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# ON THE INTERIOR SPIKE LAYER SOLUTIONS TO A SINGULARLY PERTURBED NEUMANN PROBLEM

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**Abstract.** In this paper, we construct interior spike layer solutions for a class of semilinear elliptic Neumann problems which arise as stationary solutions of Keller-Segel model in chemotaxis and also as limiting equations for the Gierer-Meinhardt system in biological pattern formation. We also classify the locations of single interior peaks. We show exactly how the geometry of the domain affects the spike solutions.

**1. Introduction.** Consider the problem

$$(1.1)\qquad \begin{cases} \varepsilon^2\Delta u-u+u^p=0 & \text{in } \Omega\,,\\ u>0 & \text{in } \Omega\,,\\ \partial u/\partial\nu=0 & \text{on } \partial\Omega\,,\end{cases}$$

where $\Omega\subset R^N$, $\varepsilon>0$, $1<p<(N+2)/(N-2)$ when $N\geq 3$, and $1<p<\infty$ when $N=1, 2$ and $\nu$ is the outward normal vector to $\partial\Omega$.

Equation (1.1) is known as the stationary equation of the Keller-Segel system in chemotaxis. It can also be seen as the limiting stationary equation of the so-called Gierer-Meinhardt system in biological pattern formation. (See [11] for more details.)

In the pioneering papers of [7], [9] and [10], Lin, Ni and Takagi established the existence of least-energy solutions and showed that for $\varepsilon$ sufficiently small the least-energy solution has only one local maximum point $P_\varepsilon$ and $P_\varepsilon\in\partial\Omega$. Moreover, $H(P_\varepsilon)\to\max_{P\in\partial\Omega}H(P)$ as $\varepsilon\to 0$, where $H(P)$ is the mean curvature of $\partial\Omega$ at $P$. Ni and Takagi [11] constructed boundary spike solutions for axially symmetric domains while in [21], the author studied the general domain case. When $p=(N+2)/(N-2)$, similar results for the boundary spike layer solutions have been obtained by [1], [2], [3], [8], [18], etc.

In all the above papers, only *boundary* spike layer solutions are obtained and studied. It remains to see whether or not *interior* spike layer solutions exist for the problem (1.1). In this paper, we shall study this question and give an affirmative answer.

To state our results, we need to introduce some notation.

By the results of [5] and [6], we know that the solution of the problem

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$$(1.2)\qquad \begin{cases} \Delta w - w + w^p = 0 & \text{in } R^N, \\ w > 0, \quad w(z) \to 0 & \text{as } |z| \to \infty, \\ w(0) = \max_{z \in R^N} w(z) \end{cases}$$

is radial and unique. We denote this solution by $w$.

Let $u \in H^1(\Omega)$ and

$$(1.3)\qquad I_\varepsilon(u) = \frac{\varepsilon^2}{2} \int_\Omega |\nabla u|^2 + \frac{1}{2} \int_\Omega u^2 - \frac{1}{p+1} \int_\Omega u^{p+1}.$$

Put

$$(1.4)\qquad I(w) = \frac{1}{2} \int_{R^N} (|\nabla w|^2 + w^2) - \frac{1}{p+1} \int_{R^N} w^{p+1}.$$

We assume that $\Omega$ is $C^3$. Moreover for each $P \in \Omega$, we assume that the set $\bar{B}_{d(P,\partial\Omega)}(P) \cap \partial\Omega$ has only finitely many connected components.

For each $P \in \Omega$, we associate $P$ with the set

$$(1.5)\qquad \Lambda_P = \left\{ d\mu_P \in M(\partial\Omega) \;\middle|\; \begin{array}{l} \text{there exists } \varepsilon_k \to 0 \text{ such that} \\ \dfrac{e^{-|z-P|/\varepsilon_k} dz}{\int_{\partial\Omega} e^{-|z-P|/\varepsilon_k} dz} \to d\mu_P(z) \end{array} \right\},$$

where $M(\partial\Omega)$ is the set of bounded measures on $\partial\Omega$.

For any $d\mu_P(\Omega) \in \Lambda_P$, it is easy to see that $\mathrm{supp}(d\mu_P) \subset \partial\Omega \cap \bar{B}_{d(P,\partial\Omega)}(P)$ when $P \in \Omega$. When $P \in \partial\Omega$, we can see that $d\mu_P = \delta_P$. When $\partial\Omega \cap \bar{B}_{d(P,\partial\Omega)}(P) = \{P_1, \dots, P_l\}$, then $d\mu_P(z) = \sum_{i=1}^l c_i \delta_{P_i}(z)$ with $c_i \geq 0$ and $\sum_{i=1}^l c_i = 1$. (See Appendix D and Appendix E in [20] for more details on this set.)

Our first result is the following:

THEOREM 1.1. *If $u_\varepsilon$ is a solution of* (1.1) *and* $\lim_{\varepsilon \to 0} \varepsilon^{-N} I_\varepsilon(u_\varepsilon) = I(w)$, *then there are two possibilities*:

(i) *$u_\varepsilon$ has only two local maximum points $P_\varepsilon^1$ and $P_\varepsilon^2$ satisfying $P_\varepsilon^1 \in \partial\Omega$, $P_\varepsilon^2 \in \partial\Omega$ and $|P_\varepsilon^1 - P_\varepsilon^2|/\varepsilon \to \infty$ as $\varepsilon \to 0$.*

(ii) *$u_\varepsilon$ has only one local maximum point $P_\varepsilon \in \Omega$ and $d(P_\varepsilon, \partial\Omega)/\varepsilon \to \infty$ as $\varepsilon \to 0$, where $d(P, \partial\Omega)$ denotes the distance from $P$ to $\partial\Omega$.*

*In the second case, if we assume that $\Omega$ is convex, we have $P_\varepsilon \to P \in \Omega$ and there exist $a \in R^n$ and $d\mu_P \in \Lambda_P$ such that*

$$(1.6)\qquad \int_{\partial\Omega} e^{\langle z-P, a \rangle} (z-P) d\mu_P(z) = 0.$$

*Therefore the set $\bar{B}_{d(P,\partial\Omega)}(P) \cap \partial\Omega$ contains at least two points.*

Our second result is a converse of Theorem 1.1.

THEOREM 1.2. *Assume that $\Omega$ is convex. Let $P \in \Omega$. Suppose there exist $a \in R^n$ and*

*$d\mu_P(z) \in \Lambda_P$ such that*

(i) $\int_{\partial\Omega} e^{\langle z-P,a\rangle}(z-P)d\mu_P(z)=0$,

(ii) *the matrix* $(\int_{\partial\Omega} e^{\langle z-P,a\rangle}(z_i-P_i)(z_j-P_j)d\mu_P(z))$ *is nonsingular.*

*Then we can construct a family of solutions $u_\varepsilon$ to* (1.1) *such that $\varepsilon^{-N}I_\varepsilon(u_\varepsilon)\to I(w)$ and* (ii) *of Theorem* 1.1 *applies. Furthermore, $P_\varepsilon \to P$ as $\varepsilon\to 0$.*

We suspect that the convexity condition on $\Omega$ in Theorems 1.1 and 1.2 is not needed (this is only needed to study a linear problem, see Lemma 2.1).

A special example is when $\Omega$ is a ball centered at the origin. In this case, $P=a=0$, $d\mu_P(z)=cdz$ for some $c>0$ and it is easy to see that $\int_{\partial\Omega} e^{\langle z-P,a\rangle}(z_i-P_i)(z_j-P_j)d\mu_P(z)=c\delta_{ij}$, hence Theorem 1.2 applies.

A point $P\in\Omega$ which satisfies the conditions (i) and (ii) of Theorem 1.2 is called a *nondegenerate peak point.* A geometric characterization of nondegenerate peak points is given in Section 5. In particular, $P$ is a nondegenerate peak point if and only if

$$P\in \operatorname{int}(\operatorname{co}(\operatorname{supp}(d\mu_P)))\,,$$

where int $(\operatorname{co}(\operatorname{supp}(d\mu_P)))$ is the interior of the convex hull of the support of $d\mu_P$. Furthermore for each nondegenerate peak point $P$ there is a unique $a\in R^N$ satisfying the conditions (i) and (ii).

If $\Omega$ is convex, we can say more. The following Corollary will be proved in Section 5.

COROLLARY 1.3. (1) *If $\Omega$ is convex and $P\in\Omega$ is a point satisfying condition* (1.6), *then $d(P,\partial\Omega)=\max_{Q\in\Omega} d(Q,\partial\Omega)$.*

(2) *If $\Omega$ is convex, then there is at most one nondegenerate peak point.*

We note that there is a remarkable connection between the *interior* spike solutions of (1.1) and the *single-peaked* solutions of the corresponding Dirichlet problem

$$\begin{cases} \varepsilon^2\Delta u-u+u^p=0 & \text{in } \Omega\,,\\ u>0 & \text{in } \Omega\,,\\ u=0 & \text{on } \partial\Omega\,. \end{cases} \tag{1.7}$$

In [12], Ni and the author studied the problem (1.7) and showed that for $\varepsilon$ sufficiently small the problem (1.7) has a least-energy solution which possesses a single spike-layer with its unique peak in the interior of $\Omega$. Moreover this unique peak must be located near the "most centered" part of $\Omega$, .i.e. where the distance function $d(P,\partial\Omega)$, $P\in\Omega$ assumes its maximum. Later in [20], the author studied the *single-peaked* solutions of (1.7) and showed that the limit of the maximum points of single-peaked solutions satisfy the same condition (1.6).

We remark that other concentration phenomena are found in [12], [13], [14], [15], [16], [17], [19], [21], [22], etc.

Our basic idea is similar to that of [20]. Namely, we decompose our solution into two parts: one part carries the peak information, the other part is very small. The crucial observations are Lemmas 2.1 and 2.3. We shall frequently use the results of [20].

The organization of our paper is as follows. In Section 2, we introduce the projection of $w$ in $H^1(\Omega)$ and set up the technical framework. We then reduce the problem to a finite dimensional problem and then solve it in Section 3. Section 4 is devoted to the proofs of Theorems 1.1 and 1.2. We study some geometric meanings of the conditions (i) and (ii) in Theorem 1.2 and give in Section 5 examples of convex domains for which there exists a nondegenerate peak point. We also prove Corollary 1.3 in Section 5. In Appendix A, a decomposition lemma is proved. Appendix B is devoted to the study of the linearization problem. We include all the estimates in Appendix C.

In this paper we denote various generic constants by $C$. The notation $O(A)$, $o(A)$ means that $|O(A)|\le C|A|$, $|o(A)|/|A|\to 0$ as $|A|\to 0$, respectively.

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**2. Preliminaries.** In this section, we introduce the projection of $w$ in $H^1(\Omega)$ and show the connection between the problems (1.1) and (1.2). Finally, we set up a technical framework for the problem (1.1).

Let $P\in\Omega$. Let $w$ be the unique solution of (1.2). For a domain $U$ in $R^N$, we set $P_U^N w$ to be the unique solution of

$$\begin{cases} \Delta u-u+w^p=0 & \text{in } U, \\ \partial u/\partial \nu=0 & \text{on } U. \end{cases} \tag{2.1}$$

Then by the maximum principle, $P_U^N w>0$.

Let $\Omega_{\varepsilon,P}=\{y\,|\,\varepsilon y+P\in\Omega\}$ and $P_{\Omega_{\varepsilon,P}}^N w$ be the projection of $w$ on $\Omega_{\varepsilon,P}$.

Let $\varphi_{\varepsilon,P}^N=w-P_{\Omega_{\varepsilon,P}}^N w$. To analyze $P_{\Omega_{\varepsilon,P}}^N w$, we need to introduce another notation. If we change the boundary condition of (2.1) to the Dirichlet condition, then we get $P_{\Omega_{\varepsilon,P}}^D w$ and $\varphi_{\varepsilon,P}^D=w-P_{\Omega_{\varepsilon,P}}^D w$ (see [12] and [20] for details).

We set $V_{\varepsilon,P}^N(y)=\varphi_{\varepsilon,P}^N(P+\varepsilon y)/\varphi_{\varepsilon,P}^N(P)$. Then $0<V_{\varepsilon,P}^N(y)$ in $\Omega_{\varepsilon,P}$ and it satisfies

$$\Delta u-u=0 \quad \text{in } \Omega_{\varepsilon,P}\,, \quad u(0)=1\,.$$

The following key result connects $\varphi_{\varepsilon,P}^D$ and $\varphi_{\varepsilon,P}^N$.

LEMMA 2.1. *Assume that $\Omega$ is convex. Let $a_0>0$. Then there exist $\nu_0$, $\varepsilon_0>0$ such that for $\varepsilon\le\varepsilon_0$, $d(P,\partial\Omega)>a_0$, we have*

$$-(1+\nu_0\varepsilon)\varphi_{\varepsilon,P}^D\le\varphi_{\varepsilon,P}^N\le-(1-\nu_0\varepsilon)\varphi_{\varepsilon,P}^D\,. \tag{2.2}$$

PROOF. Assume that $\Omega$ is convex. We just need to show that

$$(1-\nu_0\varepsilon)\varphi_{\varepsilon,P}^D\le-\varphi_{\varepsilon,P}^N\,. \tag{2.3}$$

The proof of the other inequality is the same.

To this end, we note that on $\partial\Omega$, $w'((x-P)/\varepsilon)/w((x-P)/\varepsilon)=-1+O(\varepsilon)$. Let $\psi^D_{\varepsilon,P}(x)=-\varepsilon\log\varphi^D_{\varepsilon,P}$. By the results of [20, Section 3], we have that

$$\frac{\partial\varphi^D_{\varepsilon,P}}{\partial\nu}=-(1+O(\varepsilon))\frac{\langle x-P,\nu\rangle}{|x-P|}.$$

Hence on $\partial\Omega$, we have

$$\begin{aligned}\frac{\partial\varphi^D_{\varepsilon,P}}{\partial\nu}&=-\frac{1}{\varepsilon}w\frac{\partial\psi^D_{\varepsilon,P}}{\partial\nu}\\&=-(1+O(\varepsilon))\frac{1}{\varepsilon}w'\frac{\langle x-P,\nu\rangle}{|x-P|}\\&\geq-(1+\nu_0\varepsilon)\frac{\partial\varphi^N_{\varepsilon,P}}{\partial\nu},\end{aligned}$$

since $\langle x-P,\nu\rangle/|x-P|\geq 0$ for $x\in\partial\Omega$ ($\Omega$ is convex). The proof of Lemma 2.1 is completed. □

By Lemma 2.1, we have

$$\varphi^N_{\varepsilon,P}(x)=(-1+O(\varepsilon))\varphi^D_{\varepsilon,P}(x),$$

so we can now use the results of [12] and [20] for the Dirichlet case to treat the Neumann case.

Let $a_0>0$ be a fixed positive number (to be determined later). As in [20], for each $a>0$, we define

$$F_{a_0}=\left\{P^N_{\Omega_\varepsilon,P}w\left(\frac{x-P}{\varepsilon}\right)\bigg|\, d(P,\partial\Omega)>a_0\right\}, \tag{2.4}$$

$$V_a=\{(\alpha,P)\in R\times\Omega\,|\,|\alpha-1|<a,\, d(P,\partial\Omega)>a_0\}. \tag{2.5}$$

For each $u, v\in H^1(\Omega)$, we define

$$\langle u,v\rangle_{\varepsilon,\Omega}=\varepsilon^{-N}\int_\Omega(\varepsilon^2\nabla u\cdot\nabla v+uv).$$

We denote $\langle u,u\rangle_{\varepsilon,\Omega}$ as $\|u\|^2_{\varepsilon,\Omega}$. Sometimes we omit the index $\varepsilon$, $\Omega$ when there is no confusion.

Set $d(u,F_{a_0})=\inf_{v\in F_{a_0}}\|u-v\|_{\varepsilon,\Omega}$.

The following decomposition lemma will be proved in Appendix A.

LEMMA 2.2. *If $u\in H^1(\Omega)$ such that $d(u,F_{a_0})$ is small enough and $a$ is small, then the problem*

$$minimize \ \|u-\alpha P^N_{\Omega_\varepsilon,P}w\|^2_{\varepsilon,\Omega}$$

*with respect to* $(\alpha, P)$ *has a unique solution in the open set* $V_a$.

Next we state an important lemma about the location of the interior maximum points of the solutions of (1.1).

LEMMA 2.3. *Let* $u_\varepsilon$ *be a solution of* (1.1) *such that* $\lim_{\varepsilon\to 0}\varepsilon^{-N}I_\varepsilon(u_\varepsilon)\to I(w)$. *Then there is an* $a_0$ *such that if* $u_\varepsilon$ *has only one local maximum point* $P_\varepsilon\notin\partial\Omega$ *then* $d(P_\varepsilon, \partial\Omega)\geq a_0$.

PROOF. Suppose that $u_\varepsilon$ satisfies the conditions of Lemma 2.3. Then an argument similar to that in the proof of Theorem 2.1 in [9] show that $\rho_\varepsilon := d(P_\varepsilon, \partial\Omega)/\varepsilon\to\infty$. Suppose now that $d_\varepsilon := d(P_\varepsilon, \partial\Omega)\to 0$ and so $P_\varepsilon\to P_0\in\partial\Omega$.

Let $\delta>0$ be a fixed small number. Set

$$I^1_\varepsilon=\int_{\partial\Omega}\left(\varepsilon^2|\nabla u_\varepsilon|^2+u_\varepsilon^2-\frac{2}{p+1}u_\varepsilon^{p+1}\right), \tag{2.6}$$

$$\partial_1\Omega=B_\delta(P_0)\cap\partial\Omega\,,\quad \partial_2\Omega=\partial\Omega\setminus\partial_1\Omega\,.$$

Let $K(r)$ be the Green function of $-\Delta+1$ on $R^n$. Since $\Omega$ is convex, $\partial K/\partial\nu((x-P_\varepsilon)/\varepsilon)<0$ on $\partial\Omega$. Hence

$$u_\varepsilon\geq CK\left(\frac{x-P_\varepsilon}{\varepsilon}\right)\quad\text{on } \bar\Omega\setminus B_{R_\varepsilon}(P_\varepsilon) \tag{2.7}$$

if $u_\varepsilon\geq CK((x-P_\varepsilon)/\varepsilon)$ on $\partial B_{R_\varepsilon}(P_\varepsilon)$ for a positive number $R$. Therefore

$$u_\varepsilon\geq Ce^{-\rho_\varepsilon}\quad\text{on } \partial_1\Omega\,. \tag{2.8}$$

Next we shall obtain an upper bound for $u_\varepsilon$ on $\partial_2\Omega$. To this end, fix a point $P_1\in\Omega$ so that $|P_1-P_0|=\delta_1$ where $\delta_1=\delta/8$. Let $\psi^D_{\varepsilon,P_1}$ be the unique solution of

$$\varepsilon\Delta v-|\nabla v|^2+1=0\,,\quad v=|x-P_1|\quad\text{on } \partial\Omega\,. \tag{2.9}$$

By the result of Section 3 in [20], $\psi^D_{\varepsilon,P_1}(x)\to\inf_{z\in\partial\Omega}(|z-P_1|+|z-x|)$ uniformly as $\varepsilon\to 0$ and $\partial\psi^D_{\varepsilon,P_1}/\partial\nu=-\langle x-P_1, \nu\rangle/|x-P_1|+O(\varepsilon)$ for $x\in\partial\Omega$.

Let $v_\varepsilon=\exp(-(-2\delta_1+\psi^D_{\varepsilon,P_1})/\varepsilon)$. Since $\Omega$ is convex, $\langle x-P_1, \nu\rangle/|x-P_1|\geq c>0$. Hence

$$-\frac{\partial\varphi^N_{\varepsilon,P_\varepsilon}}{\partial\nu}\leq\frac{\partial v_\varepsilon}{\partial\nu}\quad\text{on } \partial\Omega$$

if $|P_\varepsilon-P_1|<2\delta_1$. Therefore by the comparison principle, we have

$$-\varphi^N_{\varepsilon,P_\varepsilon}\leq v_\varepsilon\,. \tag{2.10}$$

So

$$P^N_{\Omega_\varepsilon,P_\varepsilon}w\leq w+\exp(-(\psi^D_{\varepsilon,P_1}-2\delta_1)/\varepsilon)\quad\text{on } \partial\Omega\,.$$

On the other hand, we note that $v_\varepsilon(y)=u_\varepsilon(\varepsilon y+P_\varepsilon)$ satisfies

$$\Delta v_\varepsilon-\frac{1}{(1+\delta_1)^2}v_\varepsilon\geq 0 \quad \text{in } \Omega_{\varepsilon,P_\varepsilon}\setminus B_{R_1}(0)\,,$$

where $R_1>0$ is a sufficiently large number depending on $\delta_1>0$. Observe that $P^N_{\Omega_{\varepsilon,P_\varepsilon}}w(y/(1+\delta_1))$ satisfies

$$\Delta P_{\Omega_{\varepsilon,P_\varepsilon}}w-\frac{1}{(1+\delta_1)^2}P_{\Omega_{\varepsilon,P_\varepsilon}}w\leq 0 \quad \text{in } \Omega_{\varepsilon,P_\varepsilon}\,.$$

It is easy to see by comparison that $u_\varepsilon(x)\leq CP^N_{\Omega_{\varepsilon,P_\varepsilon}}w(y/(1+\delta_1))$ for $\varepsilon$ small. Hence we have

$$u_\varepsilon\leq Ce^{-2\delta/\varepsilon} \quad \text{on } \bar{\Omega}\setminus B_\delta(P_0)\,. \tag{2.11}$$

By elliptic regularity theory (to see this, we look at a tubular neighborhood of $\partial_2\Omega$ and then apply interior $L^p$-estimates in that neighborhood), we have

$$|u_\varepsilon|\,,\quad |\nabla u_\varepsilon|\leq Ce^{-\delta/\varepsilon} \quad \text{on } \delta_2\Omega\,. \tag{2.12}$$

We can now finish the proof of our lemma as follows. Let $v_0$ be the outward unit normal at $P_0$. Hence $|v_0|=1$. By the Pohozaev identity, we have for any $y\in R^N$

$$\int_\Omega\left(\frac{N}{p+1}-\frac{N-2}{2}\right)u_\varepsilon^{p+1}-u_\varepsilon^2 \tag{2.13}$$
$$=\frac{1}{2}\int_{\partial\Omega}(x-y,\, v)\left(\varepsilon^2|\nabla u_\varepsilon|^2+u_\varepsilon^2-\frac{2}{p+1}u_\varepsilon^{p+1}\right).$$

Therefore we have

$$J_\varepsilon=\int_{\partial\Omega}v\left(\varepsilon^2|\nabla u_\varepsilon|^2+u_\varepsilon^2-\frac{2}{p+1}u_\varepsilon^{p+1}\right)=0\,. \tag{2.14}$$

We write

$$\begin{aligned}J_\varepsilon&=(v_0)I_\varepsilon^1+\int_{\partial\Omega}(v-v_0)\left(\varepsilon^2|\nabla u_\varepsilon|^2+u_\varepsilon^2-\frac{2}{p+1}u_\varepsilon^{p+1}\right)\\&=(v_0)I_\varepsilon^1+\int_{\partial_1\Omega}(v-v_0)\left(\varepsilon^2|\nabla u_\varepsilon|^2+u_\varepsilon^2-\frac{2}{p+1}u_\varepsilon^{p+1}\right)\\&\quad+\int_{\partial_2\Omega}(v-v_0)\left(\varepsilon^2|\nabla u_\varepsilon|^2+u_\varepsilon^2-\frac{2}{p+1}u_\varepsilon^{p+1}\right)\\&=v_0I_\varepsilon^1+I_2+I_3\,,\end{aligned}$$

where $I_2$ and $I_3$ are defined by the last equality.

For $I_\varepsilon^1$, by (2.8) we have

$$I_\varepsilon^1 \geq ce^{-2\rho_\varepsilon}.$$

For $I_2$ we have

$$|I_2| \leq \delta I_\varepsilon^1.$$

For $I_3$, by (2.11) we have

$$\frac{|I_3|}{I_\varepsilon^1} \leq Ce^{-(\delta - 2\varepsilon\rho_\varepsilon)/\varepsilon} \to 0.$$

Thus

$$J_\varepsilon / I_\varepsilon^1 = v_0 + O(\delta) + o(\varepsilon) = 0,$$

a contradiction. □

By Lemma 2.2, there exists a diffeomorphism between a neighborhood of the possible *single interior peak* solutions of (1.1) we are interested in and the open set

$$M_\eta =: \left\{ m = (\alpha, P, v) \,\middle|\, \begin{array}{l} (\alpha, P, v) \in R_+ \times \Omega \times H^1(\Omega), \, |\alpha - 1| < \eta, \\ d(P, \partial\Omega) > a_0, \, v \in E_{\varepsilon,P}, \, \|v\|_{\varepsilon,\Omega} < \eta \end{array} \right\}$$

with $\eta > 0$ being a suitable constant and

$$E_{\varepsilon,P} =: \left\{ v \in H^1(\Omega) \,\middle|\, \langle v, P_{\Omega_\varepsilon,P}^N w \rangle_{\varepsilon,\Omega} = \left\langle v, \frac{\partial}{\partial P_i} P_{\Omega_\varepsilon,P}^N w \right\rangle_{\varepsilon,\Omega} = 0, \, i = 1, \ldots, N \right\}.$$

Let us now define a functional

$$K_\varepsilon : M_\eta \to R, \quad m = (\alpha, P, v) \mapsto \varepsilon^{-N} I_\varepsilon(\alpha P_{\Omega_\varepsilon,P}^N w + v).$$

From Lemma 2.2 we have:

PROPOSITION 2.4. *$m = (\alpha, P, v) \in M_\eta$ is a critical point of $K_\varepsilon$ if and only if $u = \alpha P_{\Omega_\varepsilon,P}^N w + v$ is a critical point of $K_\varepsilon$, i.e. if and only if there exists $(A, B) \in R \times R^N$ such that*

$$\text{(2.15)} \qquad (\mathrm{E}_\alpha) \qquad \frac{\partial K_\varepsilon}{\partial \alpha} = 0,$$

$$\text{(2.16)} \qquad (\mathrm{E}_P) \qquad \frac{\partial K_\varepsilon}{\partial P_i} = \sum_{j=1}^N B_j \left\langle \frac{\partial^2 P_{\Omega_\varepsilon,P}^N w}{\partial P_i \partial P_j}, v \right\rangle_{\varepsilon,\Omega},$$

$$\text{(2.17)} \qquad (\mathrm{E}_v) \qquad \frac{\partial K_\varepsilon}{\partial v} = A P_{\Omega_\varepsilon,P}^N w + \sum_{j=1}^N B_j \frac{\partial P_{\Omega_\varepsilon,P}^N w}{\partial P_j}.$$

**3. Reduction to a finite dimensional problem.** In this section, we analyze the equations $(\mathrm{E}_\alpha)$, $(\mathrm{E}_P)$ and $(\mathrm{E}_v)$. We first analyze $(\mathrm{E}_v)$ and solve $v$. Then we take care of $(\mathrm{E}_\alpha)$. Finally we solve $(\mathrm{E}_P)$. Since it is very similar to Section 4 of [20], we omit most of the details.

We first deal with the $v$-part of $u$, in order to show that it is negligible with respect to the concentration phenomenon.

PROPOSITION 3.1. *There exist $\eta_0>0$, $\varepsilon_0>0$ such that for $\varepsilon\le\varepsilon_0$, $\eta\le\eta_0$ there exists a smooth map which to any $(\varepsilon, \alpha, P)$ with $(\alpha, P, 0)\in M_{\eta_0}$ associates $\bar{v}\in E_{\varepsilon,P}$, $\|\bar{v}\|_{\varepsilon,\Omega}<\eta$ such that $(E_v)$ is satisfied for some $(A, B_1, \ldots, B_N)\in R\times R^N$. Such a $\bar{v}$ is unique, minimizes $K_\varepsilon(\alpha, P, v)$ with respect to $v$ in $\{v\in E_{\varepsilon,P} \mid \|v\|_{\varepsilon,\Omega}<\eta\}$ and we have the estimate*

$$\frac{\partial\bar{v}}{\partial\nu}=0 \quad on\ \partial\Omega,\ \|\bar{v}\|_{\varepsilon,\Omega}\le O((\varphi^N_{\varepsilon,P}(P))^{(1+\sigma)/2}), \tag{3.1}$$

*where $\sigma=\min(p-1, 1)$.*

For the proof, see Appendix B.

Once $\bar{v}$ is obtained, we can estimate $A$ and $B$. Indeed, we have

$$\left\langle\frac{\partial K_\varepsilon}{\partial v}, P^N_{\Omega_{\varepsilon,P}}w\right\rangle_{\varepsilon,\Omega}=A\langle P^N_{\Omega_{\varepsilon,P}}w, P^N_{\Omega_{\varepsilon,P}}w\rangle_{\varepsilon,\Omega}+\sum_{j=1}^N B_j\left\langle\frac{\partial P^N_{\Omega_{\varepsilon,P}}w}{\partial P_j}, P^N_{\Omega_{\varepsilon,P}}w\right\rangle_{\varepsilon,\Omega},$$

$$\left\langle\frac{\partial K_\varepsilon}{\partial v}, \frac{\partial P^N_{\Omega_{\varepsilon,P}}w}{\partial P_i}\right\rangle_{\varepsilon,\Omega}=A\left\langle P^N_{\Omega_{\varepsilon,P}}w, \frac{\partial}{\partial P_i}P^N_{\Omega_{\varepsilon,P}}w\right\rangle_{\varepsilon,\Omega}+\sum_{j=1}^N B_j\left\langle\frac{\partial P^N_{\Omega_{\varepsilon,P}}w}{\partial P_j}, \frac{\partial P^N_{\Omega_{\varepsilon,P}}w}{\partial P_i}\right\rangle_{\varepsilon,\Omega}$$

and

$$\langle P^N_{\Omega_{\varepsilon,P}}w, P^N_{\Omega_{\varepsilon,P}}w\rangle_{\varepsilon,\Omega}=\int_{R^N}w^{p+1}+O(\varphi^N_{\varepsilon,P}(P)),$$

$$\left\langle\frac{\partial P^N_{\Omega_{\varepsilon,P}}w}{\partial P_j}, P^N_{\Omega_{\varepsilon,P}}w\right\rangle_{\varepsilon,\Omega}=O(\varphi^N_{\varepsilon,P}(P)/\varepsilon),$$

$$\left\langle\frac{\partial P^N_{\Omega_{\varepsilon,P}}w}{\partial P_j}, \frac{\partial P^N_{\Omega_{\varepsilon,P}}w}{\partial P_i}\right\rangle_{\varepsilon,\Omega}=\frac{1}{\varepsilon^2}\left(\frac{p}{N}\int_{R^N}w^{p-1}(w')^2\delta_{ij}+O(\varphi^N_{\varepsilon,P}(P))\right),$$

$$\left\langle\frac{\partial K_\varepsilon}{\partial v}, P^N_{\Omega_{\varepsilon,P}}w\right\rangle_{\varepsilon,\Omega}=\frac{\partial K_\varepsilon}{\partial\alpha},$$

$$\left\langle\frac{\partial K_\varepsilon}{\partial v}, \frac{\partial P^N_{\Omega_{\varepsilon,P}}w}{\partial P_i}\right\rangle_{\varepsilon,\Omega}=\frac{1}{\alpha}\frac{\partial K_\varepsilon}{\partial P_i}.$$

On the other hand, we have by Appendix C

$$\frac{\partial K_\varepsilon}{\partial\alpha}=\int_{R^N}w^{p+1}(\alpha-\alpha^p)+O(\varphi^N_{\varepsilon,P}(P)), \tag{3.2}$$

$$\frac{\partial K_\varepsilon}{\partial P_i}=\frac{1}{\varepsilon}\left[O((\alpha-1)\varphi^N_{\varepsilon,P}(P))+O(\varphi^N_{\varepsilon,P}(P))\right].$$

Combining all these, by Appendix C we have

$$A=O(|\alpha-1|+\varphi_{\varepsilon,P}^N(P))\,,\quad B=\varepsilon O(|\alpha-1|+\varphi_{\varepsilon,P}^N(P))\,.$$

We can now estimate the equation $(E_P)$.

$$\tag{3.3}\begin{aligned}\frac{\partial K_\varepsilon}{\partial P_i}&=\sum_{j=1}^N B_j\left\langle\frac{\partial^2 P_{\Omega_{\varepsilon,P}}^N w}{\partial P_i\partial P_j},v\right\rangle_{\varepsilon,\Omega}\\ &\le\sum_{j=1}^N|B_j|\left\|\frac{\partial^2 P_{\Omega_{\varepsilon,P}}^N w}{\partial P_i\partial P_j}\right\|_{\varepsilon,\Omega}\|v\|_{\varepsilon,\Omega}\\ &=\frac{1}{\varepsilon}O((\varphi_{\varepsilon,P}^N(P))^{(1+\sigma)/2}(|\alpha-1|+\varphi_{\varepsilon,P}^N(P)))\,.\end{aligned}$$

We have by the same arguments as in [20, Lemma 4.2].

LEMMA 3.2. *Supposed* $d(P,\partial\Omega)>a_0>0$. *Let* $v=\bar{v}$ *and* $(A,B_1,\dots,B_N)$ *be defined in Proposition* 3.1 *for* $\varepsilon<\varepsilon_0$. *Then the* $N$ *equations of* $(E_P)$ *are equivalent to*

$$\tag{3.4} p\int_{\Omega_{\varepsilon,P}} w^{p-1}w'\frac{y_i}{|y|}V_{\varepsilon,P}^N(y)+V_{P_i}(\varepsilon,\alpha,P)=0\,,$$

*where* $V_{P_i}$ *is a smooth function such that*

$$\tag{3.5} V_{P_i}=O(|\alpha-1|\varphi_{\varepsilon,P}^N(P)+(\varphi_{\varepsilon,P}^N(P))^{1+\sigma})\,.$$

**4. Proofs of Theorems 1.1 and 1.2.** We can now prove Theorems 1.1 and 1.2.

Let $u_\varepsilon$ be a solution of (1.1) such that $\lim_{\varepsilon\to 0}\varepsilon^{-N}I_\varepsilon(u_\varepsilon)=I(w)$. Combining the arguments of Section 3 in [9] and Section 3 in [12], we see that $u_\varepsilon$ can have at most two local maximum points and there are two possibilities:

(i) either $u_\varepsilon$ has two local maximum points $P_\varepsilon^1$ and $P_\varepsilon^2$, then $|P_\varepsilon^1-P_\varepsilon^2|/\varepsilon\to\infty$ and $P_\varepsilon^1$, $P_\varepsilon^2\in\partial\Omega$ by the results of Section 3 in [9],

(ii) or $u_\varepsilon$ has only one local maximum points $P_\varepsilon$ and $d(P_\varepsilon,\partial\Omega)/\varepsilon\to\infty$ by the result of Section 3 in [12].

Suppose (ii) occurs. Then by Lemma 2.3, $d(P_\varepsilon,\partial\Omega)\ge a_0$. It is easy to see that $\|u_\varepsilon-P_{\Omega_{\varepsilon,P_\varepsilon}}^N w\|_{\varepsilon,\Omega}\to 0$ as $\varepsilon\to 0$. Hence $d(u_\varepsilon,F_{a_0})\to 0$ as $\varepsilon\to 0$. By Lemma 2.2,

$$\tag{4.1} u_\varepsilon=\alpha_\varepsilon P_{\Omega_{\varepsilon,P_\varepsilon}}^N w+v$$

and by Proposition 2.3, $(\alpha_\varepsilon,P_\varepsilon,v)$ is a critical point of $K_\varepsilon$, i.e. satisfies $(E_\alpha)$, $(E_P)$ and $(E_v)$. So $v=\bar{v}(\varepsilon,\alpha_\varepsilon,P_\varepsilon)$.

By the equation $(E_\alpha)$, we have

$$\tag{4.2} \alpha_\varepsilon=1+O(\varphi_{\varepsilon,P_\varepsilon}^N(P_\varepsilon))\,.$$

Substituting this into the equation $(E_P)$, by the same argument as in [20, Lemma 5.1], we have:

LEMMA 4.1. *Suppose that* $P_\varepsilon\to P_0$. *Then we have*

$$P_0 \notin \partial\Omega;\ |P_\varepsilon - P_0| \le C_\varepsilon\,. \tag{4.3}$$

Suppose now that $(P_\varepsilon - P_0)/\varepsilon \to b_0 \in R^N$. Then we have

$$|z-P_\varepsilon| = |z-P_0| + \varepsilon\left(\left\langle b_0, \frac{z-P_0}{|z-P_0|}\right\rangle + o(1)\right). \tag{4.4}$$

Letting $a = b_0/d(P_0, \partial\Omega)$, using Lemma 3.2 and by the calculations of Appendix A in [20], we conclude the proof of Theorem 1.1.

Next we prove Theorem 1.2.

Let $P_0 \in \Omega$ satisfy (i) $\int_\Omega e^{\langle z-P_0,a\rangle}(z_i - P_{0,i})d\mu_{P_0}(z) = 0$, $i=1,\dots,N$ and (ii) the matrix $(\int_\Omega e^{\langle z-P_0,a\rangle}(z_i - P_{0,i})(z_j - P_{0,j})d\mu_{P_0}(z))$ is nonsingular. We set

$$\alpha = 1 + \beta\,, \tag{4.5}$$

$$P = P_0 + \varepsilon(\xi - ad(P_0, \partial\Omega))\,, \tag{4.6}$$

where $\beta \in R$, $\xi \in R^N$ are to be determined. With these changes of variables, the system (E) turns out to be equivalent to (with $v = \bar v(\varepsilon, \alpha, P)$):

$$\beta = V_\beta(\varepsilon, \beta, \xi)\,, \tag{4.7}$$

$$L_i(\varepsilon, \xi) = V_{P_i}(\varepsilon, \beta, \xi)\,, \qquad i = 1, \dots, N\,, \tag{4.8}$$

where for $i = 1, \dots, N$,

$$L_i(\varepsilon, \xi) = \frac{p\int_{\Omega_{\varepsilon,P}} w^{p-1} w'(y_i/|y|) V^N_{\varepsilon,P}(y)}{\varphi^N_{\varepsilon,P_0}(P_0)} \tag{4.9}$$

and $V_\beta$, $V_i$ are smooth functions satisfying

$$V_\beta = O(\varphi^N_{\varepsilon,P_0}(P_0))\,, \tag{4.10}$$

$$V_{P_i} = O(|\beta| + (\varphi^N_{\varepsilon,P_0}(P_0))^\sigma)\,. \tag{4.11}$$

Then we have:

LEMMA 4.2. *$L_i$ can be written as*

$$L = A_\varepsilon \xi + \bar L\,, \tag{4.12}$$

*where $A_\varepsilon$ is a matrix such that $A_\varepsilon \to A = \gamma(\int_\Omega e^{\langle z-P_0,a\rangle_{\varepsilon,\Omega}}(z_i - P_{0,i})(z_j - P_{0,j})d\mu_{P_0}(z))$ as $\varepsilon \to 0$ for some positive constant $\gamma > 0$ and $\bar L$ is a smooth function satisfying*

$$\bar L = O(|\xi|^2 + |\beta| + (\varphi^N_{\varepsilon,P_0}(P_0))^\sigma)\,. \tag{4.13}$$

Thus the system (4.7) and (4.8) may also be written as

$$\begin{cases} \beta = V(\varepsilon, \beta, \xi)\,, \\ A_\varepsilon \xi = W(\varepsilon, \beta, \xi)\,, \end{cases} \tag{4.14}$$

where $V$, $W$ are smooth functions satisfying

$$V(\varepsilon, \beta, \xi) = O(\varphi^N_{\varepsilon,P_0}(P_0)), \tag{4.15}$$

$$W(\varepsilon, \beta, \xi) = O(\varepsilon + |\xi|^2 + |\beta| + (\varphi^N_{\varepsilon,P_0}(P_0))^\sigma). \tag{4.16}$$

Since the matrix $A$ is invertible, we choose $(-r, r) \times B_r(0)$ where $r$ is so small that $(V, A_\varepsilon^{-1}W)$ is a continuous map from $(-r, r) \times B_r(0)$ to itself. By Brouwer's fixed point theorem, there exists $(\beta_\varepsilon, \xi_\varepsilon)$ such that $\beta_\varepsilon = O(\varphi^N_{\varepsilon,P_0}(P_0))$, $\xi_\varepsilon = O(\varepsilon)$, that is (E) is satisfied. Therefore we have a critical point $u_\varepsilon = \alpha_\varepsilon P^N_{\Omega_{\varepsilon,P_\varepsilon}} w + \bar{v}(\varepsilon, \alpha_\varepsilon, P_\varepsilon)$ of $I_\varepsilon$.

By construction, the corresponding $u_\varepsilon \in H^1(\Omega)$ is a critical point of $J_\varepsilon$, i.e. $u_\varepsilon$ satisfies

$$\varepsilon^2 \Delta u_\varepsilon - u_\varepsilon + |u_\varepsilon|^{p-1} u_\varepsilon = 0 \quad \text{in } \Omega, \quad \frac{\partial u_\varepsilon}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \tag{4.17}$$

Multiplying this equation with $u_\varepsilon^- = \max(0, -u_\varepsilon)$ and integrating the result over $\Omega$, we get

$$\int_{\Omega_{\varepsilon,P}} |\nabla u_\varepsilon^-|^2 + (u_\varepsilon^-)^2 = \int_{\Omega_{\varepsilon,P}} |u_\varepsilon^-|^{p+1}. \tag{4.18}$$

By Sobolev's imbedding theorem, we have

$$\left( \int_{\Omega_{\varepsilon,P}} |u_\varepsilon^-|^{p+1} \right)^{2/(p+1)} \leq C \int_{\Omega_{\varepsilon,P}} |u_\varepsilon^-|^{p+1}. \tag{4.19}$$

(4.19) implies that either $u_\varepsilon^- \equiv 0$ or $\int_{\Omega_{\varepsilon,P}} |u_\varepsilon^-|^{p+1} \geq C > 0$. Since by construction $\int_{\Omega_{\varepsilon,P}} |u_\varepsilon^-|^{p+1} \to 0$, we have $u_\varepsilon^- \equiv 0$ for $\varepsilon$ sufficiently small. By the maximum principle, $u_\varepsilon > 0$. Moreover since $I_\varepsilon(u_\varepsilon)/\varepsilon^N \to I(w)$, we know that $u_\varepsilon$ is a single-peaked solution. By the results of [12], $u_\varepsilon$ has a unique local maximum (hence global) point $\tilde{P}_\varepsilon$. It is not difficult to see that $P_\varepsilon - \tilde{P}_\varepsilon = o(1)$. Hence $\tilde{P}_\varepsilon \to P_0$.

**5. Analysis of the Conditions (i) and (ii) in Theorem 1.2 and proof of Corollary 1.3.** In this section, we discuss the geometric meanings of the condition (1.6) in Theorem 1.1 and the conditions (i) and (ii) in Theorem 1.2. We first prove the following:

THEOREM 5.1. *$P$ is a nondegenerate peak point if and only if*

$$P \in \operatorname{int}(\operatorname{co}(\operatorname{supp}(d\mu_P)))$$

*where* $\operatorname{int}(\operatorname{co}(\operatorname{supp}(d\mu_P)))$ *is the interior of the convex hull of the support of $d\mu_P$. Furthermore for each nondegenerate peak point $P$ there is a unique $a \in R^N$ satisfying the conditions* (i) *and* (ii).

PROOF. The following proof is essentially due to an observation of Dancer and Noussair [4].

Let $P \in \Omega$ be fixed. Without loss of generality we may assume that $P = 0$. We shall write $d\mu_0$ as $d\mu$ and $K = \operatorname{supp}(d\mu)$. Notice that $d\mu$ is a positive measure with compact support.

We note that the function

$$F(a)=\int_K e^{\langle x,a\rangle} d\mu(x) \qquad \text{for} \quad a\in R^N$$

is real analytic and for all $a$ in $R^N$,

$$\nabla F(a)=\int_K e^{\langle x,a\rangle} x\, d\mu(x)$$

and

$$HF(a)_{ij}=\int_K e^{\langle x,a\rangle} x_i x_j\, d\mu(x)\,.$$

Then 0 is a nondegenerate peak point means that $\nabla F(a)=0$ and that $HF(a)$ is nonsingular for some $a\in R^N$.

If $a\in R^N$ and $b\in S^{N-1}$, then

$$\langle HF(a)b,\, b\rangle=\int_K e^{\langle x,a\rangle}\langle b,x\rangle^2 d\mu(x)\geq 0$$

and the equality holds if and only if

$$K\subset\{x\in R^N : \langle b,\, x\rangle=0\}=: b^\perp\,.$$

We also notice that if $K\subset b^\perp$ for some $b\in S^{N-1}$, then $F(a)=F(a+tb)$ for all $a\in R^N$ and $t\in R$ and $HF$ is singular for all $a\in R^N$. Conversely if there is no $b\in S^{N-1}$ for which $K\subset b^\perp$, then

$$\langle HF(a)b,\, b\rangle>0$$

for all $b\in S^{N-1}$ and hence $HF(a)$ is nonsingular. It is easy to see that the following statements are equivalent:

(i) $K\subset b^\perp$ for some $b\in S^{N-1}$;

(ii) $\mathrm{co}(K)\subset b^\perp$ for some $b\in S^{N-1}$;

(iii) $\mathrm{co}(K)$ has trivial interior.

Henceforth, we assume that $\mathrm{co}(K)$ has nontrivial interior. We show next that the conditions:

(a) $F$ attains its minimum at some $a\in R^N$;

(b) $K\not\subset\{x\in R^N;\ \langle x,\, b\rangle\geq 0\}$ for all $b\in S^{N-1}$;

(c) $0\in\mathrm{int}(\mathrm{co}(K))$

are equivalent.

Suppose (b) does not hold. Then there is a $b\in S^{N-1}$ such that $\langle x,\, b\rangle\geq 0$ for all $x\in K$. Then for any $a\in R^N$,

$$\frac{\partial}{\partial t} F(a+tb)\big|_{t=0}=\int_K e^{\langle x,a\rangle}\langle x,\, b\rangle d\mu(x)>0\,,$$

since we have excluded the possibility that $K \subset b^{\perp}$. Therefore $\nabla F(a) \neq 0$ for all $a \in R^N$. Conversely suppose (b) holds. Then by a simple compactness argument we can show that for some $\delta$ and $\eta$ in $R^+$, $\mu(\{x \in K : \langle x, b\rangle \geq \delta\}) \geq \eta$ for all $b \in S^{N-1}$. It then follows that

$$F(tb) \geq \int_{\{x \in K; \langle x, b\rangle \geq \delta\}} e^{\langle x, tb\rangle} d\mu(x) \geq \eta e^{t\delta}$$

for all $t \in R^+$ and $b \in S^{N-1}$. It follows that $F$ has an attained minimum in $R^N$, i.e., there exists $a$ such that $\nabla F(a) = 0$. The equivalence of (b) and (c) is obvious.

Finally, we notice that $F$ is a convex function which implies the uniqueness of $a$. This completes the proof of Theorem 5.1. □

The following Corollary follows easily from Theorem 5.1.

COROLLARY 5.2. *Let $P$ be a nondegenerate peak point. Then* $\operatorname{supp}(d\mu_P)$ *contains at least three points.*

We are now ready to prove Corollary 1.3 in Section 1.

PROOF OF COROLLARY 1.3. We always assume that $\Omega$ is convex. Hence for any two points $Q_1 \in \bar{\Omega}$, $Q_2 \in \bar{\Omega}$ we have $tQ_1 + (1-t)Q_2 \in \bar{\Omega}$ for $0 \leq t \leq 1$.

(1) Let $P \in \Omega$ be a point satisfying condition (1.6). Then $\bar{B}_{d(P,\partial\Omega)}(P) \cap \partial\Omega$ contains at least two points $Q_1$, $Q_2$. We claim that $d(P, \partial\Omega) = \max_{Q \in \Omega} d(Q, \partial\Omega)$. Suppose not. Let $P'$ be a point attaining the maximum of the distance function. Consider the convex hull of $B_{d(P,\partial\Omega)}(P) \cup B_{d(P',\partial\Omega)}(P')$. Let $L$ be the hyperplane passing through $P$ and perpendicular to the line $\overrightarrow{PP'}$. Since $d(P, \partial\Omega) \neq d(P', \partial\Omega)$, the set $\bar{B}_{d(P,\partial\Omega)}(P) \cap \partial\Omega$ must lie strictly on one side of $L$, which is impossible by condition (1.6).

(2) Let $P \in \Omega$ be an nondegenerate peak point. Then $\operatorname{supp}(d\mu_P) \subset \bar{B}_{d(P,\partial\Omega)} \cap \partial\Omega$ contains at least three points $Q_1$, $Q_2$, $Q_3$. By (1), $d(P, \partial\Omega) = \max_{Q \in \Omega} d(Q, \partial\Omega)$. Let $P' \neq P$ be another nondegenerate peak point. Then $d(P, \partial\Omega) = d(P', \partial\Omega)$. Consider again the convex hull of $B_{d(P,\partial\Omega)}(P) \cup B_{d(P',\partial\Omega)}(P')$. Let $L$ be the hyperplane passing through $P$ and perpendicular to the line $\overrightarrow{PP'}$. Since $\Omega$ is convex, the set $\bar{B}_{d(P,\partial\Omega)}(P) \cap \partial\Omega$ must lie on one side of $L$ (including $L$). Condition (1.6) implies that $\bar{B}_{d(P,\partial\Omega)}(P) \cap \partial\Omega \subset L$ and hence the interior of the convex hull of $d\mu_P$ is empty since $L$ is a submanifold, a contradiction. □

Finally we construct a convex domain $\Omega$ and a point $P_0 \in \Omega$ which satisfies the conditions in Theorem 1.2.

Indeed, we may assume that $N=2$. In Figure 1, $\bar{B}_{d(P_0,\partial\Omega)}(P_0) \cap \partial\Omega = \{P_1, P_2, P_3\}$. Hence $d\mu_{P_0} = c_1\delta_{P_1} + c_2\delta_2 + c_3\delta_3$. Since $N=2$, the three vectors $P_0 - P_1$, $P_0 - P_2$, $P_0 - P_3$ are linearly dependent. Therefore $\operatorname{int}(\operatorname{co}(\operatorname{supp}(d\mu_{P_0}))) \neq \emptyset$. By Theorem 5.1, $P_0$ is a nondegenerate peak point.

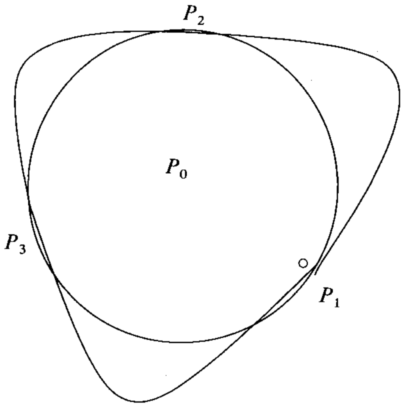


FIGURE. A convex domain.

**6. Appendix A. Decomposition lemma.** In this appendix, we shall prove the decomposition lemma (Lemma 2.2) in Section 3. We start with a lemma.

LEMMA 6.1. *Let* $(\varepsilon_k)$ *be a sequence with* $\varepsilon_k>0$, $\lim_{k\to\infty}\varepsilon_k=0$. *Let* $P_k$, $\tilde{P}_k\in\Omega_{a_0}$, $\alpha_k$, $\tilde{\alpha}_k\in(1/2, 2)$ *be such that*

$$\lim_{k\to\infty}\|\alpha_k P^N_{\Omega_{\varepsilon_k},P_k}w-\tilde{\alpha}_k P^N_{\Omega_{\varepsilon_k},\tilde{P}_k}w\|_{\varepsilon_k,\Omega}=0\,. \tag{6.1}$$

*Then we have*

$$\lim_{k\to\infty}|\alpha_k-\tilde{\alpha}_k|=0\,, \tag{6.2}$$

$$\lim_{k\to\infty}\left|\frac{P_k-\tilde{P}_k}{\varepsilon_k}\right|=0\,. \tag{6.3}$$

PROOF. We have

$$\|\alpha_k P^N_{\Omega_{\varepsilon_k},P_k}w-\tilde{\alpha}_k P^N_{\Omega_{\varepsilon_k},\tilde{P}_k}w\|_{\varepsilon_k,\Omega}=\left\|\alpha_k P^N_{\Omega_{\varepsilon_k},P_k}w-\tilde{\alpha}_k P^N_{\Omega_{\varepsilon_k},\tilde{P}_k}w\left(\cdot-\frac{\tilde{P}_k-P_k}{\varepsilon_k}\right)\right\|_{\varepsilon_k,\Omega}$$
$$=\left\|(\alpha_k-\tilde{\alpha}_k)P^N_{\Omega_{\varepsilon_k},P_k}w+\tilde{\alpha}_k\left(P^N_{\Omega_{\varepsilon_k},P_k}w-P^N_{\Omega_{\varepsilon_k},\tilde{P}_k}w\left(\cdot-\frac{\tilde{P}_k-P_k}{\varepsilon_k}\right)\right)\right\|_{\varepsilon_k,\Omega}$$
$$\geq|\alpha_k\|P^N_{\Omega_{\varepsilon_k},P_k}w\|_{\varepsilon_k,\Omega}-\tilde{\alpha}_k\|P_{\Omega_{\varepsilon_k},P_k}w\|_{\varepsilon_k,\Omega}|\,.$$

Since both $P_k$, $\tilde{P}_k\in\Omega_{a_0}$, we have $\|P^N_{\Omega_{\varepsilon_k},P_k}w\|_{\varepsilon_k,\Omega}\to\|w\|_{H^1(R^N)}$ and $\|P^N_{\Omega_{\varepsilon_k},\tilde{P}_k}w\|_{\varepsilon_k,\Omega}\to\|w\|_{H^1(R^N)}$. Hence $|\alpha_k-\tilde{\alpha}_k|=o(1)$.

On the other hand, suppose (6.3) is not satisfied, we have that by passing to a subsequense, $|P_k-\tilde{P}_k|/\varepsilon_k\to a$ with $0<a\leq\infty$.

If $0<a<\infty$, then $\|P^N_{\Omega_{\varepsilon_k,P_k}}w-P^N_{\Omega_{\varepsilon_k,\tilde{P}_k}}w(\cdot-(\tilde{P}_k-P_k)/\varepsilon_k)\|_{\varepsilon_k,\Omega}\to\|w-w(\cdot-a)\|_{H^1(R^N)}\neq 0$.

If $a=\infty$, then $\|P^N_{\Omega_{\varepsilon_k,P_k}}w-P^N_{\Omega_{\varepsilon_k,\tilde{P}_k}}w(\cdot-(\tilde{P}_k-P_k)/\varepsilon)\|_{\varepsilon_k,\Omega}\to 2\|w\|_{H^1(R^N)}\neq 0$. In any case, we reach a contradiction to (6.1). □

We now prove Lemma 2.2. We will follow closely Appendix B of [20]. We argue by contradiction. Suppose, there exist $\varepsilon_k\to 0$, $\eta_k\to 0$ such that

$$\inf_{v\in F_{a_0}}\|u_k-v\|_{\varepsilon_k,\Omega}<\eta_k\,,$$

and $(\alpha_k, P_k)$, $(\tilde{\alpha}_k, \tilde{P}_k)\in F_{a_0}$, such that if $v^k=u_k-\alpha_k P^N_{\Omega_{\varepsilon_k,P_k}}w$, $\tilde{v}^k=u_k-\tilde{\alpha}_k P^N_{\Omega_{\varepsilon_k,\tilde{P}_k}}w$,

$$\left\langle v^k, P^N_{\Omega_{\varepsilon_k,P_k}}w\right\rangle_{\varepsilon_k,\Omega}=0\,, \tag{6.4}$$

$$\left\langle v^k, \frac{\partial}{\partial P_i}P^N_{\Omega_{\varepsilon_k,P_k}}w\right\rangle_{\varepsilon_k,\Omega}=0\,, \tag{6.5}$$

$$\left\langle \tilde{v}^k, P^N_{\Omega_{\varepsilon_k,\tilde{P}_k}}w\right\rangle_{\varepsilon_k,\Omega}=0\,, \tag{6.6}$$

$$\left\langle \tilde{v}^k, \frac{\partial}{\partial \tilde{P}_i}P^N_{\Omega_{\varepsilon_k,\tilde{P}_k}}w\right\rangle_{\varepsilon_k,\Omega}=0\,. \tag{6.7}$$

Let $a_k=(P_k-\tilde{P}_k)/\varepsilon_k$, $\mu_k=\alpha_k-\tilde{\alpha}_k$. Then by Lemma 6.1, $|a_k|=o(1)$, $|\mu_k|=o(1)$.

We denote $C$ as various constants which do not depend on $k$. We first observe that

$$|w^p(y)-w^p(y-a_k)|\le C|a_k|w^p(y)\,. \tag{6.8}$$

By the maximum principle

$$|P^N_{\Omega_{\varepsilon_k,P_k}}w-P^N_{\Omega_{\varepsilon_k,\tilde{P}_k}}w|\le C|a_k|w(y)\,. \tag{6.9}$$

The rest of the proof is exactly the same as those of Appendix B in [20]. We omit the details. □

## 7. Appendix B. Analysis of $(E_v)$.

In this appendix, we prove Proposition 3.1.

PROOF. We first expand $K_\varepsilon(\alpha, P, v)$ at $(\alpha, P, 0)$ with respect to $v$, and we have

$$K_\varepsilon(\alpha, P, v)=K_\varepsilon(\alpha, P, 0)+f_{\alpha,\varepsilon,P}(v)+Q_{\alpha,\varepsilon,P}(v)+R_{\varepsilon,\alpha,P}(v)\,, \tag{7.1}$$

where

$$K_\varepsilon(\alpha, P, 0)=I_\varepsilon(\alpha P_{\Omega_{\varepsilon,P}}w)$$

$$f_{\varepsilon,\alpha,P}(v)=-\varepsilon^{-N}\alpha^p\int_\Omega (P^N_{\Omega_{\varepsilon,P}}w)^p v$$

$$Q_{\varepsilon,\alpha,P}(v)=\frac{1}{2\varepsilon^N}\left\{\varepsilon^2\int_\Omega|\nabla v|^2+\int_\Omega v^2-p\int_\Omega(\alpha P^N_{\Omega_{\varepsilon,P}}w)^{p-1}v^2\right\}$$

and $R_{\varepsilon,\alpha,P}$ satisfies

$$R_{\varepsilon,P}(v)=O(\|v\|_{\varepsilon,\Omega}^{\min(3,p+1)}),\tag{7.2}$$

$$R'_{\varepsilon,P}(v)=O(\|v\|_{\varepsilon,\Omega}^{\min(2,p)}),\tag{7.3}$$

$$R''_{\varepsilon,P}(v)=O(\|v\|_{\varepsilon,\Omega}^{\min(1,p-1)}).\tag{7.4}$$

Since $f_{\varepsilon,\alpha,P}(v)$ is a continuous linear functional on $E_{\varepsilon,P}$ equipped with the scalar product $\langle\cdot,\cdot\rangle_{\varepsilon,\Omega}$, we may write

$$f_{\varepsilon,\alpha,P}(v)=\langle F_{\varepsilon,\alpha,P},v\rangle_{\varepsilon,\Omega}\quad\text{for some } F_{\varepsilon,\alpha,P}\in E_{\varepsilon,P}.\tag{7.5}$$

Since $Q_{\varepsilon,\alpha,P}$ is a continuous quadratic form on $E_{\varepsilon,P}$, there exists a continuous and symmetric operator $L_{\varepsilon,\alpha,P}\in L(E_{\varepsilon,P})$ (the space of bounded linear operators on $E_{\varepsilon,P}$) such that

$$Q_{\varepsilon,\alpha,P}(v)=\langle L_{\varepsilon,\alpha,P}v,v\rangle_{\varepsilon,\Omega}.\tag{7.6}$$

Moreover, we have by the same argument as in [19, Lemma 4.2], there exists $\rho>0$ such that for $\varepsilon$ and $\eta$ small enough, we have

$$Q_{\varepsilon,\alpha,P}(v)\geq\rho\|v\|^2_{\varepsilon,\Omega},\quad\text{for all } v\in E_{\varepsilon,P}.\tag{7.7}$$

Therefore $L_{\varepsilon,\alpha,P}$ is a coercive operator whose modulus of coercivity is bounded from below independently of $\varepsilon$, $P$.

The derivative of $K_\varepsilon$ with respect to $v$ on $E_{\varepsilon,P}$ may be written as

$$F_{\varepsilon,\alpha,P}+2L_{\varepsilon,\alpha,P}v+O(\|v\|^2_{\varepsilon,\Omega}).$$

Using the implicit function theorem, we derive the existence of a $C^2$ map $T$ which to each $(\varepsilon,\alpha,P)$ associates $v_{\varepsilon,\alpha,P}\in E_{\varepsilon,P}$ such that

$$\frac{\partial K_\varepsilon}{\partial v}(\alpha,P,v)|_{E_{\varepsilon,P}}=0$$

and

$$\|v_{\varepsilon,\alpha,P}\|_{\varepsilon,\Omega}=O(\|F_{\varepsilon,\alpha,P}\|_{\varepsilon,\Omega}).\tag{7.8}$$

Moreover, since $v_{\varepsilon,\alpha,P}$ minimizes $K_\varepsilon$ over $E_{\varepsilon,P}$, $\partial v_{\varepsilon,\alpha,P}/\partial\nu=0$ on $\partial\Omega$.

We now claim that

$$\|f_{\varepsilon,\alpha,P}(v)\|_{\varepsilon,\Omega}\leq O((\varphi^N_{\varepsilon,P}(P))^{(1+\sigma)/2})\|v\|_{\varepsilon,\Omega},\tag{7.9}$$

which by (7.5) and (7.8), proves Proposition 3.1.

Indeed, for $v\in E_{\varepsilon,P}$, we have

$$f_{\varepsilon,P}(v) = -\varepsilon^{-N}\left[\int_{\Omega}(\alpha P^N_{\Omega_{\varepsilon,P}}w)^p v\right]$$

$$= -\varepsilon^{-N}\alpha^p \int_{\Omega}((P^N_{\Omega_{\varepsilon,P}}w)^p - w^p)v .$$

We now calculate,

$$\varepsilon^{-N}\int_{\Omega} |(P^N_{\Omega_{\varepsilon,P}}w)^p - w^p|^2 \le O((\varphi^N_{\varepsilon,P}(P))^{1+\sigma}) .$$

Thus (7.9) is established. □

**8. Appendix C. Various estimates.** In this appendix, we provide all the estimates we stated before. The proofs are similar to those in [20, Appendix C]. We omit the details.

(C.1)
$$\langle P^N_{\Omega_{\varepsilon,P}}w, P^N_{\Omega_{\varepsilon,P}}w\rangle_{\varepsilon,\Omega} = \int_{R^N} w^{p+1} + O(\varphi^N_{\varepsilon,P}(P)) ,$$

(C.2)
$$\int_{\Omega_{\varepsilon,P}} (w^p - (P^N_{\Omega_{\varepsilon,P}}w)^p)\frac{\partial P^N_{\Omega_{\varepsilon,P}}w}{\partial P_i}$$
$$= -p\frac{1}{\varepsilon}\int_{\Omega_{\varepsilon,P}} w^{p-1}w'\frac{y_i}{|y|}V^N_{\varepsilon,P}(y) + O((\varphi^N_{\varepsilon,P}(P))^{1+\sigma}) ,$$

(C.3)
$$\varepsilon\int_{\Omega_{\varepsilon,P}} (P^N_{\Omega_{\varepsilon,P}}w)^{p-1}\frac{\partial P^N_{\Omega_{\varepsilon,P}}w}{\partial P_i}\bar{v} = O((\varphi^N_{\varepsilon,P}(P))^{1+\sigma}) ,$$

(C.4)
$$\left\langle \frac{\partial P^N_{\Omega_{\varepsilon,P}}w}{\partial P_i}, \frac{\partial P^N_{\Omega_{\varepsilon,P}}w}{\partial P_j}\right\rangle_{\varepsilon,\Omega} = \frac{1}{\varepsilon^2}p\int_{R^N} w^{p-1}(w')^2\delta_{ij} + O((\varphi^N_{\varepsilon,P}(P))^{1+\sigma}) ,$$

(C.5)
$$\int_{\Omega_{\varepsilon,P}} (P^N_{\Omega_{\varepsilon,P}}w)^p\bar{v} = O((\varphi^N_{\varepsilon,P}(P))^{1+\sigma}) ,$$

(C.6)
$$\int_{\Omega_{\varepsilon,P}} |P^N_{\Omega_{\varepsilon,P}}w + \bar{v}|^{p+1} = \int_{R^N} w^{p+1} + O(\varphi^N_{\varepsilon,P}(P)) ,$$

(C.7)
$$\int_{\Omega_{\varepsilon,P}} |P^N_{\Omega_{\varepsilon,P}}w + \bar{v}|^{p-1}(P^N_{\Omega_{\varepsilon,P}}w + \bar{v})P^N_{\Omega_{\varepsilon,P}}w = \int_{R^N} w^{p+1} + O(O(\varphi^N_{\varepsilon,P}(P))) ,$$

(C.8)
$$\int_{\Omega_{\varepsilon,P}} |P^N_{\Omega_{\varepsilon,P}}w + \bar{v}|^{p-1}(P^N_{\Omega_{\varepsilon,P}}w + \bar{v})\frac{\partial P^N_{\Omega_{\varepsilon,P}}w}{\partial P_i}$$
$$= \int_{\Omega_{\varepsilon,P}} (P^N_{\Omega_{\varepsilon,P}}w)^p\frac{\partial P^N_{\Omega_{\varepsilon,P}}w}{\partial P_i} + O((\varphi^N_{\varepsilon,P}(P))^{1+\sigma}) ,$$

(C.9) $$\frac{\partial K_\varepsilon}{\partial \alpha} = (\alpha - \alpha^p) \int_{R^N} w^{p+1} + O(\varphi^N_{\varepsilon,P}(P)) ,$$

(C.10) $$\varepsilon \frac{\partial K_i}{\partial P_i} = -p \int_{\Omega_{\varepsilon,P}} w^{p-1} w' \frac{y_i}{|y|} V^N_{\varepsilon,P}(y) + O((\alpha-1)O(\varphi^N_{\varepsilon,P}(P)) + (\varphi^N_{\varepsilon,P}(P))^{1+\sigma}) .$$

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