Notre Dame Journal of Formal Logic
Volume XI, Number 4, October 1970

ON THE INTERPRETABILITY OF ARITHMETIC IN SET THEORY

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In 1950, Wanda Szmielew and Alfred Tarski [1] announced that the theory **Q**, a finitely axiomatizable essentially undecidable fragment of arithmetic, is interpretable in a small fragment **S** of set theory. The fragment **S** is so small that it is easily interpretable in any of the known formalizations of class or set theory with or without urelements and remains so interpretable even if all axioms of infinity are removed (most other axioms can be deleted also.) Furthermore, **S** is finitely axiomatized, it has three axioms, and even though its non-logical constants consist of one unary and one binary predicate symbol, the modification resulting from simple deletion of the unary symbol gives a stronger theory and hence gives another proof that first order predicate logic with a binary predicate symbol is undecidable, as is remarked in [2] (p. 34).

In 1964, the first author became interested in the result and no proof being available in the literature, the two of us devised a proof of it, an outline of which we communicated to Professor Tarski. Subsequently, Professor Tarski encouraged us to publish the proof which we do herewith.*

The proof we give appears to have some value beyond establishing the interpretability of **Q** in **S**. For instance one can prove from the definition of + in **S** that $0 + \{\{1\}\} \neq \{\{1\}\} + 0$; hence the commutative law for addition is not provable in **Q**. This raises a question, alien to the original motivation but we believe interesting in a technical sense. Can one interpret the theory **Q**, enriched by the addition of some or all of the commutative, associative and distributive laws, in the theory **S**?

The theories **Q** and **S** are the first order theories whose axioms are as follows, ([2] pp. 51 and 34):

Theory **Q**:

Q1. $Sx = Sy \rightarrow x = y$

Q2. $0 \neq Sy$

Q3. $x \neq 0 \rightarrow (\exists y)(x = Sy)$

Q4. $x + 0 = x$

Q5. $x + Sy = S(x + y)$

Q6. $x \cdot 0 = 0$

Q7. $x \cdot Sy = (x \cdot y) + x$

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*The second named author received support from NSF grant SP8457 during the preparation of this manuscript.

Received November 11, 1969

Theory **S**: *S1*. $[Ex \wedge Ey \wedge (z)(z \in x \leftrightarrow z \in y)] \rightarrow x = y$
S2. $(\exists x)[Ex \wedge (y)(y \notin x)]$
S3. $Ex \wedge Ey \rightarrow (\exists z)[Ez \wedge (w)(w \in z \leftrightarrow w \in x \vee w = y)]$

The intended interpretation of Ex is "x is a set." Thus *S1* is the axiom of extensionality for sets, *S2* asserts the existence of the empty set and *S3* guarantees the existence of $x \cup \{y\}$ for sets x and y.

We prove that **Q** is interpretable in **S** in the sense of [2] p. 21. We will not give direct definitions of S, +, ·, in **S** but instead will gradually extend **S** by definitions. Roughly the idea is to look at the usual way of interpreting **P** (Peano arithmetic) in **Z. F.** (Zermelo Frankel set theory). This is accomplished by developing the natural numbers in set theory. This development makes use of set theoretic axioms not available in **S** two of which are the axioms of regularity and infinity. We mention these axioms because between them they typify our method of handling usage of the others. The interpretability of **Q** does not require usage of the axiom of infinity—mainly because **Q** has no axioms of induction. In the usual development regularity is used to show that any natural number is well-ordered by "ϵ". This property we need and we obtain it in our development by building it into the definition of the predicate "x is a natural number." We proceed to extend **S** by definitions. Since extensionality, *S1*, pervades the whole development, usually we will omit mention of it in giving justifications.

D1. $x = 0 \leftrightarrow Ex \wedge (y)(y \notin x)$.
D2. $z = x \cup \{y\} \leftrightarrow Ex \wedge Ey \wedge Ez \wedge (w)(w \in z \leftrightarrow w \in x \vee w = y) \vee (\sim Ex \vee \sim Ey) \wedge z = 0)$.

Caution: *S3* justifies this definition but not the choice of notation. In particular, we see no way to define $x \cup y$ in **S**. Thus, whereas the notation would indicate that we have defined a composite operation, the operation cannot be so regarded in **S**.

D3. $\{x\} = 0 \cup \{x\}$.
D4. $\{x, y\} = \{x\} \cup \{y\}$.
D5. $x' = x \cup \{x\}$.
D6. $x \subset y \leftrightarrow Ex \wedge (u)(u \in x \rightarrow u \in y)$.
D7. $\text{Comp}\, x \leftrightarrow Ex \wedge (u)(u \in x \rightarrow Eu \wedge u \subseteq x)$ (x is a *complete* set).
D8. $\text{Trans}\, x \leftrightarrow Ex \wedge (u)(u \in x \rightarrow \text{Comp}\, u)$ (x is a *transitive* set).
D9. $Ix \leftrightarrow Ex \wedge (y)(z)[y \subseteq x \rightarrow (\exists w)[Ew \wedge (u)[u \in w \leftrightarrow u \in y \wedge u \in z]]]$
(x has the *intersection* property. Since w is unique we will denote it by $y \cap z$).

Corollary. $Ix \wedge y \subseteq x \rightarrow Iy$.

D10. $Cx \leftrightarrow (z)(\exists w)[Ew \wedge (u)(u \in w \leftrightarrow u \in x \wedge u \in z)]$
(x has the *complement* property. Since w is unique we will denote it by $x - z$).
D11. $Bx \leftrightarrow Ex \wedge Ix \wedge Cx$
(x has the *Boolean* property.).

Corollary. $Bx \wedge y \subseteq x \rightarrow By$.

Proof: We have $y \subseteq x \rightarrow \mathrm{E}y$ and $\mathrm{I}x \wedge y \subseteq x \rightarrow \mathrm{I}y$. To prove $\mathrm{C}y$ note that for any z, $y - z = y \cap (x - z)$.

D12. $\mathrm{W}x \leftrightarrow (u)[u \in x \rightarrow \sim(u \in v \wedge v \in u) \wedge$
$(y)[y \subseteq x \wedge (\exists z)(z \in y) \rightarrow (\exists u)(u \in y \wedge$
$(v)(v \in y \rightarrow u \in v \vee u = v)] \wedge$
$(y)[y \subseteq x \wedge (\exists z)(z \in y) \rightarrow (\exists u)(u \in y \wedge$
$(v)(v \in y \rightarrow v \in u \vee v = u)]]$.
(In the presence of Trans x, $\mathrm{W}x$ means x is *well-ordered* by $\in$ and $\breve{\in}$).

D13. $\mathrm{N}x \leftrightarrow \mathrm{B}x \wedge \mathrm{Comp}\ x \wedge \mathrm{Trans}\ x \wedge \mathrm{W}x$.

THEOREM 1. $\mathrm{E}x \rightarrow [\mathrm{Comp}\ x \leftrightarrow \mathrm{Comp}\ x'] \wedge [\mathrm{Trans}\ x \leftrightarrow \mathrm{Trans}\ x']$.

Proof: Just a corollary of the definitions.

LEMMA 2. $\mathrm{E}x \wedge \mathrm{E}y \rightarrow [\mathrm{I}x \rightarrow \mathrm{I}(x \cup \{y\})]$.

Proof: $\mathrm{I}x \rightarrow \mathrm{I}(x \cup \{y\})$: Let $z \subseteq x \cup \{y\}$ and consider any w. We want to prove the existence of $z \cap w$. If $y \in z$ then $z \subseteq x$ and $z \cap w$ exists by $\mathrm{I}x$. Assume $y \in z$. From $\mathrm{I}x$ it follows that $(z \cap x) \cap w$ exists and hence from *S3* that $((z \cap x) \cap w) \cup \{y\}$ exists. But the latter is just $z \cap w$.

$\mathrm{I}(x \cup \{y\}) \rightarrow \mathrm{I}x$: *S3* assures that $x \subseteq x \cup \{y\}$ and hence $\mathrm{I}x$ follows.

LEMMA 3. $\mathrm{E}x \wedge \mathrm{E}y \rightarrow [\mathrm{C}x \leftrightarrow \mathrm{C}(x \cup \{y\})]$.

Proof: $\mathrm{C}x \rightarrow \mathrm{C}(x \cup \{y\})$. Given any z we must prove the existence of $x \cup \{y\} - z$. If $y \in z$ this is just $x - z$; if $y \notin z$; this is just $(x - z) \cup \{y\}$. $\mathrm{C}x$ and *S3* guarantee the existence of these two sets.

THEOREM 4. $\mathrm{E}x \rightarrow [\mathrm{B}x \leftrightarrow \mathrm{B}x']$.

Proof: An immediate consequence of Lemmas 2 and 3 and $\mathrm{E}x \rightarrow \mathrm{E}x'$.

THEOREM 5. $\mathrm{I}x \wedge \mathrm{W}x \rightarrow \mathrm{W}x'$.

Proof: We consider the three conjuncts of $\mathrm{W}x'$. To establish the first conjunct we note that $(u)[u \in x \rightarrow u \notin u] \rightarrow x \notin x$. Hence $\mathrm{E}x$ and the first conjunct of $\mathrm{W}x$ imply the first conjunct of $\mathrm{W}x'$. The remaining conjuncts of $\mathrm{W}x'$ involve arbitrary subsets $y \subseteq x'$. If $y \subseteq x'$ then $y \subseteq x$ or $x \in y$. The instances of these conjuncts for $y \subseteq x$ are immediate consequences of $\mathrm{W}x$. Hence assume $x \in y$. In this case the third conjunct is immediate, x is an $\in$-last element of y. If $y = \{x\}$ the second conjunct is trivial. Thus suppose $\{x\} \subset y$. Then $x \cap y$, whose existence is assured by $\mathrm{I}x$, is nonempty. Let w be a first element of $x \cap y$. Then $w \in x$ also. Hence the second conjunct of $\mathrm{W}x'$ is established.

THEOREM 6. $\mathrm{E}x \rightarrow (\mathrm{N}x \leftrightarrow \mathrm{N}x')$.

Proof: Immediate from Theorems 1, 4, and 5.

THEOREM 7. $\mathrm{N}(0)$.

Proof: Immediate from the definitions of N and 0.

THEOREM 8. $\mathrm{E}x \wedge \mathrm{E}y \wedge \mathrm{Comp}\ y \wedge y \notin y \wedge x' = y' \rightarrow x = y$.

Proof: From $\mathrm{E}x \wedge \mathrm{E}y \wedge x' = y'$ we have

$$(u)[u \in x \vee u = x \leftrightarrow u \in y \vee u = y]$$

which together with the assumption $x \neq y$ implies $y \in x \wedge x \in y$. However the latter together with Comp y implies $y \in y$, contradicting the assumption $y \notin y$.

THEOREM 9. $\mathrm{N}x \wedge \mathrm{N}y \wedge x' = y' \rightarrow x = y$.

The following three lemmas are immediate consequences of the definitions.

LEMMA 10. Trans $x \wedge y \in x \rightarrow$ Comp y.

LEMMA 11. Trans $x \wedge y \subseteq x \rightarrow$ Trans y.

LEMMA 12. $\mathrm{W}x \wedge y \subseteq x \rightarrow \mathrm{W}y$.

THEOREM 13. $\mathrm{N}x \wedge y \in x \rightarrow \mathrm{N}y$.

Proof: From the assumptions it follows that $y \in x \wedge y \subseteq x$. The conclusion follows from $\mathrm{B}x \wedge y \subseteq x \rightarrow \mathrm{B}y$ and Lemmas 10, 11, 12.

LEMMA 14. $\mathrm{E}x \wedge \mathrm{Comp}\, x \wedge \mathrm{I}x \wedge \mathrm{W}x \wedge x \neq 0 \rightarrow (\exists u)(\mathrm{E}u \wedge x = u')$.

Proof: $\mathrm{E}x \wedge x \neq 0 \wedge \mathrm{W}x \rightarrow x$ has an ϵ-last element, u. From Comp x it follows that $u \subseteq x$ and $\mathrm{E}u$. Also $\mathrm{I}u$ by the corollary to *D9*. Hence $u' \subseteq x$. On the other hand, since u is an ϵ-last element of x we have $x \subseteq u'$. Thus by extensionality $x = u'$.

D14. $y = \mathrm{S}x \leftrightarrow (\mathrm{N}x \wedge y = x') \vee (\sim \mathrm{N}x \wedge y = x)$.

THEOREM *Q1*. $\mathrm{S}x = \mathrm{S}y \rightarrow x = y$.

Proof: Case 1. $\mathrm{N}x \wedge \mathrm{N}y$. An immediate consequence of Theorem 8.
Case 2. $\sim \mathrm{N}x \wedge \sim \mathrm{N}y$. Trivial. The other cases are impossible since $\mathrm{N}x \wedge \mathrm{S}x = \mathrm{S}y \rightarrow \mathrm{N}y$ by Theorem 6.

THEOREM *Q2*. $0 \neq \mathrm{S}y$.

Proof. If $\mathrm{N}y$ then $\mathrm{S}y = y' \neq 0$. If $\sim \mathrm{N}y$ then $\mathrm{S}y = y$ and $y \neq 0$ since $\mathrm{N}(0)$.

THEOREM *Q3*. $x \neq 0 \rightarrow (\exists y)[x = \mathrm{S}y]$.

Proof: If $\sim \mathrm{N}x$ then $x = \mathrm{S}x$; if $\mathrm{N}x$ the result is an immediate consequence of Lemma 14 and Theorem 6.

D15. $\langle x, y\rangle = \{\{x\}, \{x, y\}\}$.

COROLLARY. $\mathrm{E}x \wedge \mathrm{E}y \rightarrow [\mathrm{E}\langle x,y\rangle \wedge (u)(u \in \langle x,y\rangle \leftrightarrow u = \{x\} \vee u = \{x,y\})]$.

LEMMA 15. $(\mathrm{E}x \wedge \mathrm{E}y \wedge \mathrm{E}u \wedge \mathrm{E}v \wedge \langle x,y\rangle = \langle u,v\rangle) \rightarrow (x = u \wedge y = v)]$.

D16. $\mathrm{Rel}\, x \leftrightarrow \mathrm{E}x \wedge (w)[w \in x \rightarrow (\exists u,v)(\mathrm{E}u \wedge \mathrm{E}v \wedge w = \langle u,v\rangle)]$.

COROLLARY. $[\mathrm{Rel}\, x \wedge \langle u,v\rangle \in x] \rightarrow [\mathrm{E}u \wedge \mathrm{E}v]$.

D17. $\mathrm{Funct}\, x \leftrightarrow \mathrm{Rel}\, x \wedge (u,v,w)\,[\langle u,v\rangle \in x \wedge \langle u,w\rangle \in x \rightarrow v = w]$.

D18. $y\,\mathrm{D}x \leftrightarrow \mathrm{E}y \wedge (u)[u \in y \leftrightarrow \mathrm{E}u \wedge (\exists v)(\mathrm{E}v \wedge \langle u,v\rangle \in x)]$
(y is the *domain of* x.).

D19. $Dx \leftrightarrow (z)[z \subseteq x \rightarrow (\exists w)(w\,D\,z)]$
(x has the *domain* property.).

LEMMA 16. $Ex \wedge Ey \rightarrow [Bx \wedge Dx \leftrightarrow B(x \cup \{y\}) \wedge D(x \cup \{y\})]$.

Proof: Assume $Bx \wedge Dx$. By Lemmas 2 and 3 we have $B(x \cup \{y\})$. Let $z \subseteq x \cup \{y\}$. We prove $(\exists w)(Ew \wedge wDz)$ as follows: we have

$$I(x \cup \{y\}) \wedge Ex \rightarrow E(x \cap z)$$

and

$$Dx \rightarrow (\exists w_1)(Ew_1 \wedge w_1 D\,(x \cap z)),$$

so we take $w = w_1$ unless $y \in z \wedge (\exists u,v)[Eu \wedge Ev \wedge y = \langle u,v\rangle]$ in which case we take $w = w_1 \cup \{u\}$. The converse is immediate from Lemmas 2, 3 and the definition of Dx.

D20. $R(x,y,z) \leftrightarrow Nx \wedge Ny \wedge Nz \wedge (\exists w)\,[\mathrm{Funct}\,w \wedge y'Dw \wedge \langle 0,x\rangle \in w \wedge (u)(v)(\langle u,v\rangle \in w \wedge u \in y \rightarrow \langle u',v'\rangle \in w) \wedge \langle y,z\rangle \in w \wedge Bw \wedge Dw]$.

LEMMA 17. $Nx \rightarrow R(x,0,x)$.

Proof: Let $w = \{\langle 0,x\rangle\}$.

THEOREM 18. $R(x,y,z_1) \wedge R(x,y,z_2) \rightarrow z_1 = z_2$.

Proof: Let w_1 be a function which establishes $R(x,y,z_1)$ and let w_2 be a function which establishes $R(x,y,z_2)$. From Bw_1 it follows that $w_1 \cap w_2$ exists. The proof will be completed by showing that $y'D(w_1 \cap w_2)$. Let $tD(w_1 \cap w_2)$. (The existence of t is a consequence of Dw_1). Since $t \subseteq y'$ we need only prove that $y' - t = 0$. ($y' - t$ exists since $N(y) \rightarrow N(y') \rightarrow Cy'$.) If $y' - t \neq 0$, Wy', which follows from Ny via Theorem 6, implies the existence of an ϵ-first element u of $y' - t$. Theorem 13 gives us Nu. Since $0 \in t$, we have $u \neq 0$. From Lemma 14 and Theorem 6 we conclude the existence of u_1 such that $u = u_1'$ and Nu_1. We will obtain the desired contradiction by showing first that $u_1 \in t$ and then $u \in t$:

$$u_1 \in y' \text{ since } u_1 \in u \text{ and } u \in y' \text{ and Comp } y'.$$

(This also proves $u_1 \in y$.) But

$$u_1 \notin y' - t \text{ since } u \notin u_1$$

(because $u_1 \in u \wedge u \notin u \wedge \mathrm{Comp}\ u$) and $u \neq u_1$. Hence $u_1 \in t$, that is,

$$(\exists v)\,[Ev \wedge \langle u_1,v\rangle \in w_1 \cap w_2].$$

Since $u_1 \in y$ also we have $\langle u_1',v'\rangle \in w_1 \cap w_2$. Hence $u \in t$ contradicting $u \in y' - t$. Thus $t = y'$. Since $\mathrm{Funct}\,w_1$ and $\mathrm{Funct}\,w_2$ we have $z_1 = z_2$.

THEOREM 19. $R(x,y,z) \supset R(x,y',z')$.

Proof: Let w be a function establishing $R(x,y,z)$. Then $w_1 = w \cup \{\langle y',z'\rangle\}$ establishes $R(x,y',z')$ (using Lemma 16).

THEOREM 20. $Ny \wedge R(x,y',z_1) \rightarrow (\exists z)\,[z_1 = z' \wedge R(x,y,z)]$.

Proof: We first prove $(\exists z)(z_1 = z')$. Let w_1 be a function establishing

$R(x,y',z_1)$. For some z, $\langle y,z\rangle \in w_1$ and hence $\langle y',z'\rangle \in w_1$. From Funct $w_1 \wedge \langle y',z_1\rangle \in w_1$ we have $z_1 = z'$. It remains to prove $R(x,y,z)$. Let $w = w_1 - \{\langle y',z_1\rangle\}$. Then $w_1 = w \cup \{\langle y',z_1\rangle\}$, and hence $Bw \wedge Dw$ by Lemma 16. The other properties desired of w are immediate.

D21. $1 = \{0\}$.

LEMMA 21. $\sim N(\{1\})$.

Proof: $\sim$ Comp $(\{1\})$ since $1 \notin \{1\}$.

D22. $z = x + y \leftrightarrow R(x,y,z) \vee (Nx \wedge \sim(\exists z)R(x,y,z) \wedge z = \{1\}) \vee (\sim Nx \wedge z = x)$. (Since this definition works our proof shows $x + y = y + x$ not provable in **Q**.)

THEOREM *Q4*: $x + 0 = x$.

Proof: If Nx then $R(x,0,x)$ by Lemma 17. If $\sim Nx$ then $x + 0 = x$ by *D22*.

THEOREM *Q5*. $x + Sy = S(x + y)$.

Proof: If $\sim Nx$ the result is immediate from *D14*, *D21*, and *D22*. Thus assume Nx.
Case: $\sim Ny$. Then $Sy = y$ and $\sim(\exists z)R(x,y,z)$. Hence $x + Sy = x + y = \{1\}$. Since $\sim N(\{1\})$, $S(\{1\}) = \{1\}$, that is, $x + Sy = S(x + y)$.
Case: Ny. Then $Sy = y'$. If $(\exists z)R(x,y,z)$ then by Theorem 19, $R(x,y',z')$; also $z' = Sz$ and $y' = Sy$. Hence $x + Sy = S(x + y) = z'$. Suppose $\sim(\exists z)R(x,y,z)$. By Theorem 20 $\sim(\exists z)R(x,y',z)$. Hence $x + y = \{1\}$ and $x + Sy = \{1\}$. Also $S(x + y) = x + y$.

D23. $P(x,y,z) \leftrightarrow Nx \wedge Ny \wedge (\exists w)[\text{Funct } w \wedge Bw \wedge Dw \wedge y'Dw \wedge \langle 0,0\rangle \in w \wedge (u)(v)(\langle u,v\rangle \in w \wedge u \in y \rightarrow \langle u', v + x\rangle \in w \wedge \langle y,z\rangle \in w)]$.

LEMMA 22. $Nx \rightarrow P(x,0,0)$

Proof: Let $w = \{\langle 0,0\rangle\}$.

THEOREM 23. $P(x,y,z) \wedge P(x,y,z_1) \rightarrow z = z_1$.

Proof: Identical to that of Theorem 18.

THEOREM 24. $P(x,y,z) \rightarrow P(z,y', z + x)$.

Proof: Identical to that of Theorem 19.

THEOREM 25. $Ny \wedge P(x,y',z_1) \rightarrow (\exists z)[z_1 = z + x \wedge P(x,y,z)]$.

Proof: Identical to that of Theorem 20.

D24. $z = x \cdot y \leftrightarrow P(x,y,z) \vee [Nx \wedge \sim(\exists z)P(x,y,z) \wedge z = \{1\}] \vee [\sim Nx \wedge y \neq 0 \wedge z = \{1\}] \vee [\sim Nx \wedge y = 0 \wedge z = 0]$.

THEOREM *Q6*. $x \cdot 0 = 0$.

Proof: Immediate from *D24* and Lemma 22.

THEOREM *Q7*. $x \cdot Sy = x \cdot y + x$.

Proof: Case: $\sim Nx$. Since $Sy \neq 0$ we have $x \cdot Sy = \{1\}$. Also $x \cdot y = 0$ or

$x \cdot y = \{1\}$. If $x \cdot y = 0$ then the second clause of *D21* gives $x \cdot y + x = \{1\}$. If $x \circ y = \{1\}$ then Lemma 21 and the third clause of *D22* gives $x \cdot y + x = \{1\}$. Thus in either case $x \cdot Sy = x \cdot y + x$.

Case: Nx. If $\sim$Ny then $Sy = y$ and $x \circ Sy = \{1\}$. But $\{1\} + x = \{1\}$, hence $x \circ Sy = x \cdot y + x$. If N$y$ then $Sy = y'$. Suppose for some z, P(x,y,z). By Theorem 24, P$(x,y',z+x)$. Hence $x \cdot y + x = z + x = x \cdot y' = x \cdot Sy$. Finally suppose $\sim(\exists z)$P(x,y,z). By Theorem 25, $\sim(\exists z)$P(x,y',z). Hence $x \cdot Sy = \{1\}$ and $x \cdot y = \{1\}$. Again from *D22* we have $x \cdot y + x = \{1\} = x \cdot Sy$.

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