ON THE INTERPOLATION CONSTANT FOR ORLICZ SPACES

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ABSTRACT. In this paper we deal with the interpolation from Lebesgue spaces L^p and L^q, into an Orlicz space L^φ, where $1 \le p < q \le \infty$ and $\varphi^{-1}(t) = t^{1/p}\rho(t^{1/q-1/p})$ for some concave function ρ, with special attention to the interpolation constant C. For a bounded linear operator T in L^p and L^q, we prove modular inequalities, which allow us to get the estimate for both the Orlicz norm and the Luxemburg norm,

$$\|T\|_{L^\varphi \to L^\varphi} \le C \max\left\{ \|T\|_{L^p \to L^p}, \|T\|_{L^q \to L^q} \right\},$$

where the interpolation constant C depends only on p and q. We give estimates for C, which imply $C < 4$. Moreover, if either $1 < p < q \le 2$ or $2 \le p < q < \infty$, then $C < 2$. If $q = \infty$, then $C \le 2^{1-1/p}$, and, in particular, for the case $p = 1$ this gives the classical Orlicz interpolation theorem with the constant $C = 1$.

1. INTRODUCTION

The classical Riesz-Thorin interpolation theorem says that (L^p, L^q) are interpolation spaces for linear operators between (L^{p_0}, L^{q_0}) and (L^{p_1}, L^{q_1}), where

$$(1.1) \qquad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1},$$

with the estimation of the norm

$$\|T\|_{L^p \to L^q} \le C \left(\|T\|_{L^{p_0} \to L^{q_0}} \right)^{1-\theta} \left(\|T\|_{L^{p_1} \to L^{q_1}} \right)^{\theta}$$

where $C \le \sqrt{2}$. The constant C is 1 when either the spaces are complex or the spaces are real and $p_i \le q_i, i = 0, 1$ (see, e.g., [5, Section 1.7]).

After the Riesz-Thorin interpolation theorem several results have been proved about the interpolation of Orlicz spaces. The problem was the following: *if T is any bounded linear operator from L^{φ_i} into $L^{\psi_i}, i = 0, 1$, then under what conditions on φ and ψ is it true that T is also bounded from L^φ to L^ψ?*

The assumption corresponding to (1.1) which appears naturally here is

$$(1.2) \qquad \varphi^{-1} = \varphi_0^{-1}\rho\left(\frac{\varphi_1^{-1}}{\varphi_0^{-1}}\right), \quad \psi^{-1} = \psi_0^{-1}\rho\left(\frac{\psi_1^{-1}}{\psi_0^{-1}}\right),$$

for some concave function ρ.

The first result on interpolation of Orlicz spaces appeared in the case when $\rho(t) = t^\theta, 0 \leq \theta \leq 1$, and we mention here Ya. B. Rutickii (1963), A. P. Calderón (1964), M. M. Rao (1966) as the precursors of such results. The constants found by Ya. B. Rutickii were 4 in the complex case and 8 in the real case. The corresponding constants in two other papers are 2 and 4, respectively. Theorems with general concave ρ (and the interpolation theorems for, in fact, the Calderón-Lozanovskii construction) were done by several authors; among others, we mention V. I. Ovchinnikov (1976, 1984), J. Gustavsson-J. Peetre (1977), E. I. Berezhnoi (1980), V. A. Shestakov (1981), E. I. Pustylnik (1983), P. Nilsson (1985) and L. Maligranda (1985, 1989). For the precise references and the proofs, see [16] and also [5, 8, 11, 17, 18].

Only some of these results on interpolation of Orlicz (or Caldéron-Lozanovskii) spaces take care about the estimate of the operator norm. One known result is the following:

Theorem 1.1. *If (1.2) holds for some concave function ρ, then (L^φ, L^ψ) are interpolation spaces for linear operators between $(L^{\varphi_0}, L^{\psi_0})$ and $(L^{\varphi_1}, L^{\psi_1})$, and*

$$\|T\|_{L^\varphi \to L^\psi} \leq C \max\{M_0, M_1\}, \quad M_i := \|T\|_{L^{\varphi_i} \to L^{\psi_i}}, \quad i = 0, 1,$$

where $C \leq 26$ and all norms in Orlicz spaces are the Luxemburg norms.

The proof of this statement can be found in [16, Theorem 14.2]. The careful analysis of the proof shows that, in fact,

$$(1.3) \qquad C \leq 2(3 + 2\sqrt{2})C_\psi < 12C_\psi, \text{ where } C_\psi = \sup_{t>0} \frac{\psi^{-1}(2t)}{\psi^{-1}(t)} \leq 2.$$

In the general case we have only the fairly rough estimate of the norm $C < 24$, but for the "diagonal case" and Lebesgue spaces, i.e., when $L^{\varphi_0} = L^{\psi_0} = L^p$ and $L^{\varphi_1} = L^{\psi_1} = L^q$, one can obtain more precise estimates. Moreover, for most of the operators, we have information about boundedness between L^p spaces and we would like to get estimates in more general spaces, for example, in Orlicz spaces. Our problem here starts with the bounded linear operators T from L^p into L^p which are also bounded from L^q into L^q (with $1 \leq p < q \leq \infty$) and we want to have boundedness of T from the Orlicz space L^φ into itself with the best possible estimate of the norm

$$(1.4) \quad \|T\|_{L^\varphi \to L^\varphi} \leq C \max\{M_0, M_1\}, \quad M_0 := \|T\|_{L^p \to L^p}, \ M_1 := \|T\|_{L^q \to L^q},$$

where $\varphi^{-1}(u) = u^{1/p}\rho(u^{1/q-1/p})$ for some concave function ρ.

Besides the above-mentioned authors working in the general case, there were also others working either with weak type operators or the diagonal case and Lebesgue spaces. I. B. Simonenko [21], among others, showed in 1964 that if

$$(1.5) \qquad 1 \leq p < a_\varphi := \inf_{u>0} \frac{u\varphi'(u)}{\varphi(u)} \leq b_\varphi := \sup_{u>0} \frac{u\varphi'(u)}{\varphi(u)} < q < \infty,$$

then the Orlicz space L^φ is an interpolation space between L^p and L^q but the constant C in the estimation of the norm can be large: $C \approx \left((a_\varphi - p)(q - b_\varphi)\right)^{-1}$. B. W. Boyd [4] extended in 1967 this theorem to rearrangement-invariant spaces and his constant in the estimation of the norm increases to infinity when one of the Boyd indices of the space is going either to p or to q.

Let us also mention that interpolation theorems of Marcinkiewicz type (operators are of weak type, i.e., maps L^p into weak L^p and L^q into weak L^q) in Orlicz spaces were done by A. Zygmund (1956), A. Torchinsky (1976), A. Cianchi (1998) (see [6] and the references given there) and in symmetric spaces by E. M. Semenov (1968), D. W. Boyd (1969), M. Zippin (1971), S. G. Krein and E. M. Semenov (1973) (see [10, 14] and references given there). Observe that the constants in these theorems are still large.

The paper is organized as follows. Section 2, called preliminaries, contains necessary definitions and some auxiliary results from the theory of Orlicz spaces and interpolation theory. In Section 3 we consider useful properties of concave and convex functions on $[0, \infty)$. In particular, the equivalence between representations $\varphi^{-1}(u) = u^{1/p}\rho(u^{1/q-1/p})$ and $\varphi(u) = u^p\rho(u^{q-p})$ is proved. In Section 4 we prove modular estimates of bounded linear operators in L^p and L^q spaces which are the keys for the estimates in the norms, both the Luxemburg one and the Orlicz one. In the case $q < \infty$, the idea of the proof goes back to J. Peetre [20]. In the proof we use essentially the exact estimation of the Sparr functional $K_{p,q}^*$ [22]. In the case $q = \infty$ our proof is based on the Krée formula, the Hardy-Littlewood-Pólya majorization theorem and the convexity of $\varphi(u^{1/p})$. Finally, in Section 5, we put all our pieces of results together and prove our main Theorem 5.1, which shows that the interpolation constant C in the estimate (1.4) is always less than 4. Moreover, if either $1 < p < q \leq 2$ or $2 \leq p < q < \infty$, then $C < 2$. If $q = \infty$, then $C \leq 2^{1-1/p}$, and, in particular, for the case $p = 1$ this gives the classical Orlicz interpolation theorem with the constant $C = 1$.

2. PRELIMINARIES

Let (Ω, μ) be a σ-finite measure space. Let $\varphi : [0, \infty) \to [0, \infty]$ be a convex nondecreasing function such that $\varphi(0) = 0$ and $\lim_{u \to 0+} \varphi(u) = 0$ but not identically zero or infinity on $(0, \infty)$. For a measurable real or complex-valued function x, define a functional (*modular*)

$$(2.1) \qquad I_\varphi(x) := \int_\Omega \varphi(|x(s)|)d\mu(s) = \int_0^\infty \varphi(x^*(t))dt,$$

where x^* is the nonincreasing rearrangement of x (see [1, 10, 12]). The *Orlicz space* $L^\varphi = L^\varphi(\Omega, \mu)$ is the space of all equivalence classes of measurable functions on Ω such that $I_\varphi(\lambda x) < \infty$ for some $\lambda = \lambda(x) > 0$. This space is a Banach space with two norms: the *Luxemburg norm*

$$\|x\|_\varphi := \inf \{\lambda > 0 \ : \ I_\varphi(x/\lambda) \leq 1\}$$

and the *Orlicz norm* (in the Amemiya form)

$$\|x\|_\varphi^0 := \inf_{k>0} \frac{1}{k}\Big(1 + I_\varphi(kx)\Big).$$

It is well known that $\|x\|_\varphi \leq \|x\|_\varphi^0 \leq 2\|x\|_\varphi$, and $\|x\|_\varphi \leq 1$ if and only if $I_\varphi(x) \leq 1$ (cf. [9]). The Orlicz space L^φ with each of the above two norms is a rearrangement-invariant space (= symmetric space with the Fatou property) (see [1, 10]). If φ satisfies the Δ_2-condition, then the dual of the Orlicz space L^φ is an Orlicz space L^{φ^*} generated by the conjugate function φ^*, defined by

$$\varphi^*(u) := \sup_{v>0} \Big(uv - \varphi(v)\Big).$$

Moreover, $(\|\cdot\|_\varphi)^* = \|\cdot\|^0_{\varphi^*}$ and $(\|\cdot\|^0_\varphi)^* = \|\cdot\|_{\varphi^*}$.

Let (X_0, X_1) be a couple of Banach spaces and let X be an intermediate Banach space between X_0 and X_1, that is, $X_0 \cap X_1 \subset X \subset X_0 + X_1$ with continuous imbeddings ($X_0 \cap X_1, X_0 + X_1$ are standard spaces; see [1, 3, 5, 10, 12]). By $\mathcal{A}(X_0, X_1)$ we denote the class of all admissible operators, i.e., linear operators $T : X_0 + X_1 \to X_0 + X_1$ which restriction to X_i is bounded from X_i into itself for $i = 0, 1$. We denote

$$M := \max\{M_0, M_1\}, \quad \text{where} \quad M_i := \|T|_{X_i}\|_{X_i \to X_i}, \quad i = 0, 1.$$

The space X is said to be an *interpolation space* between X_0 and X_1 if every admissible operator $T \in \mathcal{A}(X_0, X_1)$ maps X into itself and

$$\|T\|_{X \to X} \leq C \max\{M_0, M_1\}$$

for some $C > 0$.

For $0 < p, q < \infty, t > 0$ and $x \in X_0 + X_1$, we define a functional $K_{p,q}$ known as the Peetre L-functional (Peetre [20]; see also [3, Th. 5.2.2], [5, Definition 3.1.22]):

$$K_{p,q}(t, x; X_0, X_1) := \inf\left\{\|x_0\|^p_{X_0} + t\|x_1\|^q_{X_1} \; : \; x = x_0 + x_1, \; x_0 \in X_0, x_1 \in X_1\right\}.$$

In the case $p = q = 1$ this is the classical Peetre K-functional, which we shortly denote by $K(t, x; X_0, X_1)$.

Proposition 2.1. *If $T \in \mathcal{A}(X_0, X_1)$, then*

$$K_{p,q}\left(t, \frac{Tx}{M}; X_0, X_1\right) \leq K_{p,q}(t, x; X_0, X_1) \quad \text{for all} \quad t > 0.$$

The proof is standard.

Following [22], we consider the functional $K^*_{p,q}$ on the couple of Lebesgue spaces defined by

$$K^*_{p,q}(t, x; L^p, L^q) := \int_\Omega \min\left(|x(s)|^p, t|x(s)|^q\right) d\mu(s).$$

Lemma 2.2 (G. Sparr [22, Lemma 5.1, Example 5.3]). *Suppose $1 \leq p < q < \infty$. If $x, y \in L^p + L^q$ and*

$$K_{p,q}(t, x; L^p, L^q) \leq K_{p,q}(t, y; L^p, L^q) \quad \text{for all} \quad t > 0,$$

then

$$K^*_{p,q}(t, x; L^p, L^q) \leq \gamma_{p,q} K^*_{p,q}(t, y; L^p, L^q) \quad \text{for all} \quad t > 0,$$

where

$$\gamma_{p,q} := \inf\left\{\gamma > 0 \; : \; \inf_{\substack{x + y = \gamma, \\ x, y \geq 0}} \left(x^p + y^q\right) = 1\right\}$$

is the Sparr constant, which cannot be replaced by any smaller constant, and which satisfies the inequalities $1 < \gamma_{p,q} < 2$.

3. SOME PROPERTIES OF CONCAVE AND CONVEX FUNCTIONS

We need some properties of concave and convex functions. More information about concave and convex functions and their properties can be found in [5, Ch. 3], [9], [10, Ch. 2, Section 1].

We denote by $\mathcal{P}$ the set of all quasi-concave functions $\rho : [0, \infty) \to [0, \infty)$ which are continuous, positive on $\mathbf{R}_+ := (0, \infty)$ and such that $\rho(st) \le \max\{1, s\}\rho(t)$ for all $s, t > 0$. Let $\widetilde{\mathcal{P}}$ denote the subset of all concave functions in $\mathcal{P}$. Note that if $\rho \in \mathcal{P}$, then $\tilde{\rho}$ defined by

$$\tilde{\rho}(t) := \inf_{s>0} \left(1 + \frac{t}{s}\right) \rho(s)$$

belongs to $\widetilde{\mathcal{P}}$ and

(3.1) $$\rho(t) \le \tilde{\rho}(t) \le 2\rho(t) \quad \text{for all} \quad t > 0.$$

Later on p' will always denote the conjugate number to p, $1 \le p \le \infty$, that is, $1/p + 1/p' = 1$ ($1/\infty$ means 0).

Lemma 3.1. *Let $1 < p < \infty$ and φ be a convex function on $\mathbf{R}_+$. The function $u^{-p}\varphi(u)$ is increasing (decreasing) on $\mathbf{R}_+$ if and only if the function $u^{-p'}\varphi^*(u)$ is decreasing (increasing) on $\mathbf{R}_+$.*

Proof (cf. [7, Lemma 6.1.4]). Let $u^{-p}\varphi(u)$ be increasing. If $0 < u_1 \le u_2$, then

$$
\begin{aligned}
\varphi^*(u_2) &= \sup_{v>0}\left(vu_2 - \varphi(v)\right) = \sup_{v>0}\left(\left(\frac{u_2}{u_1}\right)^{p'-1} vu_2 - \varphi\left(\left(\frac{u_2}{u_1}\right)^{p'-1} v\right)\right) \\
&\le \sup_{v>0}\left(\left(\frac{u_2}{u_1}\right)^{p'} vu_1 - \left(\frac{u_2}{u_1}\right)^{(p'-1)p} \varphi(v)\right) \\
&= \left(\frac{u_2}{u_1}\right)^{p'} \sup_{v>0}\left(vu_1 - \varphi(v)\right) = \left(\frac{u_2}{u_1}\right)^{p'} \varphi^*(u_1),
\end{aligned}
$$

which means that $u^{-p'}\varphi^*(u)$ is decreasing. The remaining implications can easily be proved by also using the fact that for convex function φ we have $\varphi^{**} = \varphi$. $\qquad\square$

Now we derive relations between some representations of φ and φ^{-1}.

Lemma 3.2. *Suppose that $1 \le p < q \le \infty$ and, for some $\rho \in \widetilde{\mathcal{P}}$,*

$$\varphi^{-1}(u) = u^{1/p}\rho(u^{1/q-1/p}) \quad \text{for all} \quad u > 0.$$

(a) *Then φ is convex.*

(b) *If $q < \infty$, then there exists a function $h \in \mathcal{P}$ such that*

$$\varphi(u) = u^q h(u^{p-q}) \quad \text{for all} \quad u > 0.$$

(c) *If $1 < p < q < \infty$, then there exists a function $k \in \mathcal{P}$ such that*

$$\varphi^*(u) = u^{p'} k(u^{q'-p'}) \quad \text{for all} \quad u > 0.$$

(d) *If $q = \infty$ and $\rho_*(t) := t\rho(1/t)$ satisfies $\rho_*(\mathbf{R}_+) = \mathbf{R}_+$, then $\varphi(u) = \rho_*^{-1}(u)^p$ and $\psi(u) = \varphi(u^{1/p})$ is a convex function.*

Proof. Proposition (a) follows from [16, Lemma 14.2].

(b) Since $\rho(t)$ is increasing and $\rho(t)/t$ is decreasing, for all $s, t > 0$ we have

$$\min\left\{s^{1/p}, s^{1/q}\right\}\varphi^{-1}(t) \leq \varphi^{-1}(st) \leq \max\left\{s^{1/p}, s^{1/q}\right\}\varphi^{-1}(t).$$

Hence, φ^{-1} is strictly increasing on $\mathbf{R}_+$ and $\varphi^{-1}(\mathbf{R}_+) = \mathbf{R}_+$. It is easy to see that the following statements are equivalent:

(i) $\varphi^{-1}(st) \leq \max\left\{s^{1/p}, s^{1/q}\right\}\varphi^{-1}(t)$ for all $s, t > 0$;

(ii) $\varphi(st) \leq \max\left\{s^p, s^q\right\}\varphi(t)$ for all $s, t > 0$;

(iii) $u^{-p}\varphi(u)$ is increasing and $u^{-q}\varphi(u)$ is decreasing;

(iv) $h \in \mathcal{P}$, where h is given by $\varphi(u) = u^q h(u^{p-q})$.

Now we prove (c). If $1 < p < q < \infty$, then, by Lemma 3.1, (iii) is equivalent to each of the following properties:

(v) $u^{-p'}\varphi^*(u)$ is decreasing and $u^{-q'}\varphi^*(u)$ is increasing;

(vi) $\varphi^*(st) \leq \max\left\{s^{p'}, s^{q'}\right\}\varphi^*(t)$ for all $s, t > 0$;

(vii) $k \in \mathcal{P}$, where k is given by $\varphi^*(u) = u^{p'}k(u^{q'-p'})$.

Let us prove (d). First observe that ρ_* is concave (see, e.g., [16, Lemma 8.7]). By the assumption $\rho_*(\mathbf{R}_+) = \mathbf{R}_+$, we can see that $\varphi^{-1}(\mathbf{R}_+) = \mathbf{R}_+$ and φ^{-1} is concave. Hence, φ is a finite valued convex function vanishing only at zero, $\varphi(u) = \rho_*^{-1}(u)^p$. We show that even $\psi(u) = \varphi(u^{1/p})$ is convex. Since ρ_* is concave, $\varphi_0 := \rho_*^{-1}$ is convex. Hence,

$$\varphi_0(u) = \varphi_0^{**}(u) = \sup_{v \geq 0}\left(uv - \varphi_0^*(v)\right).$$

To prove that

$$\varphi(u^{1/p}) = \varphi_0(u^{1/p})^p = \sup_{v > 0}\left(u^{1/p}v - \varphi_0^*(v)\right)^p$$

is convex, it is enough to show that $f(u) := \left(u^{1/p}v - \varphi_0^*(v)\right)^p$ is a convex function. Since

$$f'(u) = \left(u^{1/p}v - \varphi_0^*(v)\right)^{p-1}u^{1/p-1}v = \left(\frac{u^{1/p}v - \varphi_0^*(v)}{u^{1/p}}\right)^{p-1}v =: g(u)^{p-1}v$$

and $g(u)$ is increasing, it follows that f is convex. □

Note that the previous lemma guarantees only that $h, k \in \mathcal{P}$ (and h and k need not necessarily to be concave).

Example 3.3. If $1 \leq p < q < \infty$ and $\varphi^{-1}(u) = u^{1/p}\rho(u^{1/q-1/p})$ with $\rho(t) = \min\left\{1, t\right\}$, then $\varphi(u) = u^q h(u^{p-q})$ with $h(t) = \max\left\{1, t\right\}$. Obviously, $\rho \in \tilde{\mathcal{P}}$, but $h \in \mathcal{P} \setminus \tilde{\mathcal{P}}$. In particular, if $p = 2$ and $q = 3$, then $\varphi^*(u) = u^{3/2}k(u^{2-3/2})$ with

$$k(t) = \begin{cases} t/4, & 0 \leq t \leq \sqrt{2}, \\ t^{-1} - t^{-3}, & \sqrt{2} \leq t \leq \sqrt{3}, \\ 2 \cdot 3^{-3/2} & t \geq \sqrt{3}. \end{cases}$$

One can prove that $k \in \tilde{\mathcal{P}}$.

4. Modular estimates

For the proof of the first modular estimate, we need the following representation of concave functions, which goes back to J. Peetre [19] (see also [3, Lemma 5.4.3]).

Lemma 4.1. *Every function $h \in \widetilde{\mathcal{P}}$ can be represented in the following form:*

$$(4.1) \qquad h(u) = a_h + b_h u + \int_0^\infty \min(u, t) \, dm(t), \quad \text{for all} \quad u > 0,$$

where

$$a_h := \lim_{u \to 0+} h(u), \quad b_h := \lim_{u \to \infty} \frac{h(u)}{u},$$

and $m : \mathbf{R}_+ \to \mathbf{R}_+$ is a nondecreasing function (in fact, $m(t) = -h'(t)$).

Now we are ready to prove some modular inequalities, which are the keys for the above-mentioned estimations of norms.

Theorem 4.2. *Let $1 \le p < q \le \infty$ and $T \in \mathcal{A}(L^p, L^q)$.*
   (a) *If $q < \infty$ and $\varphi(u) = u^q h(u^{p-q})$ for some $h \in \widetilde{\mathcal{P}}$ (φ not necessarily convex), then*

$$I_\varphi \left(\frac{Tx}{M} \right) \le \gamma_{p,q} I_\varphi(x) \quad \text{for all} \quad x \in L^p \cap L^q.$$

   (b) *If $q = \infty$ and $\psi(u) = \varphi(u^{1/p})$ is convex, then*

$$I_\varphi \left(\frac{Tx}{2^{1-1/p} M} \right) \le I_\varphi(x) \quad \text{for all} \quad x \in L^p + L^\infty.$$

Proof. (a) Due to Lemma 4.1, h can be represented in the form (4.1). Hence,

$$(4.2) \qquad \varphi(u) = u^q h(u^{p-q}) = a_h u^q + b_h u^p + \int_0^\infty \min(u^p, t u^q) \, dm(t), \quad u \in \mathbf{R}_+.$$

Consequently,

$$(4.3) \qquad I_\varphi \left(\frac{Tx}{M} \right) = \int_\Omega \varphi \left(\frac{|Tx(s)|}{M} \right) d\mu(s) = a_h \left\| \frac{Tx}{M} \right\|_q^q + b_h \left\| \frac{Tx}{M} \right\|_p^p$$

$$+ \int_\Omega \left[\int_0^\infty \min \left(\left(\frac{|Tx(s)|}{M} \right)^p, t \left(\frac{|Tx(s)|}{M} \right)^q \right) dm(t) \right] d\mu(s).$$

Since the operator T is bounded in L^p and L^q, we get

$$(4.4) \qquad a_h \left\| \frac{Tx}{M} \right\|_q^q + b_h \left\| \frac{Tx}{M} \right\|_p^p \quad \le \quad a_h \left(\frac{M_1}{M} \right)^q \|x\|_q^q + b_h \left(\frac{M_0}{M} \right)^p \|x\|_p^p$$

$$\le \quad a_h \|x\|_q^q + b_h \|x\|_p^p$$

and, according to Proposition 2.1,

$$K_{p,q} \left(t, \frac{Tx}{M}; L^p, L^q \right) \le K_{p,q}(t, x; L^p, L^q) \quad \text{for all} \quad t > 0.$$

By using Sparr's Lemma 2.2 we obtain

$$K_{p,q}^* \left(t, \frac{Tx}{M}; L^p, L^q \right) \le \gamma_{p,q} K_{p,q}^*(t, x; L^p, L^q) \quad \text{for all} \quad t > 0.$$

Hence, by the Fubini theorem and in view of the definition of $K_{p,q}^*$, we get

$$(4.5) \quad \int_\Omega \left[\int_0^\infty \min\left(\left(\frac{|Tx(s)|}{M} \right)^p, t \left(\frac{|Tx(s)|}{M} \right)^q \right) dm(t) \right] d\mu(s)$$

$$= \int_0^\infty K_{p,q}^* \left(t, \frac{Tx}{M}; L^p, L^q \right) dm(t)$$

$$\leq \gamma_{p,q} \int_0^\infty K_{p,q}^*(t, x; L^p, L^q) dm(t)$$

$$= \gamma_{p,q} \int_\Omega \left[\int_0^\infty \min(|x(s)|^p, t|x(s)|^q) dm(t) \right] d\mu(s).$$

Combining (4.4)–(4.5) and taking into account that $\gamma_{p,q} > 1$ we obtain

$$I_\varphi \left(\frac{Tx}{M} \right) \leq a_h \|x\|_q^q + b_h \|x\|_p^p + \gamma_{p,q} \int_\Omega \left[\int_0^\infty \min(|x(s)|^p, t|x(s)|^q) \, dm(t) \right] d\mu(s)$$

$$\leq \gamma_{p,q} \left(a_h \|x\|_q^q + b_h \|x\|_p^p + \int_\Omega \left[\int_0^\infty \min(|x(s)|^p, t|x(s)|^q) \, dm(t) \right] d\mu(s) \right)$$

$$= \gamma_{p,q} \int_\Omega \varphi(|x(s)|) \, d\mu(s) = \gamma_{p,q} I_\varphi(x).$$

(b) For all $x \in L^p + L^\infty$ and $t > 0$, according to the Krée formula (see [3, Theorem 5.2.1]),

$$(4.6) \quad \left(\int_0^t x^*(s)^p ds \right)^{1/p} \leq K(t^{1/p}, x; L^p, L^\infty) \leq 2^{1-1/p} \left(\int_0^t x^*(s)^p ds \right)^{1/p}.$$

The constant $2^{1-1/p}$ cannot be improved (see [2]). Due to Proposition 2.1,

$$(4.7) \quad K \left(t, \frac{Tx}{M}; L^p, L^\infty \right) \leq K(t, x; L^p, L^\infty) \quad \text{for all} \quad t > 0.$$

From (4.6) and (4.7) it follows that

$$\int_0^t \left(\frac{(Tx)^*(s)}{2^{1-1/p}M} \right)^p ds \leq \int_0^t x^*(s)^p \, ds \quad \text{for all} \quad t > 0.$$

Since $\psi(u) = \varphi(u^{1/p})$ is convex, by the Hardy-Littlewood-Pólya majorization theorem (see, e.g., [1, p. 88]),

$$\int_0^\infty \varphi \left(\frac{(Tx)^*(s)}{2^{1-1/p}M} \right) ds = \int_0^\infty \psi \left(\left[\frac{(Tx)^*(s)}{2^{1-1/p}M} \right]^p \right) ds$$

$$\leq \int_0^\infty \psi(x^*(s)^p) \, ds = \int_0^\infty \varphi(x^*(s)) \, ds.$$

Since the modular I_φ is rearrangement-invariant (see (2.1)), the latter inequality gives (b). □

The method of the proof of part (a) is due to J. Peetre [20]. In this proof the estimation of the functional $K_{p,q}^*$ ("$K_{p,q}^*$-monotonicity" property) is very essential. G. Sparr [22] proved that $\gamma_{p,q}$ is the best possible constant in the estimation of the functional $K_{p,q}^*$ (cf. Lemma 2.2). He also proved that $1 < \gamma_{p,q} < 2$ for $1 < p, q < \infty$. Now we give more precise information about this constant.

Proposition 4.3. *Suppose* $1 \le p, q < \infty$.
   (a) *Then* $\gamma_{p,q} = \gamma_{q,p}$ *and* $\gamma_{1,1} = 1$.
   (b) *If* $q > 1$, *then*

$$(4.8) \qquad \gamma_{p,q} = \inf \left\{ x + \left(\frac{p}{q} x^{p-1} \right)^{1/(q-1)} \; : \; x^p + \left(\frac{p}{q} x^{p-1} \right)^{q/(q-1)} = 1 \right\}.$$

In particular,

$$\gamma_{q,q} = 2^{1-1/q}, \quad \gamma_{1,q} = 1 + q^{1/(1-q)} - q^{q/(1-q)}.$$

   (c) $\gamma_{p,q}$ *continuously increases in* p *and* q.
   (d) *If* $p \le q$, *then* $2^{1-1/p} \le \gamma_{p,q} \le 2^{1-1/q}$.

Proof. Property (a) is obvious. Suppose $q > 1$ and rewrite $\gamma_{p,q}$ in the form

$$\gamma_{p,q} = \left\{ \gamma > 0 \; : \; \min_{0 \le x \le \gamma} F(\gamma, x, p, q) = 1 \right\},$$

where $F(\gamma, x, p, q) = x^p + (\gamma - x)^q$ and $\gamma \in (0, 2), x \in [0, \gamma], p \in [1, \infty), q \in (1, \infty)$. Obviously, for $x \in (0, \gamma)$,

$$(4.9) \quad \frac{\partial F}{\partial x} = px^{p-1} - q(\gamma - x)^{q-1}, \frac{\partial^2 F}{\partial x^2} = p(p-1)x^{p-2} + q(q-1)(\gamma - x)^{q-2} > 0.$$

Hence, $F(\gamma, x, p, q)$ is strictly convex in x and it has a unique minimum on $[0, \gamma]$ at a point $x(\gamma, p, q) \in (0, \gamma)$, which is the solution of the equation

$$\frac{\partial F}{\partial x} = px^{p-1} - q(\gamma - x)^{q-1} = 0.$$

Clearly,

$$(4.10) \qquad\qquad 0 < \gamma - 1 < x(\gamma, p, q) < 1,$$

and

$$\gamma = x(\gamma, p, q) + \left(\frac{p}{q} x(\gamma, p, q)^{p-1} \right)^{1/(q-1)},$$

$$\min_{0 \le x \le \gamma} F(\gamma, x, p, q) = x(\gamma, p, q)^p + \left(\frac{p}{q} x(\gamma, p, q)^{p-1} \right)^{q/(q-1)}.$$

So, (4.8) is proved. Using (4.8), one can easy calculate $\gamma_{1,q}$ and $\gamma_{q,q}$. The proof of (b) is finished.

   Let us prove (c). Due to (a), it is sufficient to prove that $\gamma_{p,q}$ continuously increases in p. Consider $\gamma = \gamma(p)$ such that

$$\min_{0 \le x \le \gamma} F(\gamma, x, p, q) = 1.$$

From conditions

$$(4.11) \qquad \begin{cases} \min\limits_{0 \le x \le \gamma} F(\gamma, x, p, q) - 1 = x^p + (\gamma - x)^q - 1 = 0, \\ \frac{\partial F}{\partial x}(\gamma, x, p, q) = px^{p-1} - q(\gamma - x)^{q-1} = 0, \end{cases}$$

taking into account that $\frac{\partial^2 F}{\partial x^2} > 0$ (see (4.9)), one can find the derivative of the implicit function $\gamma(p)$:

$$\frac{d\gamma}{dp} = -\frac{x^p \log x}{q(\gamma - x)^{q-1}}.$$

Taking into account (4.10), we see that $\frac{d\gamma}{dp} > 0$ for all $\gamma(p)$ satisfying (4.11). Hence, such $\gamma(p)$ continuously increases in p.

On the other hand, $\gamma_{p,q}$ is the smallest $\gamma(p)$ satisfying (4.11) (note that conditions (4.11) depend on p). Thus, $\gamma_{p,q}$ continuously increases in p. Property (c) is proved.

Property (d) follows from the monotonicity of $\gamma_{p,q}$:

$$2^{1-1/p} = \gamma_{p,p} \leq \gamma_{p,q} \leq \gamma_{q,q} = 2^{1-1/q}. \quad \square$$

5. THE MAIN INTERPOLATION THEOREM

Our main result reads:

Theorem 5.1. *Suppose $1 \leq p < q \leq \infty$ and $\rho \in \widetilde{\mathcal{P}}$. If $q = \infty$ assume in addition that $\rho_*(\mathbf{R}_+) = \mathbf{R}_+$, where $\rho_*(t) := t\rho(1/t)$. If $\varphi^{-1}(u) = u^{1/p}\rho(u^{1/q-1/p})$, then the Orlicz space L^φ (with both the Luxemburg and the Orlicz norm) is an interpolation space for linear operators between L^p and L^q, and*

$$\|T\|_{L^\varphi \to L^\varphi} \leq C \max\{\|T\|_{L^p \to L^p}, \|T\|_{L^q \to L^q}\},$$

where

    (a) $C \leq 2\gamma_{1,q} = 2(1 + q^{1/(q-1)} - q^{q/(q-1)}) \leq 2^{2-1/q} < 4$, *when* $1 = p < q < \infty$.

    (b) $C \leq \min\left\{(2\gamma_{p,q})^{1/p}, (2\gamma_{q',p'})^{1/q'}\right\} \leq 2^{1/(pq')+\min\{1/p,1/q'\}} < 4$, *when* $1 < p < q < \infty$.

    (c) $C \leq 2^{1-1/p} < 2$, *when* $1 \leq p < q = \infty$.

In particular, if either $1 < p < q \leq 2$ or $2 \leq p < q \leq \infty$, then $C < 2$.

Proof. First observe that the function φ is convex, due to Lemma 3.2(a). Hence, φ generates an Orlicz space L^φ.

Let $1 \leq p < q < \infty$. By Lemma 3.2(b), there is a function $h \in \mathcal{P}$ such that $\varphi(u) = u^q h(u^{p-q})$. From (3.1) we see that $\tilde{h} \in \widetilde{\mathcal{P}}$ and

$$(5.1) \qquad \varphi(u) \leq u^q \tilde{h}(u^{p-q}) \leq 2\varphi(u) \quad \text{for all } u > 0.$$

Applying Theorem 4.2(a) to the function $\psi(u) = u^q \tilde{h}(u^{p-q})$ and taking into account (5.1), we obtain

$$(5.2) \quad I_\varphi\left(\frac{Tx}{M}\right) \leq I_\psi\left(\frac{Tx}{M}\right) \leq \gamma_{p,q} I_\psi(x) \leq 2\gamma_{p,q} I_\varphi(x) \quad \text{for all} \quad x \in L^p \cap L^q.$$

Note that if $A > 0$, then

$$(5.3) \qquad \varphi(Au) = (Au)^q h((Au)^{p-q}) \leq A^q u^q \max\left\{1, A^{p-q}\right\} h(u^{p-q})$$

$$= \max\left\{A^p, A^q\right\} \varphi(u).$$

In particular, φ satisfies the Δ_2-condition for all $u \geq 0$. If $A = (2\gamma_{p,q})^{-1/p}$, then from (5.2) and (5.3) we conclude

$$I_\varphi\left(\frac{Tx}{(2\gamma_{p,q})^{1/p}M}\right) \leq \frac{1}{2\gamma_{p,q}} I_\varphi\left(\frac{Tx}{M}\right) \leq I_\varphi(x) \quad \text{for all} \quad x \in L^p \cap L^q,$$

which gives

$$(5.4) \qquad \|Tx\|_\varphi \leq (2\gamma_{p,q})^{1/p}M\|x\|_\varphi, \quad \|Tx\|_\varphi^0 \leq (2\gamma_{p,q})^{1/p}M\|x\|_\varphi^0$$

for all $x \in L^p \cap L^q$. Since φ satisfies the Δ_2-condition for all $u \geq 0$, it follows that $L^p \cap L^q$ is dense in L^φ (see [9, Ch. 2] for the case of N-functions and a finite

measure, in the general case this result can be obtained analogously). Hence, (5.4) is true for all $x \in L^\varphi$. This fact and Proposition 4.3(d) show that

$$(5.5) \qquad C \leq (2\gamma_{p,q})^{1/p} \leq 2^{(2-1/q)/p} < 4.$$

Now we will prove the second estimate in (b) by using duality arguments. Suppose $1 < p < q < \infty$ and $T \in \mathcal{A}(L^p, L^q)$. Then T maps $L^p + L^q$ into itself, but T maps also $L^p \cap L^q$ into itself (see, e.g., [10, Ch. 1, Lemma 4.1]). Since $L^p \cap L^q$ is dense in L^p and L^q, then $(L^p \cap L^q)^* = L^{p'} + L^{q'}$ (see, e.g., [10, Ch. 1, Th. 3.1]). Therefore, $T^* \in \mathcal{A}(L^{p'}, L^{q'})$. Due to Lemma 3.2(c), there is a function $k \in \mathcal{P}$ such that $\varphi^*(u) = u^{p'} k(u^{q'-p'})$ for all $u > 0$. As above one can prove that

$$(5.6) \qquad \|T^*x\|_{\varphi^*} \leq (2\gamma_{q',p'})^{1/q'} M \|x\|_{\varphi^*}, \quad \|T^*x\|_{\varphi^*}^0 \leq (2\gamma_{q',p'})^{1/q'} M \|x\|_{\varphi^*}^0$$

for all $x \in L^{p'} \cap L^{q'}$, and φ^* satisfies the Δ_2-condition for all $u \geq 0$. Consequently, (5.6) holds for all $x \in L^{\varphi^*}$. Taking into account duality of the Orlicz and Luxemburg norms and Proposition 4.3, this gives

$$(5.7) \qquad C \leq (2\gamma_{q',p'})^{1/q'} \leq 2^{(2-1/p')/q'} < 4.$$

Since

$$\min\left\{2^{(2-1/q)/p}, 2^{(2-1/p')/q'}\right\} = 2^{1/(pq')+\min\{1/p,1/q'\}},$$

we obtain from (5.5) and (5.7) that $C < 2$ in the cases $1 < p < q \leq 2$ or $2 \leq p < q < \infty$. The proof of (b) is complete.

Let $1 \leq p < q = \infty$. In view of Lemma 3.2(d), the function $\psi(u) = \varphi(u^{1/p})$ is convex. Hence, by Theorem 4.2(b), we obtain the modular estimate

$$I_\varphi\left(\frac{Tx}{2^{1-1/p}M}\right) \leq I_\varphi(x) \quad \text{for all} \quad x \in L^\varphi,$$

which implies

$$\|Tx\|_\varphi \leq 2^{1-1/p} M \|x\|_\varphi, \quad \|Tx\|_\varphi^0 \leq 2^{1-1/p} M \|x\|_\varphi^0$$

for all $x \in L^\varphi$. This gives (c). $\qquad\square$

Remark 5.2. Obviously Theorem 5.1 (a), (b) can, in fact, be reformulated in terms of Simonenko indices (1.5): if

$$1 \leq p \leq a_\varphi \leq b_\varphi \leq q < \infty,$$

then the Orlicz space L^φ is an interpolation space between L^p and L^q with the same interpolation constant as in Theorem 5.1 (a), (b).

Remark 5.3. If $1 \leq p < q < \infty$ and $\varphi(u) = u^q h(u^{p-q})$, where $h \in \widetilde{\mathcal{P}}$, then from the proof of the above theorem it follows that L^φ is an interpolation space between L^p and L^q, and we have a better estimate of the interpolation constant

$$C \leq (\gamma_{p,q})^{1/p} \leq 2^{1/(q'p)} < 2.$$

We illustrate Remark 5.3 with the following example:

Example 5.4. If $r \geq (3+\sqrt{5})/2$, then $\varphi(u) = u^r(1+|\log u|)$ is convex. For every p and q such that $1 \leq p < r < q < \infty$ and $r - p = q - r$, we have $\varphi(u) = u^q h(u^{p-q})$ with $h(u) = \sqrt{u}\left(1 + (q-p)^{-1}|\log u|\right)$. One can prove that $h \in \widetilde{\mathcal{P}}$.

Remark 5.5. In connection with Theorem 5.1(c) let us mention that for the case $p = 1$ and $q = \infty$ it coincides with the well-known Orlicz interpolation theorem. More precisely, Orlicz proved it in 1934 with certain constant $C > 1$ but from the Calderón-Mitjagin interpolation theorem it follows with the constant 1 (see [10, Ch. 2, Th. 4.9]; cf. also [15] for the direct proof). Moreover, G. G. Lorentz and T. Shimogaki [13, Theorem 7] observed that for the function

$$\varphi(u) = \int_0^u (u - t)^p dm(t)$$

with increasing function $m : \mathbf{R}_+ \to \mathbf{R}_+$, the interpolation constant C is 1.

Acknowledgements

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