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On the Internal and Boundary Stabilization of Timoshenko Beams

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Abstract. In this paper we consider Timoshenko systems with either internal or boundary feedbacks. We establish explicit and generalized decay results, without imposing restrictive growth assumption near the origin on the damping terms.

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1. Introduction

Timoshenko [16] gave the following system of coupled hyperbolic equations

$$\begin{aligned} \rho u_{tt} &= \big(K(u_x-\varphi)\big)_x\,, \quad \text{in} \quad (0,L)\times(0,+\infty) \\ I_\rho \varphi_{tt} &= (EI\varphi_x)_x + K(u_x-\varphi)\,, \quad \text{in} \quad (0,L)\times(0,+\infty)\,, \end{aligned} \tag{1.1}$$

as a simple model describing the transverse vibration of a beam. Here t denotes the time variable, x is the space variable along the beam of length L, in its equilibrium configuration, u is the transverse displacement of the beam, and φ is the rotation angle of the filament of the beam. The coefficients ρ, I_ρ, E, I and K are respectively the mass per unit length, the polar moment of inertia of a cross section, Young's modulus of elasticity, the moment of inertia of a cross section, and the shear modulus.

Kim and Renardy [5] considered (1.1) together with boundary controls of the form

$$\begin{aligned} K\varphi(L,t) - K\frac{\partial u}{\partial x}(L,t) &= \alpha\frac{\partial u}{\partial t}(L,t)\,, \quad \forall t\geq 0 \\ EI\frac{\partial \varphi}{\partial x}(L,t) &= -\beta\frac{\partial \varphi}{\partial t}(L,t)\,, \quad \forall t\geq 0 \end{aligned}$$

and used the multiplier techniques to establish an exponential decay result for the energy of (1.1). In addition, a polynomial decay result for the energy of (1.1) was established by Yan [18] when considering the boundary conditions

$$\begin{aligned} K\left(\varphi(L,t)-\frac{\partial u}{\partial x}(L,t)\right) &= f_1\left(\frac{\partial u}{\partial t}(L,t)\right), \quad \forall t\geq 0\\ -EI\frac{\partial \varphi}{\partial x}(L,t) &= f_2\left(\frac{\partial \varphi}{\partial t}(L,t)\right), \quad \forall t\geq 0\,, \end{aligned}$$

and f_1, f_2 having polynomial growth near the origin. Soufyane and Wehbe [15] established the uniform stability of (1.1), using a unique locally distributed feedback. Precisely, they considered

$$\begin{aligned} \rho u_{tt} &= \big(K(u_x-\varphi)\big)_x\,, \quad \text{in} \quad (0,L)\times(0,+\infty)\\ I_\rho \varphi_{tt} &= (EI\varphi_x)_x + K(u_x-\varphi) - b\varphi_t\,, \quad \text{in} \quad (0,L)\times(0,+\infty) \\ u(0,t) &= u(L,t) = \varphi(0,t) = \varphi(L,t) = 0\,, \quad \forall t\geq 0\,, \end{aligned} \tag{1.2}$$

where b is a positive and continuous function, which satisfies

$$b(x)\geq b_0>0\,, \quad \forall x\in[a_0,a_1]\subset[0,L]\,,$$

and proved that the uniform stability of (1.2) holds if and only if the wave speeds are equal ($\frac{K}{\rho}=\frac{EI}{I_\rho}$); otherwise only the asymptotic stability can be obtained. This result has been extended by Rivera and Racke [11] for the damping function $b=b(x)$ possibly changes sign, and for a nonlinear system in [10]. Rivera and Racke [9] also treated a nonlinear system with damping effect through heat conduction of the form

$$\begin{aligned} \rho_1 u_{tt} - \sigma(u_x,\varphi_x)_x &= 0\,, \quad \text{in} \quad (0,L)\times(0,+\infty)\\ \rho_2\varphi_{tt} - b\varphi_{xx} + K(u_x+\varphi) + \gamma\theta_x &= 0\,, \quad \text{in} \quad (0,L)\times(0,+\infty)\,,\\ \rho_3\theta_t - K\theta_{xx} + \gamma\varphi_{xt} &= 0\,, \quad \text{in} \quad (0,L)\times(0,+\infty)\,, \end{aligned}$$

where θ is the difference temperature. Under appropriate conditions on the nonlinearity, they proved an exponential decay result for the case of equal wave speeds ($\frac{K}{\rho_1}=\frac{b}{\rho_2}$). Raposo et al. [12] considered the following system

$$\begin{aligned} \rho_1 u_{tt} - K(u_x-\varphi)_x + u_t &= 0\,, \quad \text{in} \quad (0,L)\times(0,+\infty)\\ \rho_2\varphi_{tt} - b\varphi_{xx} - K(u_x-\varphi) + \varphi_t &= 0\,, \quad \text{in} \quad (0,L)\times(0,+\infty) \\ u(0,t) = u(L,t) = \varphi(0,t) = \varphi(L,t) &= 0\,, \quad \forall t\geq 0\,, \end{aligned} \tag{1.3}$$

and proved that the energy associated with (1.3) decays exponentially without imposing the equal wave speed condition. This result is expected in the presence of linear damping terms in both equations. As they mentioned, their aim was to use a method developed by Liu and Zheng [6], which is based on the semigroup theory. Ammar-Khodja et al. [1] considered a linear Timoshenko-type system with memory of the form

$$\rho_1\varphi_{tt} - K(\varphi_x + \psi)_x = 0\,, \quad \text{in} \quad (0,L)\times(0,+\infty)$$
$$\rho_2\psi_{tt} - b\psi_{xx} + \int_0^t g(t-s)\psi_{xx}(s)ds + K(\varphi_x+\psi) = 0\,, \quad \text{in} \quad (0,L)\times(0,+\infty) \tag{1.4}$$

together with homogeneous boundary conditions. They used the multiplier techniques and proved that the system is uniformly stable if and only if the wave speeds are equal ($\frac{K}{\rho_1} = \frac{b}{\rho_2}$) and g decays uniformly. Precisely, they proved an exponential decay if g decays in an exponential rate and polynomially if g decays in a polynomial rate. In [4], Guesmia and Messaoudi investigated the effect of both frictional and viscoelastic dampings. They considered the following system

$$\begin{aligned} \varphi_{tt} - (\varphi_x + \psi)_x &= 0 \\ \psi_{tt} - \psi_{xx} + \varphi_x + \psi + \int_0^t g(t-s)\big(a(x)\psi_x(s)\big)_x ds + b(x)h(\psi_t) &= 0 \end{aligned} \tag{1.5}$$

in $(0,1)\times(0,+\infty)$, together with homogeneous boundary conditions. An exponential and polynomial decay result has been established under weaker conditions on the relaxation function g than that in [1]. Santos [13] considered a Timoshenko system and showed that the presence of two feedbacks of memory type at a portion of the boundary stabilizes the system uniformly, and the rate of decay of the energy is of the same order of decay as the relaxation functions. This result has been generalized by Messaoudi and Soufyane [8], where they considered a multi-dimensional Timoshenko-type system with boundary conditions of memory type and proved energy decay results, for which the usual exponential and polynomial decay rates are only special cases. For more results concerning the controllability of Timoshenko systems, we refer the reader to [2, 3, 14, 17], and [19].

In this paper we are concerned with the following types of Timoshenko systems

$$\begin{cases} a\varphi_{tt} - k(\varphi_x+\psi)_x = 0\,, \quad (0,1)\times\mathbb{R}_+ \\ b\psi_{tt} - d\psi_{xx} + k\,(\varphi_x+\psi) + h_2(\psi_t) = 0\,, \quad (0,1)\times\mathbb{R}_+ \\ \varphi(0,t) = \psi(0,t) = \psi(1,t) = 0\,, \quad \varphi_x(1,t) = -h_1\big(\varphi_t(1,t)\big)\,, \quad t\ge 0 \\ \varphi(\,\cdot\,,0)=\varphi_0,\ \varphi_t(\,\cdot\,,0)=\varphi_1,\ \psi(\,\cdot\,,0)=\psi_0,\ \psi_t(\,\cdot\,,0)=\psi_1 \quad \text{in} \quad (0,1) \end{cases} \tag{1.6}$$

$$\begin{cases} a\varphi_{tt} - k(\varphi_x+\psi)_x + h_1(\varphi_t) = 0\,, \quad (0,1)\times\mathbb{R}_+ \\ b\psi_{tt} - d\psi_{xx} + k\,(\varphi_x+\psi) + h_2(\psi_t) = 0\,, \quad (0,1)\times\mathbb{R}_+ \\ \varphi(0,t) = \varphi(1,t) = \psi(0,t) = \psi(1,t) = 0\,, \quad t\ge 0 \\ \varphi(\,\cdot\,,0)=\varphi_0,\ \varphi_t(\,\cdot\,,0)=\varphi_1\,, \quad \psi(\,\cdot\,,0)=\psi_0,\ \psi_t(\,\cdot\,,0)=\psi_1 \quad \text{in} \quad (0,1) \end{cases} \tag{1.7}$$

and

$$\begin{cases} a\varphi_{tt} - k(\varphi_x + \psi)_x = 0\,, \quad (0,1)\times\mathbb{R}_+ \\ b\psi_{tt} - d\psi_{xx} + k\,(\varphi_x + \psi) = 0\,, \quad (0,1)\times\mathbb{R}_+ \\ \varphi(0,t) = 0\,, \quad \psi(1,t) + \varphi_x(1,t) = -h_1\big(\varphi_t(1,t)\big)\,, \quad t \geq 0 \\ \psi(0,t) = 0\,, \quad \psi_x(1,t) = -h_2\big(\psi_t(1,t)\big)\,, \quad t \geq 0 \\ \varphi(\,\cdot\,,0) = \varphi_0\,, \quad \varphi_t(\,\cdot\,,0) = \varphi_1,\ \psi(\,\cdot\,,0) = \psi_0,\ \psi_t(\,\cdot\,,0) = \psi_1 \quad \text{in} \quad (0,1) \end{cases} \tag{1.8}$$

where h_1 and h_2 are specific functions and a, b, d, k are positive constants. These systems describe the transverse vibrations of a beam subjected to a joint effect of two (internal or/and boundary) frictional mechanisms. Our aim is to establish explicit and generalized decay rate results for the energy of these systems, without imposing any restrictive growth assumption near the origin on the damping terms. The results of this paper allow a larger class of functions h_1 and h_2, from which the energy decay rates are not necessarily of exponential or polynomial types (see the examples in Section 4).

The proofs of our results are done basically in two steps. In the first step, we use the multiplier method and benefit from [2] and [8] to choose the right multipliers. In the second step, we follow, with necessary modifications dictated by the nature of our systems, the method introduced and used by Martinez [7] to study the wave equations. The paper is organized as follows. In Section 2, we present some notations and material needed for our work. The statements and proofs of the main results are given in Sections 3 and 4. In the last section, we investigate the special case of the polynomial growth.

2. Preliminaries

In this section we present some material needed for the proofs of our main results. In the sequel we assume that system (1.6) has a unique solution

$$\begin{aligned} &\varphi \in L^\infty\big(\mathbb{R}_+; H^2(0,1)\cap V\big) \cap W^{1,\infty}(\mathbb{R}_+; V) \cap W^{2,\infty}\big(\mathbb{R}_+; L^2(0,1)\big)\,, \\ &\psi \in L^\infty\big(\mathbb{R}_+; H^2(0,1)\cap H_0^1(0,1)\big) \cap W^{1,\infty}\big(\mathbb{R}_+; H_0^1(0,1)\big) \cap W^{2,\infty}\big(\mathbb{R}_+; L^2(0,1)\big)\,, \end{aligned}$$

system (1.7) has a unique solution

$$\varphi,\, \psi \in L^\infty\big(\mathbb{R}_+; H^2(0,1)\cap H_0^1(0,1)\big) \cap W^{1,\infty}\big(\mathbb{R}_+; H_0^1(0,1)\big) \cap W^{2,\infty}\big(\mathbb{R}_+; L^2(0,1)\big)\,,$$

and system (1.8) has a unique solution

$$\varphi,\, \psi \in L^\infty\big(\mathbb{R}_+; H^2(0,1)\cap V\big) \cap W^{1,\infty}(\mathbb{R}_+; V) \cap W^{2,\infty}\big(\mathbb{R}_+; L^2(0,1)\big)\,,$$

where $V = \{v \in H^1(0,1) : v(0) = 0\}$. These results can be proved, for initial data in suitable function spaces, using standard arguments such as the Galerkin method.

The following lemma will be of essential use in establishing our main results.

Lemma 2.1 ([7]). *Let $E : \mathbb{R}_+ \to \mathbb{R}_+$ be a nonincreasing function and $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ be a strictly increasing C^1-function, with $\sigma(t) \to +\infty$ as $t \to +\infty$.*

Assume that there exist $p, q \geq 0$ and $c > 0$ such that

$$\int_S^\infty \sigma'(t)E(t)^{1+p}dt \leq cE(S)^{1+p} + \frac{cE(S)}{\sigma^q} \quad 1 \leq S < +\infty .$$

Then there exist positive constants k and ω such that

$$E(t) \leq ke^{-\omega\sigma(t)} \quad \forall t \geq 1 , \quad \textit{if} \quad p = q = 0$$

$$E(t) \leq \frac{k}{\sigma(t)^{\frac{1+q}{p}}} \quad \forall t \geq 1 , \quad \textit{if} \quad p > 0 .$$

Now, we introduce the energy functional

$$E(t) := \frac{1}{2}\int_0^1 \big(a\varphi_t^2 + b\psi_t^2 + d\psi_x^2 + k(\varphi_x + \psi)^2\big)dx . \tag{2.1}$$

We will use c, throughout this paper, to denote a generic positive constant which may depend on the initial energy of the solution (see (3.12) for instance).

3. Decay of energy of system (1.6)

In this section we state and prove our main result for system (1.6). We consider the following hypothesis on h_1 and h_2

(H1) $h_i : \mathbb{R} \to \mathbb{R}$ (for $i = 1, 2$) are nondecreasing C^1 functions such that

$$H_i(|s|) \leq |h_i(s)| \leq H_i^{-1}(|s|) \quad \text{for all} \quad |s| \leq m , \quad i = 1, 2$$

$$c_1\,|s| \leq |h_1(s)| \leq c_2\,|s| \quad \text{for all} \quad |s| \geq m$$

$$c_1\,|s| \leq |h_2(s)| \leq c_2\,|s|^q \quad \text{for all} \quad |s| \geq m ,$$

where H_1 and H_2 are strictly increasing C^1 functions on $[0, +\infty)$, $H_1(0) = H_2(0) = 0$, the constants m, c_1, c_2 are positive, and $q \geq 1$.

Remark 3.1. Hypothesis (H1) implies that $sh_i(s) > 0$, for all $s \neq 0$, $i = 1, 2$.

Lemma 3.1. *Let (φ, ψ) be the solution of* (1.6). *Then the energy functional satisfies*

$$E'(t) = -k\varphi_t(1, t)h_1\big(\varphi_t(1, t)\big)dx - \int_0^1 \psi_t h_2(\psi_t)dx \leq 0 . \tag{3.1}$$

Proof. By multiplying the first two equations in (1.6) by φ_t and ψ_t respectively, integrating over $(0, 1)$, and doing some manipulations, we obtain (3.1). □

In the next lemma, we use the multiplier w, defined by

$$w(x, t) = -\int_0^x \psi(s, t)ds , \quad x \in [0, 1] . \tag{3.2}$$

Lemma 3.2. *Let* (φ, ψ) *be the solution of* (1.6) *and* $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ *be a concave nondecreasing* C^2*-function. Then, for* $T \geq S \geq 0$*, the energy functional satisfies*

$$\int_S^T \sigma'(t)E(t)^2 dt \leq cE(S)^2 + c\int_S^T \sigma' E\left(\int_0^1 \left[\psi_t^2 - c'\psi h_2(\psi_t)\right]dx\right)dt \\ + c\int_S^T \sigma' E\Big(\varphi_t^2(1,t) + h_1^2\big(\varphi_t(1,t)\big)\Big)dt\,. \qquad (3.3)$$

Proof. We multiply the first equation in (1.6) by $(x\varphi_x + Nw)\sigma' E$ and the second equation by $N\psi\sigma' E$, where $N > 0$ to be chosen later, integrate over $(0,1)\times(S,T)$, and use integration by parts to get

$$\begin{aligned}\int_S^T \sigma'(t)E(t)^2 dt = &-\left[\sigma'(t)E(t)\int_0^1 (ax\varphi_x\varphi_t + Naw\varphi_t + Nb\psi\psi_t)\,dx\right]_S^T \\ &+ \int_S^T (\sigma'' E + \sigma' E')\left(\int_0^1 [ax\varphi_x\varphi_t + Naw\varphi_t + Nb\psi\psi_t]\,dx\right)dt \\ &+ Na\int_S^T \sigma' E\left(\int_0^1 w_t\varphi_t dx\right)dt + k\int_S^T \sigma' E\left(\int_0^1 (\psi + x\psi_x)(\psi+\varphi_x)dx\right)dt \\ &+ Nk\int_S^T \sigma' E\big(w(1,t)\varphi_x(1,t)\big)dt - \int_S^T \sigma' E\left(\int_0^1 \left(Nd - \frac{d}{2}\right)\psi_x^2 dx\right)dt \qquad (3.4) \\ &+ \int_S^T \sigma' E\left(\int_0^1 \left[\left(Nb + \frac{b}{2}\right)\psi_t^2 - N\psi h_2(\psi_t)\right]dx\right)dt \\ &+ \int_S^T \sigma' E\left[\frac{a}{2}\varphi_t^2(1,t) + \frac{k}{2}h_1^2\big(\varphi_t(1,t)\big)\right]dt\,.\end{aligned}$$

We exploit Young's, Poincaré's, and Hölder's inequalities, and the fact that

$$\varphi_x^2 \leq 2(\psi+\varphi_x)^2 + 2\psi^2\,,$$

to estimate the terms in the right hand side of (3.4) as follows

- $I_1 := -\left[\sigma'(t)E(t)\int_0^1 (ax\varphi_x\varphi_t + Naw\varphi_t + Nb\psi\psi_t)\,dx\right]_S^T$.

Since

$$\begin{aligned}\int_0^1 (ax\varphi_x\varphi_t + Naw\varphi_t + Nb\psi\psi_t)\,dx &\leq c\int_0^1 \left(\varphi_x^2 + \varphi_t^2 + w^2 + \psi^2 + \psi_t^2\right)dx \\ &\leq c\int_0^1 \big((\psi+\varphi_x)^2 + \varphi_t^2 + \psi_x^2 + \psi_t^2\big)dx \\ &\leq cE(t)\,,\end{aligned}$$

then, by the properties of E and σ, we conclude that

$$I_1 \leq c\left|[\sigma'(t)E(t)^2]_S^T\right| \leq c\sigma'(S)E(S)^2\,.$$

- $I_2 := \int_S^T (\sigma'' E + \sigma' E')(\int_0^1 [ax\varphi_x\varphi_t + Naw\varphi_t + Nb\psi\psi_t]\,dx)dt$.

As in above, we conclude that

$$\begin{aligned} I_2 &\le c\left|\int_S^T \sigma'' E^2 dt\right| + c\left|\int_S^T \sigma' E' E dt\right| \\ &\le cE(S)^2\left|\int_S^T \sigma'' dt\right| + c\sigma'(S)\left|\int_S^T EE' dt\right| \\ &\le c\sigma'(S)E(S)^2 . \end{aligned}$$

- $I_3 := Na\int_S^T \sigma' E(\int_0^1 w_t\varphi_t dx)dt$

$$I_3 \le \varepsilon\int_S^T \sigma' E\left(\int_0^1 \varphi_t^2 dx\right)dt + C_\varepsilon\int_S^T \sigma' E\left(\int_0^1 \psi_t^2 dx\right)dt .$$

- $I_4 := k\int_S^T \sigma' E(\int_0^1 (\psi + x\psi_x)(\psi+\varphi_x)dx)dt$

$$I_4 \le \varepsilon\int_S^T \sigma' E\left(\int_0^1 (\psi+\varphi_x)^2 dx\right)dt + C_\varepsilon\int_S^T \sigma' E\left(\int_0^1 \psi_x^2 dx\right)dt .$$

- $I_5 := Nk\int_S^T \sigma' E w(1,t)\varphi_x(1,t)dt$

$$w^2(1,t) = \left(\int_0^1 w_x dx\right)^2 \le \int_0^1 w_x^2 dx \le c\int_0^1 \psi_x^2 dx .$$

Therefore, using the boundary condition in (1.6), we have

$$I_5 \le c\int_S^T \sigma' E\left(\int_0^1 \psi_x^2 dx\right)dt + cN^2\int_S^T \sigma' E h_1^2\big(\varphi_t(1,t)\big)dt .$$

By using our estimates for I_1–I_5 into (3.4) and taking ε small enough and N large enough, we obtain (3.3). □

We are now ready to state and prove the main result for system (1.6).

Theorem 3.3. *Assume that* (H1) *holds. Then there exists a constant* $c > 0$ *such that, for* t *large, the solution of* (1.6) *satisfies*

$$E(t) \le c\left(K^{-1}\left(\frac{1}{t}\right)\right)^2 , \tag{3.5}$$

where

$$K(s) = s(H_1^{-1} + H_2^{-1})^{-1}(s) .$$

Moreover, if H_1, H_2 *are strictly convex on* $(0,r)$*, for some* $r > 0$*, and* $H_1'(0) = H_2'(0) = 0$*, then we have the improved estimate*

$$E(t) \le c\left((H_1^{-1} + H_2^{-1})\left(\frac{1}{t}\right)\right)^2 . \tag{3.6}$$

Proof. Let $H_0 := \left(H_1^{-1} + H_2^{-1}\right)^{-1}$ and

$$\phi(t) := 1 + \int_1^t \frac{1}{H_0(1/s)} ds \quad t \geq t', \tag{3.7}$$

for some $t' > \max\{1, \frac{1}{m}\}$. Then

$$\phi'(t) = \frac{1}{H_0(1/t)} > 0 \quad \forall t \geq t', \quad \phi'(t) \to +\infty \quad \text{as} \quad t \to +\infty,$$

and $\phi'(t)$ is strictly increasing.

Thus, ϕ is a convex and strictly increasing C^2-function, with $\phi(t) \to +\infty$ as $t \to +\infty$.

If we set

$$\sigma_0 := \phi^{-1}, \quad t \geq t', \tag{3.8}$$

then it is easy to check that σ_0 is strictly increasing and $\sigma_0'(t) = H_0(1/\sigma_0(t))$ is strictly decreasing. So σ_0 is a concave C^2-function, with $\sigma_0(t) \to +\infty$ as $t \to +\infty$.

We use this particular function σ_0, and take $t_1 \geq t'$ such that $\sigma_0'(t_1) < m$, to estimate the last integrals in (3.3), for $T \geq S \geq t_1$, as follows.

1) Estimate for $\int_S^T \sigma_0' E(\int_0^1 \psi_t^2 dx) dt$

We consider the following partition of $(0, 1)$

$$\begin{aligned} \Omega_1 &= \left\{x \in (0,1) : |\psi_t| > m\right\} \\ \Omega_2 &= \left\{x \in (0,1) : |\psi_t| \leq m \text{ and } |\psi_t| \leq H_2^{-1}\left(\sigma_0'(t)\right)\right\} \\ \Omega_3 &= \left\{x \in (0,1) : |\psi_t| \leq m \text{ and } |\psi_t| > H_2^{-1}\left(\sigma_0'(t)\right)\right\}. \end{aligned} \tag{3.9}$$

Consequently, we have

$$\begin{aligned} &\sigma_0'(t) \int_{\Omega_1} \psi_t^2 dx \leq \frac{1}{c_1} \sigma_0'(t) \int_0^1 \psi_t h_2(\psi_t) dx \leq -cE'(t) \\ &\sigma_0'(t) \int_{\Omega_2} \psi_t^2 dx \leq \sigma_0'(t) \Big(H_2^{-1}\left(\sigma_0'(t)\right)\Big)^2 \leq \sigma_0'(t) \Big(H_0^{-1}\left(\sigma_0'(t)\right)\Big)^2 \\ &\sigma_0'(t) \int_{\Omega_3} \psi_t^2 dx \leq m \int_{\Omega_3} H_2(|\psi_t|)\, |\psi_t|\, dx \leq m \int_0^1 \psi_t h_2(\psi_t) dx \leq -mE'(t), \end{aligned}$$

which gives

$$\int_S^T \sigma_0' E\left(\int_0^1 \psi_t^2 dx\right) dt \leq cE(S)^2 + cE(S) \int_S^T \sigma_0'(t) \Big(H_0^{-1}\left(\sigma_0'(t)\right)\Big)^2 dt. \tag{3.10}$$

2) Estimate for $\int_S^T \sigma_0' E(\int_0^1(-c'\psi h_2(\psi_t))dx)dt$

We consider the following partition of $(0,1)$

$$\begin{aligned}\Omega^1 &= \{x\in(0,1) : |\psi_t| > m\}\\ \Omega^2 &= \{x\in(0,1) : |\psi_t|\le m \text{ and } |\psi_t|\le \sigma_0'(t)\}\\ \Omega^3 &= \{x\in(0,1) : |\psi_t|\le m \text{ and } |\psi_t| > \sigma_0'(t)\}.\end{aligned}\tag{3.11}$$

Then, using Hölder's, Young's and Poincaré's inequalities, (H1) and the embedding $H_0^1(0,1)\hookrightarrow L^r(0,1)$ for $r\ge 1$, we have

$$\begin{aligned}\sigma_0'(t)\int_{\Omega^1}\psi h_2(\psi_t)dx &\le \sigma_0'(t)\left(\int_{\Omega^1}|\psi|^{q+1}\,dx\right)^{\frac{1}{q+1}}\left(\int_{\Omega^1}|h_2(\psi_t)|^{1+\frac{1}{q}}\,dx\right)^{\frac{q}{q+1}}\\ &\le c\sigma_0'(t)\left(\int_0^1\psi_x^2dx\right)^{\frac12}\left(\int_{\Omega^1}\psi_t h_2(\psi_t)dx\right)^{\frac{q}{q+1}}\\ &\le c\sigma_0'(t)E(t)^{\frac12}\big(-E'(t)\big)^{\frac{q}{q+1}}\le c\sigma_0'(t)\left[\varepsilon E(t)^{\frac{q+1}{2}}-C_\varepsilon E'(t)\right]\\ &\le c\varepsilon\sigma_0'(t)E(t)-C_\varepsilon E'(t).\end{aligned}\tag{3.12}$$

$$\begin{aligned}\sigma_0'(t)\int_{\Omega^2}\psi h_2(\psi_t)dx &\le \varepsilon\sigma_0'(t)\int_{\Omega^2}\psi^2dx + C_\varepsilon\sigma_0'(t)\int_{\Omega^2}h_2(\psi_t)^2dx\\ &\le c\varepsilon\sigma_0'(t)E(t)+C_\varepsilon\sigma_0'(t)\Big(H_2^{-1}\big(\sigma_0'(t)\big)\Big)^2\\ &\le c\varepsilon\sigma_0'(t)E(t)+C_\varepsilon\sigma_0'(t)\Big(H_0^{-1}\big(\sigma_0'(t)\big)\Big)^2.\end{aligned}$$

$$\begin{aligned}\sigma_0'(t)\int_{\Omega^3}\psi h_2(\psi_t)dx &\le \varepsilon\sigma_0'(t)\int_{\Omega^3}\psi^2dx + C_\varepsilon\sigma_0'(t)\int_{\Omega^3}h_2(\psi_t)^2dx\\ &\le c\varepsilon\sigma_0'(t)E(t)+C_\varepsilon H_2^{-1}(m)\int_0^1\psi_t h_2(\psi_t)dx\\ &\le c\varepsilon\sigma_0'(t)E(t)-C_\varepsilon E'(t).\end{aligned}$$

A combination of all the above leads to

$$\begin{aligned}&\int_S^T\sigma_0'E\left(\int_0^1\big(-c'\psi h_2(\psi_t)\big)dx\right)dt\\ &\quad\le c\varepsilon\int_S^T\sigma_0'E^2dt+C_\varepsilon E(S)^2+C_\varepsilon E(S)\int_S^T\sigma_0'(t)\Big(H_0^{-1}\big(\sigma_0'(t)\big)\Big)^2dt.\end{aligned}\tag{3.13}$$

3) Estimate for $\int_S^T \sigma_0' E\varphi_t^2(1,t)dt$

By considering the following cases

$$\begin{aligned}&C1 : |\varphi_t(1,t)| > m\\ &C2 : |\varphi_t(1,t)|\le m \quad\text{and}\quad |\varphi_t(1,t)|\le H_1^{-1}\big(\sigma_0'(t)\big),\\ &C3 : |\varphi_t(1,t)|\le m \quad\text{and}\quad |\varphi_t(1,t)| > H_1^{-1}\big(\sigma_0'(t)\big),\end{aligned}$$

we deduce, as in the above, that

$$\int_S^T \sigma_0' E\varphi_t^2(1,t)dt \le cE(S)^2 + cE(S)\int_S^T \sigma_0'(t)\Big(H_0^{-1}\big(\sigma_0'(t)\big)\Big)^2 dt\,. \tag{3.14}$$

4) Estimate for $\int_S^T \sigma_0' Eh_1^2(\varphi_t(1,t))dt$

We consider the following cases

$$\begin{aligned} &C'1: |\varphi_t(1,t)| > m\\ &C'2: |\varphi_t(1,t)| \le m \quad \text{and} \quad |\varphi_t(1,t)| \le \sigma_0'(t),\\ &C'3: |\varphi_t(1,t)| \le m \quad \text{and} \quad |\varphi_t(1,t)| > \sigma_0'(t)\,,\end{aligned}$$

and we similarly obtain

$$\int_S^T \sigma_0' Eh_1^2\big(\varphi_t(1,t)\big)dt \le cE(S)^2 + cE(S)\int_S^T \sigma_0'(t)\Big(H_0^{-1}\big(\sigma_0'(t)\big)\Big)^2 dt\,. \tag{3.15}$$

Combining (3.3), (3.10), (3.13)–(3.15) and taking ε small enough lead to

$$\begin{aligned}\int_S^\infty \sigma_0'(t)E(t)^2dt &\le cE(S)^2 + cE(S)\int_S^\infty \sigma_0'(t)\Big(H_0^{-1}\big(\sigma_0'(t)\big)\Big)^2 dt\\ &= cE(S)^2 + cE(S)\int_{\sigma_0(S)}^\infty \left(H_0^{-1}\left(H_0\left(\frac{1}{s}\right)\right)\right)^2 ds\\ &= cE(S)^2 + \frac{cE(S)}{\sigma_0(S)}\,.\end{aligned}$$

Lemma 2.1, then gives

$$E(t) \le \frac{c}{\sigma_0(t)^2} \quad \forall t \ge t_1\,. \tag{3.16}$$

To obtain (3.5), we take $s_0 > t'$ such that $H_0(\frac{1}{s_0}) \le 1$. Since H_0 is increasing and $K(s) = sH_0(s)$, we have

$$\sigma_0^{-1}(s) \le 1 + (s-1)\frac{1}{H_0(\frac{1}{s})} \le \frac{s}{H_0(\frac{1}{s})} = \frac{1}{K(\frac{1}{s})} \quad \forall s \ge s_0\,.$$

So, with $t = \frac{1}{K(\frac{1}{s})}$, we easily see that

$$\frac{1}{\sigma_0(t)} \le K^{-1}\left(\frac{1}{t}\right) \quad \forall t \ge t'\,.$$

Therefore, using (3.16), estimate (3.5) is established.

To prove (3.6), we assume, without loss of generality, that $r = m$. In fact, if $r < m$ and $r \le |s| \le m$, then, using (H1), we have, for $i = 1, 2$, $q_1 = 1$, and $q_2 = q$,

$$|h_i(s)| \le \frac{H_i^{-1}(|s|)}{|s|^{q_i}}\,|s|^{q_i} \le \frac{H_i^{-1}(m)}{r^{q_i}}\,|s|^{q_i}$$

and

$$|h_i(s)| \ge \frac{H_i(|s|)}{|s|}\,|s| \ge \frac{H_i(r)}{m}\,|s|\,.$$

This implies that

$$H_i(|s|) \le |h_i(s)| \le H_i^{-1}(|s|) \quad \text{for all } |s| \le r,\ i = 1, 2$$

$$c_1' |s| \le |h_1(s)| \le c_2' |s| \quad \text{for all } |s| \ge r$$

$$c_1' |s| \le |h_2(s)| \le c_2' |s|^q \quad \text{for all } |s| \ge r,$$

which justifies our assumption ($r = m$).

Since $H_1(0) = H_2(0) = H_1'(0) = H_2'(0) = 0$ and, for $s > 0$,

$$0 < K_0(s) = \frac{H_0(s)}{s} = \frac{\left(H_1^{-1} + H_2^{-1}\right)^{-1}(s)}{s} \le \frac{H_i(s)}{s}, \quad i = 1, 2,$$

then $H_0'(0) = K_0(0) = 0$. Also, one can easily conclude that H_0 is strictly convex on $(0, m)$. Then, using the Mean value Theorem and the strict convexity of H_i, $i = 0, 1, 2$, on $(0, m)$, we deduce that

$$K_i(s) = \frac{H_i(s)}{s}, \quad i = 0, 1, 2,$$

are strictly increasing on $(0, m)$.

Now, we take $\sigma_0 = \phi^{-1}$, where

$$\phi(t) := 1 + \int_1^t \frac{1}{K_0(1/s)} ds \quad t \ge t'.$$

In this case, we replace (3.9) and (3.11) by

$$\begin{aligned}
\Omega_1 &= \{x \in (0,1) : |\psi_t| > m\} \\
\Omega_2 &= \Big\{x \in (0,1) : |\psi_t| \le m \text{ and } |\psi_t| \le K_2^{-1}\big(\sigma_0'(t)\big)\Big\} \\
\Omega_3 &= \Big\{x \in (0,1) : |\psi_t| \le m \text{ and } |\psi_t| > K_2^{-1}\big(\sigma_0'(t)\big)\Big\}
\end{aligned}$$

and

$$\begin{aligned}
\Omega^1 &= \{x \in (0,1) : |\psi_t| > m\} \\
\Omega^2 &= \Big\{x \in (0,1) : |\psi_t| \le m \text{ and } H_2^{-1}(|\psi_t|) \le K_2^{-1}(\sigma_0'(t))\Big\} \\
\Omega^3 &= \Big\{x \in (0,1) : |\psi_t| \le m \text{ and } H_2^{-1}(|\psi_t|) > K_2^{-1}\big(\sigma_0'(t)\big)\Big\}.
\end{aligned}$$

Consequently, we arrive at

$$\begin{aligned}
\sigma_0'(t) \int_{\Omega_3} \psi_t^2 dx &\le \int_{\Omega_3} K_2(|\psi_t|)\psi_t^2 dx \\
&= \int_{\Omega_3} H_2(|\psi_t|)\, |\psi_t|\, dx \\
&\le \int_0^1 \psi_t h_2(\psi_t) dx \le -E'(t)
\end{aligned}$$

$$\sigma_0'(t)\int_{\Omega^3} h_2(\psi_t)^2 dx \le \int_{\Omega^3} K_2\big(H_2^{-1}(|\psi_t|)\big)H_2^{-1}(|\psi_t|)\,|h_2(\psi_t)|\,dx$$
$$= \int_0^1 \psi_t h_2(\psi_t)dx \le -E'(t)\,.$$

The other cases can be dealt with similarly. Then, the same reasoning leads to (3.6). □

4. Decay of energy of systems (1.7) and (1.8)

In this section we state and prove our main results for systems (1.7) and (1.8). To achieve this goal, we consider the following hypothesis on h_1 and h_2

(H2) $h_i : \mathbb{R} \to \mathbb{R}$ (for $i = 1, 2$) are nondecreasing C^1 functions such that

$$H_i(|s|) \le |h_i(s)| \le H_i^{-1}(|s|) \quad \text{for all} \quad |s| \le m\,, \quad i = 1, 2$$
$$c_1\,|s| \le |h_i(s)| \le c_2\,|s|^q \quad \text{for all} \quad |s| \ge m\,, \quad i = 1, 2$$

where H_1 and H_2 are strictly increasing C^1 functions on $[0, +\infty)$, $H_1(0) = H_2(0) = 0$, m, c_1, c_2 are positive constants, $q \ge 1$ for system (1.7) and $q = 1$ for (1.8).

Remark 4.1. Hypothesis (H2) implies that $sh_i(s) > 0$, for all $s \ne 0$.

It is easy to check that the energy functional for system (1.7) satisfies

$$E'(t) = -\int_0^1 \varphi_t h_1(\varphi_t)dx - \int_0^1 \psi_t h_2(\psi_t)dx \le 0 \tag{4.1}$$

and for system (1.8)

$$E'(t) = -k\varphi_t(1,t)h_1\big(\varphi_t(1,t)\big) - d\psi_t(1,t)h_2\big(\psi_t(1,t)\big) \le 0\,. \tag{4.2}$$

Theorem 4.1. *Assume that* (H2) *holds. Then there exists a constant $c > 0$ such that, for t large, the solutions of* (1.7) *and* (1.8) *satisfy*

$$E(t) \le c\left(K^{-1}\left(\frac{1}{t}\right)\right)^2, \tag{4.3}$$

where $K(s) = s\left(H_1^{-1} + H_2^{-1}\right)^{-1}(s)$.

Moreover, if H_1, H_2 are strictly convex on $(0, r)$, for some $r > 0$, and $H_1'(0) = H_2'(0) = 0$, then we have the improved estimate

$$E(t) \le c\left(\left(H_1^{-1} + H_2^{-1}\right)\left(\frac{1}{t}\right)\right)^2. \tag{4.4}$$

Proof. We define σ_0 as in (3.7) and (3.8), and so σ_0 is a strictly increasing concave C^2-function, with $\sigma_0(t) \to +\infty$ as $t \to +\infty$.

A) System (1.7)

By multiplying the first two equations in (1.7) by $\sigma_0' E\varphi$ and $\sigma_0' E\psi$ respectively, integrating over $(0,1)\times(S,T)$, and using integration by parts, we obtain

$$\begin{aligned}2\int_S^T \sigma_0'(t)E(t)^2 dt = &-\left[\sigma_0'(t)E(t)\int_0^1 (a\varphi\varphi_t + b\psi\psi_t)\,dx\right]_S^T \\ &+\int_S^T (\sigma_0'' E + \sigma_0' E')\left(\int_0^1 [a\varphi\varphi_t + b\psi\psi_t]\,dx\right)dt \\ &+\int_S^T \sigma_0' E\left(\int_0^1 \left[2a\varphi_t^2 - \varphi h_1(\varphi_t) + 2b\psi_t^2 - \psi h_2(\psi_t)\right]dx\right)dt\,.\end{aligned}$$

Similar computations as in Lemma 3.2 lead to

$$\begin{aligned}\int_S^T \sigma_0'(t)E(t)^2 dt \le\, & cE(S)^2 \\ &+ c\int_S^T \sigma_0' E\left(\int_0^1 \left[2a\varphi_t^2 - \varphi h_1(\varphi_t) + 2b\psi_t^2 - \psi h_2(\psi_t)\right]dx\right)dt\,. \end{aligned}\tag{4.5}$$

B) System (1.8)

We multiply the equations in (1.8) by $[(N+1)x\varphi_x - \frac{N}{2}\varphi]\sigma_0' E$ and $[(N+1)x\psi_x + \frac{N}{2}\psi]\sigma_0' E$ respectively, where $N > 0$ to be suitably chosen, and perform some manipulations to get

$$\begin{aligned}\int_S^T \sigma_0'(t)E(t)^2 dt = &\int_S^T \sigma_0' E\left(\int_0^1 k\psi(\varphi_x+\psi)dx - N\int_0^1 (a\varphi_t^2 + d\psi_x^2)dx\right)dt \\ &-\left[\sigma_0'(t)E(t)\int_0^1 \left[(N+1)\left(ax\varphi_x\varphi_t + bx\psi_x\psi_t\right) + \frac{N}{2}(b\psi\psi_t - a\varphi\varphi_t)\right]dx\right]_S^T \\ &+\int_S^T (\sigma_0'' E + \sigma_0' E')\left(\int_0^1 \left[(N+1)\left(ax\varphi_x\varphi_t + bx\psi_x\psi_t\right)\right.\right. \\ &\left.\left.+\frac{N}{2}(b\psi\psi_t - a\varphi\varphi_t)\right]dx\right)dt \\ &+\frac{N}{2}\int_S^T \sigma_0' E\Big[k\varphi(1,t)h_1\big(\varphi_t(1,t)\big) - d\psi(1,t)h_2\big(\psi_t(1,t)\big)\Big]dt \\ &+(N+1)k\int_S^T \sigma_0' E\psi(1,t)h_1\big(\varphi_t(1,t)\big)dt \\ &+\frac{N+1}{2}\int_S^T \sigma_0' E\Big[a\varphi_t^2(1,t) + kh_1^2\big(\varphi_t(1,t)\big) \\ &+ b\psi_t^2(1,t) + dh_2^2\big(\psi_t(1,t)\big)\Big]dt\,. \end{aligned}\tag{4.6}$$

The terms in the right hand side of (4.6) can be estimated similarly as in Lemma 3.2 and we obtain

$$\begin{aligned}\int_S^T \sigma_0'(t)E(t)^2 dt \leq cE(S)^2 \\ + c\int_S^T \sigma_0' E\Big[\varphi_t^2(1,t) + h_1^2\big(\varphi_t(1,t)\big) + \psi_t^2(1,t) \\ + h_2^2\big(\psi_t(1,t)\big)\Big]dt\,.\end{aligned} \tag{4.7}$$

By repeating the same procedures as in Theorem 3.3, we estimate the integral term in (4.5) or in (4.7). Consequently, (4.3) and (4.4) are established. □

Examples. We give some examples to illustrate the energy decay rates obtained by our results.

(1) Between polynomial and exponential growth

If $H_1(s) = H_2(s) = e^{-(\ln s)^2}$ near zero. Then, we have the following energy decay rate

$$E(t) \leq ce^{-2(\ln t)^{\frac{1}{2}}}\,.$$

(2) Exponential growth

If $H_1(s) = H_2(s) = e^{-1/s}$ near zero. Then, we have the following energy decay rate

$$E(t) \leq \frac{c}{(\ln(t))^2}\,.$$

(3) Faster than exponential growth

If $H_1(s) = H_2(s) = e^{-e^{1/s}}$ near zero. Then, we have the following energy decay rate

$$E(t) \leq \frac{c}{(\ln(\ln(t)))^2}\,.$$

5. The case of the polynomial growth

As a special case of (H1) on the system (1.6), we assume that there exist constants $c_1, c_2 > 0$ and $q_1, q_2 \geq 1$ such that

$$c_1 \min\{|s|, |s|^{q_i}\} \leq |h_i(s)| \leq c_2 \max\{|s|, |s|^{1/q_i}\} \quad i = 1, 2 \tag{5.1}$$

According to Theorem 3.3, we have the following estimate

$$E(t) \leq \frac{c}{t^{2/q}}\,.$$

where $q = \max\{q_1,\, q_2\}$. However, we can obtain a better decay rate as follows.

We multiply the first equation in (1.6) by $(x\varphi_x + Nw)E^{\frac{q-1}{2}}$ and the second equation by $N\psi E^{\frac{q-1}{2}}$ for $q = \max\{q_1, q_2\}$. Consequently, by similar computations as in Lemma 3.2, for $\sigma(t) = t$, we obtain

$$\begin{aligned}\int_S^T E(t)^{1+\frac{q-1}{2}}\,dt &\le cE(S)^{1+\frac{q-1}{2}} + c\int_S^T E^{\frac{q-1}{2}}\left(\int_0^1 \big[\psi_t^2 - c'\psi h_2(\psi_t)\big]dx\right)dt \\ &\quad + c\int_S^T E^{\frac{q-1}{2}}\Big(\varphi_t^2(1,t) + h_1^2\big(\varphi_t(1,t)\big)\Big)dt \\ &\le cE(S)^{1+\frac{q-1}{2}} + \varepsilon\int_S^T E^{1+\frac{q-1}{2}}\,dt \\ &\quad + C_\varepsilon\int_S^T E^{\frac{q-1}{2}}\left(\int_0^1 \big[\psi_t^2 + h_2(\psi_t)^2\big]dx\right)dt \\ &\quad + c\int_S^T E^{\frac{q-1}{2}}\Big(\varphi_t^2(1,t) + h_1^2\big(\varphi_t(1,t)\big)\Big)dt\,.\end{aligned}$$

By choosing ε small enough and using (5.1), we infer

$$\begin{aligned}\int_S^T E(t)^{1+\frac{q-1}{2}}\,dt &\le cE(S)^{1+\frac{q-1}{2}} \\ &\quad + c\int_S^T E^{\frac{q-1}{2}}\left(\int_0^1 \Big[\big(\psi_t h_2(\psi_t)\big)^{\frac{2}{q_2+1}} + \psi_t h_2(\psi_t)\Big]dx\right)dt \\ &\quad + c\int_S^T E^{\frac{q-1}{2}}\bigg(\Big(\varphi_t(1,t)h_1\big(\varphi_t(1,t)\big)\Big)^{\frac{2}{q_1+1}} \\ &\quad + \varphi_t(1,t)h_1\big(\varphi_t(1,t)\big)\bigg)dt\,.\end{aligned} \tag{5.2}$$

Case 1: $(q_1, q_2) = (1, 1)$

In this case, we clearly have

$$\int_S^T E(t)dt \le cE(S) + c\int_S^T \big(-E'(t)\big)dt \le cE(S)\,. \tag{5.3}$$

Case 2: $(q_1, q_2) \neq (1, 1)$

The use of Hölder's and Young's inequalities, in (5.2), yields

$$\begin{aligned}\int_S^T E(t)^{1+\frac{q-1}{2}}\,dt &\le cE(S)^{1+\frac{q-1}{2}} + c\int_S^T E^{\frac{q-1}{2}}\Big[(-E')^{\frac{2}{q_1+1}} + (-E')^{\frac{2}{q_2+1}}\Big]dt \\ &\le cE(S)^{1+\frac{q-1}{2}} + C_\delta E(S) + \delta\int_S^T E^{1+\frac{q-1}{2}}\,dt\,.\end{aligned}$$

Hence, choosing δ small enough, we find

$$\int_S^T E(t)^{1+\frac{q-1}{2}}\,dt \le cE(S)^{1+\frac{q-1}{2}} + cE(S)\,. \tag{5.4}$$

Consequently, (5.3), (5.4), and Lemma 2.1 lead to

$$E(t) \le ce^{-wt}\,, \quad \text{if} \quad (q_1, q_2) = (1,1) \tag{5.5}$$

$$E(t) \le \frac{c}{t^{\frac{2}{q-1}}}\,, \quad q = \max\{q_1, q_2\}\,, \quad \text{if} \quad (q_1, q_2) \ne (1,1)\,. \tag{5.6}$$

Remark 5.1. By the same way, under the condition (5.1), we obtain (5.5)–(5.6) for the systems (1.7) and (1.8).

Remark 5.2. We note that our results allow a larger class of functions h_1, h_2. In fact, the usual exponential and polynomial decay estimates are only special cases. These results improve and generalize those established by Kim and Renardy [5], Yan [18], and Raposo et al. [12], and extend the decay results established for the wave equations by Martinez [7] to the Timoshenko systems.

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