ON THE INTERNAL CONSTITUTION OF THE PLANETS

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Summary

Using the theory of the Thomas-Fermi atom, without the correction of Fock and Dirac, the equation of state of matter under high pressure has been obtained. This equation being $p = C\rho^2$, each planet is a planet of the Laplace type. The maximum radius of a planet appears to be about one-and-a-half times the radius of Jupiter. The two largest planets are composed of hydrogen. The mass-radius relation of a planet composed of silicon is in agreement with the observed values of the minor planets.

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1. *Introduction.*—In any theory dealing with the constitution of stellar material, at least as far as the electrons are concerned, we have to use the statistics of Fermi-Dirac. The results of the Fermi-Dirac statistics, however, are not very different from those of the classical statistics if the temperature T and the density n of the electrons satisfy the inequality

$$kT \gg E, \quad \text{where} \quad E = (h^2/2m)(3n/8\pi)^{\frac{2}{3}}, \tag{1}$$

k being Boltzmann's constant and m the mass of an electron.

Saha's theory, based as it is on Maxwell-Boltzmann statistics, can therefore be used only if this condition is fulfilled, which is certainly the case in the outer layers of a star. But, as Kothari † has pointed out, Saha's theory is not applicable to the interior of the planets, the white dwarf stars and perhaps many other stars. In these cases the condition (1) is not satisfied and we can use only the Fermi-Dirac statistics.

The temperature in the interior of the planets satisfies the opposite inequality $kT \ll E$; this condition renders possible a simple treatment of the problem of their internal constitution. This inequality implies that the thermal energy of the electrons is very small in comparison with their Fermi-energy; the temperature gradient existing in the planets has therefore little or no effect on the behaviour of the electrons. In this paper this effect will be neglected; in the following we have to deal only with the effect which an increasing pressure has on the energy and the density of the material.

2. *The equation of state.*—In order to obtain this relation between the pressure p and the density ρ, Slater and Krutter ‡, and also Kothari, followed a very simple method: the pressure exerted on a body can be calculated by means of the theorem of the virial, if the energies of its particles (electrons and nuclei) are known. In a first approximation they assumed that the energy of every atom increases with increasing pressure only because the space occupied by the electrons belonging to that atom decreases; any other change in the potential energy of the atoms caused by the disturbing action of its neighbours is neglected. The energy of the electrons and of the nucleus of an atom can be found if the electron distribution is known.

* Originally submitted in different form on 1946 February 8.

† D. S. Kothari, *Proc. Roy. Soc.*, A, **165**, 486–500, 1938.

‡ J. C. Slater and H. Krutter, *Phys. Rev.*, **47**, 559–568, 1935.

Kothari simplified the problem still further by assuming the electrons to be uniformly distributed round the nucleus; Slater and Krutter used the more satisfying Thomas-Fermi distribution, which is self-consistent.

The electron density n in the Thomas-Fermi atom is given by

$$4\pi r^3 n = Z\{\phi(x)\}^{\frac{3}{2}}. \tag{2}$$

Here $x = r/\alpha$, where r is the distance to the centre of the atom, and $\alpha = a_0(9\pi^2/128Z)^{\frac{1}{3}}$, a_0 being the radius of the smallest Bohr orbit and Z the atomic number. The function ϕ satisfies the equations

$$\frac{d^2\phi}{dx^2} = \frac{\phi^{\frac{3}{2}}}{x^{\frac{1}{2}}}, \quad \phi(0) = 1. \tag{3}$$

The radius R of the atom is determined by $R = \alpha X$, where X is the value of x for which $d\phi/dx = \phi/x$. At a given value of R the function ϕ is determined by these three equations; the density n and the potential and kinetic energies of the electrons are then also known.

Substituting these values in the equation of the virial theorem, Slater and Krutter obtained

$$p = \frac{(2/15)(Z^2e^2/\alpha)\phi_X^{\frac{5}{2}}X^{\frac{1}{2}}}{(4/3)\pi(\alpha X)^3}, \tag{4}$$

where ϕ_X is the value of ϕ at $x = X$.

Since the density ρ of a material with atomic weight A is given by

$$\rho = \frac{Am_H}{(4/3)\pi R^3} \tag{5}$$

multiplied by an unknown factor of the order of unity depending on the lattice arrangement of the atoms, it is evident that the elimination of X between equations (4) and (5) results in the equation of state. It being impossible to carry out this elimination, this equation is represented graphically in Fig. 1, where $\log\rho$ is

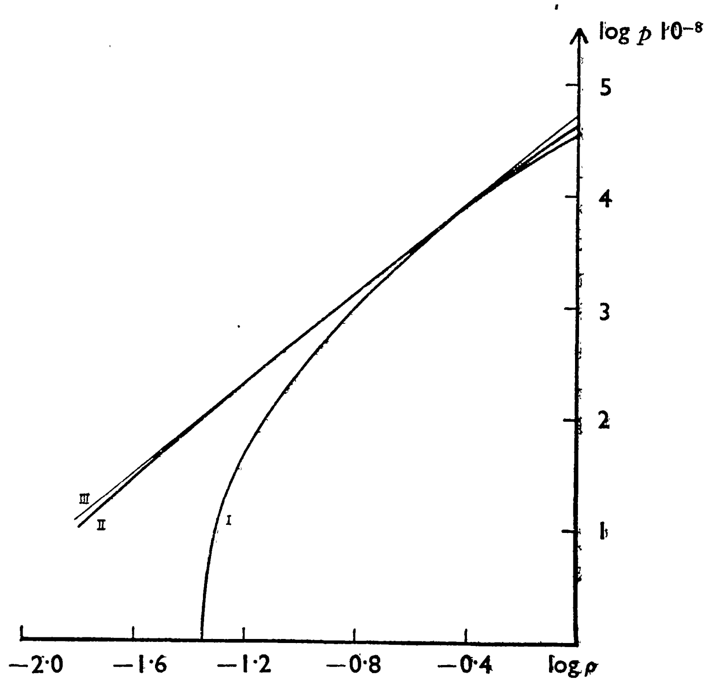

FIG. 1.—(p, ρ) function (H).

drawn as a function of $\log p$ (curve II). From this figure it is evident that $\log p$ is approximately proportional to $\log \rho$. By numerical computation, based on the tables of Slater and Krutter, it can be easily shown that the function

$$p = 5{\cdot}07 \times 10^{12} Z^{\frac{4}{3}} \rho^2 A^{-2}, \tag{6}$$

is a fairly accurate approximation to the p function of Slater and Krutter. Since it is impossible to obtain an exact analytical expression $p = f(\rho)$ valid for Thomas-Fermi matter under high pressure, it is necessary to use the slightly less exact relation (6) as the equation of state. In Fig. 1 equation (6) is represented by the straight line III; in the same figure we have drawn also the (p, ρ) relation based on Kothari's assumption (curve I).

3. *The radius-mass relation.*—The equations for a star (or planet) with zero luminosity are

$$\frac{dp}{dr} = -GM(r)\frac{\rho}{r^2} \quad \text{and} \quad \frac{dM}{dr} = 4\pi r^2 \rho.$$

Eliminating M we get

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dp}{dr}\right) = -4\pi G r^2 \rho.$$

Since $p = K\rho^2$, with $K = 5{\cdot}07 \times 10^{12} Z^{\frac{4}{3}} A^{-2}$, we obtain

$$\frac{d}{dr}\left(r^2 \frac{d\rho}{dr}\right) = -2\pi \frac{G}{K} \rho r^2.$$

The solution of this equation is

$$\rho = \rho_c \sin Cr/Cr, \quad \text{where} \quad C = (2\pi G/K)^{\frac{1}{2}};$$

ρ_c is the density at the centre of the star (or planet). The difference between this function of r and the Laplace solution * of Clairaut's equation is to be found in the value of C: the numerical value of C is here

$$C = 0{\cdot}1832 A/aZ^{\frac{2}{3}},$$

where a is the radius of the Earth, while Laplace obtained $C = 2{\cdot}5165/a$.

The maximum radius of a Laplace planet is given by

$$Cr = \pi, \quad \text{or} \quad r = \pi a Z^{\frac{2}{3}}/0{\cdot}1832 A.$$

For a planet composed of hydrogen, this gives $r = 17a$, approximately. Since for any other element this maximum radius is smaller than the radius of Saturn, it follows that, assuming the validity of the Thomas-Fermi calculations, the two largest planets consist of hydrogen. Moreover, the maximum radius of a planet (or a white dwarf) cannot be larger than $17a$. Inasmuch as equation (6) is certainly not valid if the pressure is small (say $p < 10^4$ atmospheres), these results are of course true only for the inner layers of a planet, beginning at a depth of about 100 km.

Since the mass of a Laplace planet with radius R is given by

$$M = (4\pi\rho_c/C^3)(\sin CR - CR\cos CR)$$

and the surface density by

$$\rho_R = \rho_c \sin CR/CR,$$

we have

$$M = (4\pi\rho_R/C^3)(1 - CR\cot CR). \tag{7}$$

* F. Tisserand, *Mécanique céleste*, II, 1891.

In order to obtain some numerical results which can be compared with the R and M values for the planets, we have calculated (7), assuming that the four largest planets are composed of hydrogen and that the density at a depth of 100 km. is equal to unity. (This value is somewhat arbitrarily chosen and in one case is in contradiction to the observed value of ρ. In view of the simplifying assumptions we have made we cannot expect more than rough agreement with the known (R, M) values: an attempt to get better results would be wholly misleading.)

The results, together with the (R, M) curve of Kothari (curve I) for a hydrogen planet, are plotted in Fig. 2 (curve II); the observed (R, M) values of the four major planets are also indicated. In Fig. 3 the (R, M) relation for a planet composed of silicon is represented; here we have taken $\rho_R = 3$. This figure also shows the (R, M) values of the minor planets. We have chosen silicon because

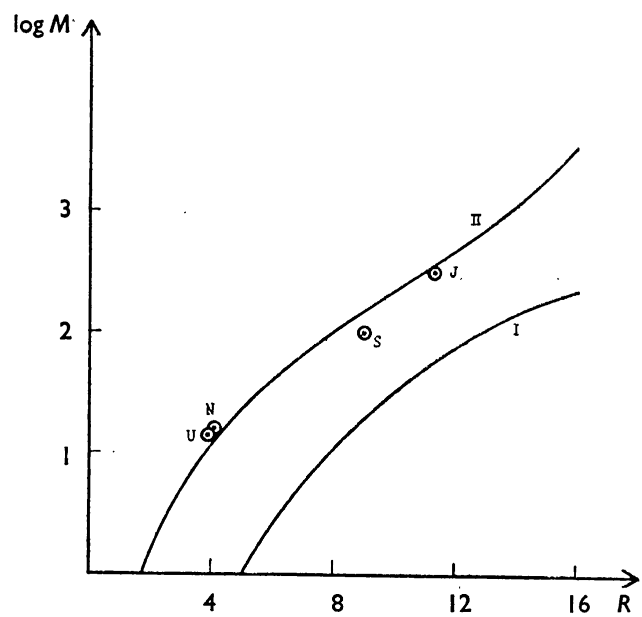

FIG. 2.—(M, R) *relation for an Hydrogen-planet, with* ρ_R *equal to unity.*

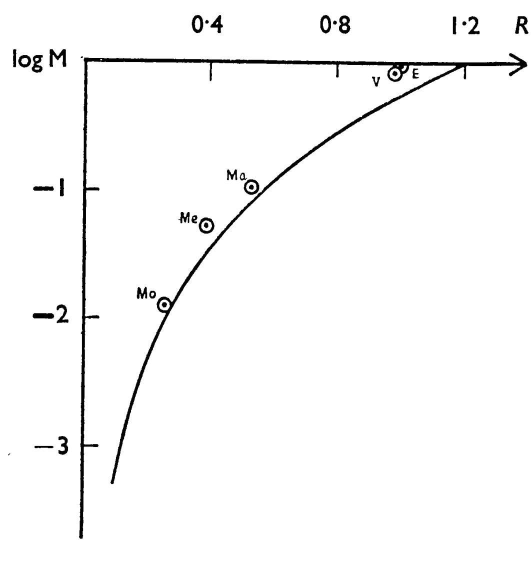

FIG. 3.—(M, R) *relation for an Silicon-planet, with* ρ_R *equal to* 3.

this element is a fair representative element of the Russell mixture. The only inference which can be drawn from Fig. 3 is that the results of the calculations are not incompatible with the existing conditions.

Finally I wish to express my thanks to Professor van der Waals for his kind interest taken in this paper and to Professor Slater for the table of numerical solutions of equation (3).

H. R. Singel 96,
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1946 *August* 31.

ADDENDUM

M.N., **106**, No. 4, A. R. Hogg's paper:

In the paper *Photo-electric Observations of V Puppis*, by A. R. Hogg (communicated by the Commonwealth Astronomer), a recently published set of spectroscopic observations on this star by D. M. Popper (*Ap. J.*, **97**, 394, 1943) was unfortunately overlooked.

It is desired to express thanks to Dr J. H. Irwin for drawing the attention of the Commonwealth Observer to this omission.

ERRATUM

M.N., **106**, No. 4, *Electromagnetic Forces in Solar Prominences*, by D. S. Evans: P. 320, line 30, for helium and ions *read* helium ions and.