY and N. WIENER, "Notes on the theory and application of Fourier transforms, VII. On the Volterra equation," *Amer. Math. Soc. Trans.*, Vol. 35 (1933), pp. 785–791.