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On the Intersection of Finitely Generated Subgroups of Free Groups

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ABSTRACT. Howson [4] proved that the intersection of two finitely generated subgroups H and K of ranks m and n respectively is finitely generated. He proved that the rank N of $H \cap K$ is at most $2mn - m - n + 1$. H. Neumann [8,9] gave a better bound of $2mn - 2n - 2m + 3$. Burns [2] further improved the general upper bound to $N \leq 2mn - 3m - 2n + 4$ (for $m \leq n$).

Imrich [6] gave shorter proof of Neumann's result and also Nickolas [10] gave simple proof for Burn's result. Servatius [12] gave graphical proof for Burn's result.

Burns [1] showed that the stronger bound $N \leq mn - n - m + 2$ holds if H or K is of finite index in F.

In this paper it is shown that stronger bound $N \leq mn - n - m + 2$ always hold.

In section 1 we gave basic concepts about free groups, graphs and cayley graphs.

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In section 2 we showed the main theorem 2.7: "If H and K are finitely generated subgroups of a free group F on generators a, b of ranks m, n respectively and if all vertices in $\Gamma^*(H)$ and $\Gamma^*(K)$ are of degree 2 and 3 only, then the rank N of $H \cap K$ satisfies $N \leq mn - n - m + 2$".

In order to prove the main Theorem 2.7, we followed and improved the techniques which were used by Nickolas [10] especially the concept of compatibelty of paths and branch points.

By this improvement, we could have an upper bound on the number of the compatible branch points in the core of $\Gamma^*(H) \tilde{\times} \Gamma^*(k)$ which is the product of $\Gamma^*(H)$ and $\Gamma^*(k)$.

Therefore we started to know the least number of typing of compatible branch points of degree 3 only in $\Gamma^*(H)$ as shown in Lemma 2.3.

In Lemma 2.4 we showed that if $\Gamma^*(H)$ has only two types of compatible branch points X_1 and X_2 then the number of branch points of type X_1 = the number of branch points of type X_2.

In Lemma 2.5 we showed that if $\Gamma^*(H)$ has more than two types of compatible branch points then the largest number of one type of compatible branch points is n, where # Br $(\Gamma^*(H) = 2n \geq 4$ or $i < n$ for all $i = 1, \ldots, r$, where r is the number of typing of compatible branch point X_1.

Therefore by above Lemmas 2.3, 3.4 and 2.5 we could have an upper bound for the number of branch points in the core of $\Gamma^*(H) \tilde{\times} \Gamma^*(K)$ which is less than or equal to $\frac{\#Br(\Gamma^*(H)) \times \#Br(\Gamma^*(K))}{2}$ as shown in Theorem 2.6.

1. INTRODUCTION

1.1. Free groups

A group F is said to be **free** on a finite subset $X \subseteq F$, where $X = \{x_1, x_2, \ldots, x_n\}$ if for any group B and any mapping $f : X \to G$ there is a unique homomorphism $\theta : F \to G$ such that $x\theta = xf$ for all $x \in X$.

The cardinality of X is called the **rank** of F and is denoted by $|X|$; and X is called a **set of generators** of the free group F. If X is finite then F is called **finitely generated** free group.

A **word** on X is a finite sequence of elements in $X^+ \cup X^-$ where $X^- = \{x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}\}$ $X = X^+ = \{x : x \in X\}$. A word is denoted by W. A word $W = x_{i_1}^{\epsilon_1} x_{i_2}^{\epsilon_2} \ldots x_{i_r}^{\epsilon_r}$ when $1 \leq i \leq n$, $r \geq 0$, $\epsilon = \pm 1$, is called a **reduced** word or a finitely reduced word if $x_{i_j}^{\epsilon_j} \neq x_{i_{j+1}}^{\epsilon_{j+1}}$.

The set of all reduced words is denoted by F_x. The **inverse** of the word $W = x_{i_1}^{\epsilon_1} x_{i_2}^{\epsilon_2} \ldots x_{i_r}^{\epsilon_r}$ is the word $x_{i_r}^{-\epsilon_r} x_{i_{r-1}}^{-\epsilon_{r-1}} \ldots x_{i_2}^{-\epsilon_2} x_{i_1}^{-\epsilon_1}$ and is denoted by W^{-1}. The **length** of the word W_1 is the length r of the finite sequence $x_{i_1}^{\epsilon_1} x_{i_2}^{\epsilon_2} \ldots x_{i_r}^{\epsilon_r}, \epsilon = \pm 1$ and is denoted by $L(W)$. The **empty** word has length zero, i.e. $L(1) = 0, 1$ is the identity element of F. It is clear that $L(W) = L(W^{-1})$.

The words W_1 and W_2 on X are called **equivalent** (denoted by $W_1 \sim W_2$) if the following operations applied a finite number of times change W_1 into W_2 or W_2 into W_1 : (1) Insertion of one of the words VV^{-1} between any two consecutive symbols of W, or before W or after W. (2) Deletion of one of the words VV^{-1} if it forms a block of consecutive symbols in W. Nielson - Schreier [11] showed that if F is a free group of rank n and H is a subgroup of F, then H is free. If $|F : H| = g$ is finite, then the rank of H is equal to $g(n-1)+1$.

1.2. Graphs

A **graph** Γ is a collection of two sets V (V is not empty set) and E called the set of vertices and edges respectively of the graph Γ, together with two functions $i : E \to V$, $t : E \to V$ (we say that edge e joins the vertex $i(e)$ to $t(e)$. The vertex $i(e)$ is called the **initial vertex** of e and $t(e)$ is called the **terminal vertex** of e). Moreover for each e in E there is an element $\bar{e} \neq e$ in E, called the **inverse** of e, such that $i(\bar{e}) = t(e)$, $t(\bar{e}) = i(e)$ and $\bar{\bar{e}} = e$.

A **subgraph** Δ of a graph Γ is a graph with $V(\Delta) \subseteq V(\Gamma)$, $E(\Delta) \subseteq E(\Gamma)$. If $e \in E(\Delta)$, then $i_\Delta(e)$, $t_\Delta(e)$ and $\bar{e}$ have the same meaning in Γ as they do in Δ. If $\Delta \neq \Gamma$ then we call Δ a **proper** subgraph.

A **path** P in a graph Γ is a finite sequence $e_1, e_2, \ldots, e_n$, where $e_i \in E(\Gamma)$, $1 \leq i \leq n-1$, $t(e_i) = i(e_{i+1})$.

The **initial vertex** of P is the initial vertex of e_1 and the **terminal vertex** of P is the terminal vertex of e_n. The path P is called **reduced** path if $e_i \neq \bar{e}_{i+1}$, for $\leq i \leq n-1$ where $P = e_1 e_2 \ldots e_n$, $e_i \in E(\Gamma)$ and

closed if $i(P) = t(p)$. If P is reduced and closed, then P is called a **circuit** or cycle.

The **length** of a path P is the number of edges in the path P. If P_1 and P_2 are paths in a graph Γ and the terminal of P_1 equals the initial vertex of P_2, they may be concatenated to form a path P_1P_2 with $L(P_1P_2) = L(P_1) + L(P_2)$ such that $i(P_1P_2) = i(P_1)$, $t(P_1P_2) = t(P_2)$, $t(P_1) = i(P_2)$.

A **trivial successions** of edges is a path of the form $e\bar{e}$, $\bar{e}e$, where $e \in E$, $\bar{e} \in E$.

If a path P contains a trivial succession of edges then by collapsing the trivial successions of edges we get a new path P'. This operation is called an **elementary reduction** and is denoted by $P \downarrow P'$.

Two paths P and P' are called **equivalent**, denoted by $P \sim P'$, if there is a finite sequence of paths $P = P_1, P_2, \ldots, P_k = P'$ such that either $P_j \downarrow P_{j+1}$ or $P_{j+1} \downarrow P_j$ for $i \leq j \leq k-1$. Therefore as in [3], (a) Each path P is equivalent to a unique reduced path. (b) the operation of composition of paths is compatible with equivalence. That is, $P \sim P'$, $S \sim S'$ implies $PS \sim P'S'$, if the compositions are defined.

A graph Γ is **connected** if $v, u \in V(\Gamma)$ implies there exists a path in Γ joining v to u. A **component** of Γ is a maximal connected subgraph of Γ.

A **tree** is a connected non-empty graph without reduced circuits.

If Δ is a subgraph of a connected graph Γ, then Δ is called **spanning** if every pair of vertices of Γ is joined by at least one path in Δ and a **spanning tree** if Δ is tree and spanning.

A **morphism** of graphs is a function $f : \Gamma \rightarrow \pi$ such that f takes each edge to an edge or a vertex and each vertex to a vertex with the following property $f(\bar{e}) = \overline{f(e)}$, where $e \in \Gamma$ and $i(f(e)) = f(i(e))$ when $i(v) = v$ for $v \in V$.

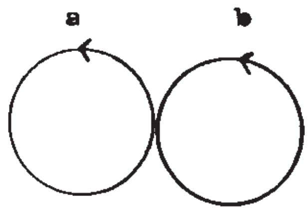
Two graphs Γ and π are called **isomorphic** is there exists a one-one mapping f of the vertices and edges of Γ onto the vertices and edges respectively of π, which preserves the relation "is the initial vertex of", "is the terminal vertex of" and "is the inverse of".

1.3. Cayley graphs

The graph $\Gamma(F, X)$ is called the **cayley graph** of the group F with respect to $X \subseteq F$. It has vertex set F and set of edges $F \times X$ (i.e. $(w, x) \in E(\Gamma(F, X))$ with $i(w, x)$ is the initial vertex of (w, x) and $t(w, x)$ the termanal vertex of (w, x) for every edge $(w, x) \in E(\Gamma(F, X))$, the inverse edge of (w, x) is $(wx, x^{-1}) \in E(\Gamma(F, X))$.

The **quotient** graph [5] or cayley coset graph $\Gamma(F, X)/H$ for a subgroup H of F has set of vertices $\{Hw : w \in F, H \leq F\}$ and set of edges $\{(Hw, x) : w \in F, x \in X\}$ such that an edge $(Hw, x) \in \Gamma(F, X)/H$ takes the vertex Hw to Hwx. It is also denoted by $\Gamma(H)$. The **core** of a coset graph $\Gamma(H)$ is the smallest subgraph containing all cycles. It is denoted by $\Gamma^*(H)$, for example, if F is a free group on generators a, b, then

$\Gamma^*(F)$:



The number of cycles in $\Gamma^*(H)$ is called the **cyclomatic number**. The cyclomatic number of $\Gamma^*(H)$ is the minimal number of edges that we can delete to make a tree.

The rank of the finitely generated subgroup H is the cyclomatic number of $\Gamma^*(H)$. The vertex V is called a **branch point** if $d(V) \geq 3$ where $V \in V(\Gamma^*(H))$ and $d(V)$ is the degree of the vertex V.

2. MAIN THEOREM

By direct calculations we can prove the following proposition.

Proposition 2.1. *If $\Gamma^*(H)$ is a core graph of finitely generated subgroup H of rank m and if all vertices of $\Gamma^*(H)$ are of degree 2 and 3 only. Then $m = 1 + \frac{\#Br(\Gamma^*(H))}{2}$, where $\#Br(\Gamma^*(H)$ is the number of branch points in $\Gamma^*(H)$.* ∎

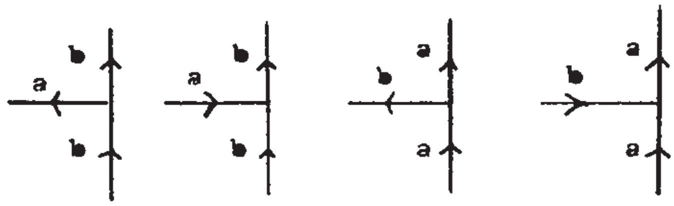
Two branch points are called **neighbours** if they are connected by a (reduced) path which does not contain any branch point.

Now free group F generated by a, b and $\Gamma^*(H)$ has vertices of degree 2, 3 and 4 as in [12]. We can reduce the degree of vertices in $\Gamma^*(H)$ into vertices of degree 2 and 3 only (by isomorphically embedding F into a free group Q on $\{u, v\}$ via the map $\theta : F \to Q$ with $\theta(a) = uv^{-1}$ and $\theta(b) = v^2$ and taking the graph into new set of labels $\{u, v\}$.

The **product of core graphs $\Gamma^*(H)$ and $\Gamma^*(k)$** (where H and K are finitely generated subgroups of the free group $F\{a, b\}$) is the graph $\Gamma^*(H) \tilde{\times} \Gamma^*(H)$ with set of vertices $V_1 \times V_2 = \{(v, u);\ v \in V_1$ and $u \in V_2\}$ and edges $\{((u, v), y);\ (v, y) \in E(\Gamma^*(H))$ and $(u, y) \in E(\Gamma^*(K))$ and $i \in X\}$. Nickolas [10] showed the following:

Proposition 2.2. *Let $\Gamma^*(H)$, $\Gamma^*(K)$ and $\Gamma^*(H \cap K)$ be core graphs of $\Gamma(H)$, $\Gamma(K)$ and $\Gamma(H \cap K)$ respectively. If $\Gamma^*(H) \tilde{\times} \Gamma^*(K)$ is the product of $\Gamma^*(H)$ and $\Gamma^*(K)$ defined above, then $\Gamma * (H \cap K)$ may be identified with core of a connected component of the graph $\Gamma^*(H) \tilde{\times} \Gamma^*(K)$. N.B. In the rest of the paper we will write core of $\Gamma^*(H) \tilde{\times} \Gamma^*(K)$ to mean core of a connected component of the graph $\Gamma^*(H) \tilde{\times} \Gamma^*(K)$.* ■

In $\Gamma^*(H)$ there are four possibilities for the branch points in the core of $\Gamma(H)$ as follows:



These branch points are called a-sources, a-sinks, b-sources and b-sinks respectively.

For any two paths P_{ij} and P_{rs} in $\Gamma^*(H) \cup \Gamma^*(K)$ (may $K = H$), we say P_{ij} and P_{rs} are **compatible** if $P_{ij} = P_{rs}$ and P_{ij} or P_{rs} or both of them may pass through several branch points such that P_{ij} starting at a branch point i and ending at a branch point j and P_{rs} starting at a branch point r and ending at a branch point s also (i, r) and (j, s) are neighbouring branch points in $\Gamma^*(H) \tilde{\times} \Gamma^*(K)$.

Therefore branch points i, r in $\Gamma^*(H) \cup \Gamma^*(K)$ (may $K = H$) are called **compatible** if all paths P_{ij} and P_{rs} are compatible in $\Gamma^*(H) \cup$

$\Gamma^*(K)$ and (i,r) and (j_t,s_t) are neighbouring branch points in $\Gamma^*(H) \tilde{\times} \Gamma^*(K)$ for each $t = 1,2,3$. (Incompatible will mean not compatible). We can write $P_{ij_t} = P_t = P_{rs_t}$ for each t. If u_1 and u_2 are compatible branch points in $\Gamma^*(H) \cup \Gamma^*(K)$ then P_t are compatible for each t. Also if $\mathcal{U}_2$ and $\mathcal{U}_3$ are compatible branch points in $\Gamma^*(H) \cup \Gamma^*(K)$ then $q_{u_2k_t} = q_t = q_{u_3v_t}$ are compatible for each $t = 1,2,3$.

Therefore $L(P_t) \le L(q_t))$, where $L(P_t)$ and $L(q_k)$ are the length of the paths P_t and q_t respectively, $t = 1,2,3$.

If $\mathcal{U}_1$ and $\mathcal{U}_2$ are compatible then $g_{u_1i_t} = g_t = g_{u_3h_t}$ are compatible for each $t = 1,2,3$.

Therefore either $P_t = q_t = g_t$ or $g_t = q_t = P_th_t$, where $t(P_t) = i(h_t)$ as in examples 1 and 3.

The following example show us that the compatibility of branch points is not transitive in general.

We see that 1 and 2 are compatible. Also 2 and 6 are compatible but 1 and 6 are incompatible.

Definition. *A consistent graph is a directed X-labelled graph (where $X = \{x_1, x_2, \ldots, x_n\}$) with no reduced paths labelled x_i, x_i^{-1} or x_i^{-1}, x_i, $x_i \in X \cup X^{-1}$, $1 \le i \le n$.*

If $X = \{a,b\}$, then no reduced paths labelled aa^{-1}, bb^{-1}, $a^{-1}a$ and $b^{-1}b$ ever occur in a consistent graph.

Lemma 2.3. *Let $\Gamma^*(H)$ be the core graph of the finitely generated subgroup H of the free group F on generators a,b. Then $\Gamma^*(H)$ has at least two types of compatible branch points, where all branch points are of degree 3 only.*

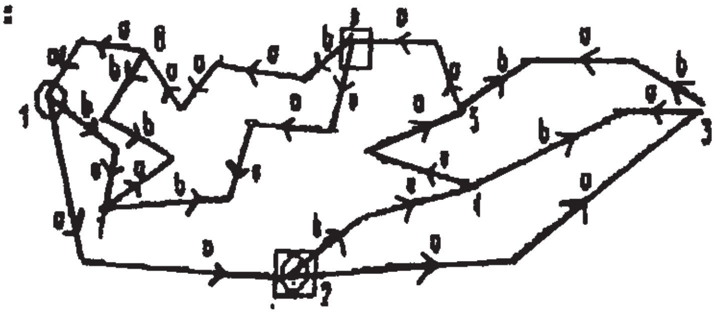
Proof. Suppose $\Gamma^*(H)$ has only one type of compatible branch points. Then all branch points are of one type say b-cources.

Thus all possibilities for paths joining neighbouring branch points are:

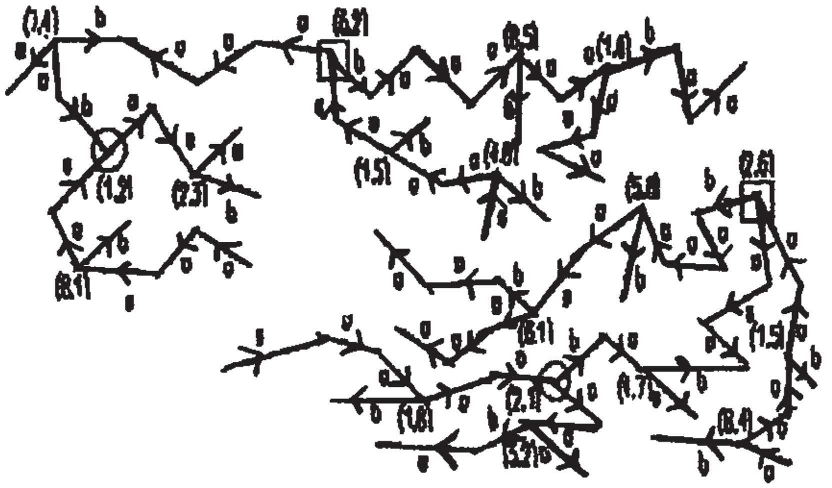
$A = e_1 \ldots e_n$ where e_1 is an edge labelled b and e_n labelled a^{-1},
$B = e_1 \ldots e_m$ where e_1 is an edge labelled b and e_m labelled a,
$C = e_1 \ldots e_k$ where e_1 is an edge labelled b and e_k labelled b^{-1}

$D = e_1 \dots e_t$ where e_1 is an edge labelled a and e_t labelled a
$E = e_1 \dots e_r$ where e_1 is an edge labelled a and e_r labelled a^{-1}
$G = e_1 \dots e_s$ where e_1 is an edge labelled a^{-1} and e_s labelled a

$\Gamma^*(H)$:



Components of $\Gamma^*(H) \tilde{\times} \Gamma^*(K)$:



Since there are exactly three reduced paths P_1, P_2 and P_3 begining at each compatible branch point.

Let P_1 be a path starting with an edge labelled b, P_2 be a path starting with an edge labelled a and P_3 be a path starting with an edge labelled a^{-1}.

Therefore all P_1 are compatible, all P_2 are compatible and all P_3 are compatible.

Since $\Gamma^*(H)$ is a consistent graph so no path of type C, E or G contains a subpath labelled $a^{-1}a$, aa^{-1}, bb^{-1} or $b^{-1}b$.

Thus each path C, E and G contains at least one subpath labelled ba. Then $\Gamma^*(H)$ does not contain paths of types C, E or G since $C \neq C^{-1}$, $E \neq E^{-1}$ and $G \neq G^{-1}$ otherwise we will have more than one type of compatible branch points.

Therefore $\Gamma^*(H)$ has only paths of types A, B and D.

Then $A = BV$ or $B = AY$

Suppose $A = BV$. Therefore $A_j = B_i V_j$ for each i, j.

Therefore P_1 ending with an edge labelled a^{-1}, a or b^{-1}. $V_j = b \ldots a^{-1}$ or $V_j = b^{-1} \ldots a^{-1}$ or $V_j = a \ldots a^{-1}$.

I) Suppose $V_j = b \ldots a^{-1}$

Therefore there exists $P_1 = u_0 B_1 u_1 A_j u_2 \ldots u_n$ and
$P_1 = u_1 A_j u_2 \ldots u_m$

Thus $A_j = B_i V_j$ and $B_i = V_j Z_i$, when $Z_i = b \ldots a$, $Z_i = b^{-1} \ldots a$ or $Z_i = a^{-1} \ldots a$.

Therefore

$$P_1 = u_0 V_j Z_i u_1 V_j Z_i V_j u_2 \ldots u_n$$
$$P_1 = u_1 V_j Z_i V_j u_2 Z_i V_j \ldots u_3 \ldots u_m \text{ and}$$
$$P_1 = u_2 Z_i V_j \ldots u_3 \ldots u_k$$

Now suppose $Z_i = b \ldots a$.

Since $L(Z_i V_j) = L(V_j Z_i)$ and $V_j Z_i$ and $Z_i V_j$ have different ending of edges so u_1 and u_2 are incompatible.

If $Z_i = b^{-1} \ldots a$ then u_2 is a branch point of type b-sink a contradiction.

Therefore $Z_i = a^{-1} \ldots a$

Thus all P_1 are compatible

Let $D_r^{-1} = Z_i V_j$ for some r, where D_r is a path of type D. Since D_r^{-1} and B_i^{-1} starting with an edge labelled a^{-1} so there exists at least two branch points u_1 and u_2 such that D_r^{-1} beginning at u_2 and B_i^{-1} begining at u_1.

Since $D_r^{-1} = Z_i V_j$ and $B_i^{-1} = Z_i^{-1} V_j^{-1}$ and
$Z_i = a^{-1} \ldots b \ldots a \neq a^{-1} \ldots b^{-1} \ldots a = Z_i^{-1}$ so
u_1 and u_2 are incompatible.

II) Suppose $V_j = a \ldots a^{-1}$
Since $A_j = B_j V_j$ so $B_i \neq V_j Z_i$.
Since $P_1 = u_0 B_i u_1 V_j u_2 \ldots u_n$ and $P_1 = u_1 B_i V_j u_2 \ldots u_m$ and B_i and V_j have different starting of edges so u_0 and u_1 are incompatible.

III) Suppose $V_j = b^{-1} \ldots a^{-1}$
Since $A_j = B_i V_j$ so $B_i \neq V_j Z_i$.
Since $P_1 = u_0 B_i u_1 B_i V_j u_2 \ldots u_n$, $P_1 = u_1 B_i V_j u_2 \ldots u_m$ so u_0 and u_1 are incompatible.
Similarly we have the same result if $B_i = A_j Y_i$.
Also all other cases will give the same result.

If $\Gamma^*(H)$ has more than or type of branch points then $\Gamma^*(H)$ has at least two types of compatible branch points. Thus $\Gamma^*(H)$ has at least two types of compatible branch points. ■

Lemma 2.4. *Let $\Gamma^*(H)$ be the core graph of finitely generated subgroup H of the free group F on generators a, b. If $\Gamma^*(H)$ has only two types of compatible branch points X_1 and X_2 then $x_1 = x_2$ where x_1 and x_2 are the number of the compatible branch points of types X_1 and X_2 respectively and $\Gamma^*(H)$ has $2n = x_1 + x_2$ branch points.*

Proof.

If $n = 2$

Then by Lemma 2.3, we have two types of compatible branch points. Thus $x_1 = x_2$.

Suppose $x_1 = x_2$ for $m = 2k < 2n$.

Now we prove $x_1 = x_2$ for $m + 2 = 2k + 2$.

Suppose u_{k+1} and u_{2k+2} are compatible branch points and of types b-sources.

I) If all branch points are of one type b - sources say, as in proof of Lemma 2.3 $\Gamma^*(H)$ has paths of type A and G only or B and E only or

C and D only or A, B and D only otherwise will have more than two types of compatible branch points.

i) If $\Gamma^*(H)$ has paths of type A and G only.
There is a path of type A or G joining two neighbouring branch points u_j and u_r in $\Gamma^*(H)$, $j, r \leq m$.
Let a path of type A joining the branch points u_j and u_{2k+1} then the path of type A joins u_{2k+2} and u_r.
Then u_{2k+1} and u_{2k+2} should join by paths of types E and G a contradiction. If the path G joins u_j and u_{2k+1} then G joins u_{2k+2} and u_r. Therefore the path P_3 beginning at u_{2k+1} and starting with a subpath of type G^{-1}, also P_3 beginning at u_{2k+2} and starting with a subpath of type G since $G \neq G^{-1}$ so u_{2k+1} and u_{2k+2} are incompatible.

Similarly

We will have a contradiction if $\Gamma^*(H)$ contains paths of types B and E or C and D only.

ii) If $\Gamma^*(H)$ has paths of types A, B and D only. Then there is a path of type A or B or D joins two neighbouring branch points u_j and u_r, $j, r \leq m$.
Let a path of type A_j joins u_j and u_{2k+1} then the path A_j joins u_{2k+2} and u_r and suppose $A_j = B_i V_i$. Therefore the path of type B_i should joins u_{2k+2} and u_{2k+1} otherwise we will have a contradiction. Therefore as in proof of Lemma 2.3 we have $A_j = B_i V_j$ and $B_i = V_i Z_i$, $Z_i = a^{-1} \dots a$, $V_j = b \dots a^{-1}$.

Thus $P_1 = u_j V_j Z_i V_j u_{2k+1} V_j Z_i u_{2k+2} V_j Z_i V_j u_r \dots u_n$,
$P_1 = u_{2k+1} V_j Z_i u_{2k+2} V_j Z_i V_j u_r \dots u_m$,
$P_1 = u_{2k+2} V_j Z_i V_j u_r Z_i V_j \dots u_k$
$P_3 = u_{2k+1} D_r^{-1} u_{2k+2} \dots u_g$
$P_3 = u_{2k+2} B_i^{-1} u_{2k+1} \dots u_h$
since $D_r^{-1} = a^{-1} \dots a^{-1}$, $B_i^{-1} = Z_i^{-1} V_j^{-1} = a^{-1} \dots a \; a \dots b^{-1}$ so $B_i^{-1} = D_r^{-1} C_i$ or $D_r^{-1} = B_i^{-1} E_r$.

(1) Suppose $D_r^{-1} = B_i^{-1} E_r$ and $E_r = a^{-1} \dots a^{-1}$
$P_3 = u_{2k+1} \; B_i^{-1} E_r u_{2k+2} B_i^{-1} u_{2k+1} E_r B_i^{-1} \dots u_g$

Since $L(B_i^{-1}E_r) = L(E_rB_i^{-1})$ and have different ending so u_{2k+1} and u_{2k+1} are incompatible.
Thus u_{2k+1} and u_{2k+2} are incompatible.

(2) Suppose $D_r^{-1} = B_i^{-1}E_r$ and $E_r = a \ldots a^{-1}$
Thus $P_3 = u_{2k+1}B_i^{-1}E_ru_{2k+2}B_i^{-1}u_{2k+1}E_rB_i^{-1} \ldots u_g$
Since A_j^{-1} starting with an edge labelled a and $E_rB_i^{-1}$ starting with an edge labelled a so $A_j^{-1} = E_rB_i^{-1}$ and then $E_r = V_j^{-1}$ a contradiction since E_r and V_j^{-1} have different ending of edges.
Thus u_{2k+1} and u_{2k+2} are incompatible.
Similarly we have some result if $V_j = a \ldots a^{-1}$ and also if $B_i^{-1} = D_r^{-1}F$.
Thus $x_1 = x_2$.

II) Suppose $\Gamma^*(H)$ has two different types of branch points b-sources and b-sinks say.
It $n = 2$ then $x_1 = x_2$.
Suppose $x_1 = x_2$ for $m = 2k < 2n + 2$.

Thus all possibilities for paths joining neighbouring branch points are:

$A = e_1 \ldots e_n$ where e_1 is an edge labelled b and e_n labelled a^{-1}
$B = e_1 \ldots e_m$ where e_1 is an edge labelled b and e_m labelled a
$D = e_1 \ldots e_t$ where e_1 is an edge labelled a and e_t labelled a
$G = e_1 \ldots e_i$ where e_1 is an edge labelled b^{-1} and e_i labelled a and
$K = e_1 \ldots e_j$ where e_1 is an edge labelled b^{-1} and e_j labelled a^{-1}.

All other possibilities of paths will give more than two types of compatible branch points.

Since all branch points of type b-sources are compatible so there are exactly three paths P_1, P_2 and P_3 defined as before. Since all branch points of type b-sinks are compatible so there are exactly three paths q_1, q_2 and q_3 beginning at b-sinks.

Let q_1 be a path starting with an edge labelled b^{-1}
q_2 be a path starting with an edge labelled a and
q_3 be a path starting with an edge labelled a^{-1}

Let $u_j \& u_r$ be two neighbouring branch points in $\Gamma^*(H)$, $j, r \leq m$.

i) If u_j and u_r are joined by paths of types A, B or D then as in proof of Lemma 2.3 we have u_{2k+1} and u_{2k+2} are incompatible.

ii) If u_j and u_r are joined by a path of type G then u_j and u_{2k+1} are joined by G and then u_{2k+2} and u_r are joined by G a contradiction. Thus u_{2k+1} and u_{2k+2} are incompatible.
Similarly we have a contradiction if u_j and u_{2k+1} are joined by a path k.
Thus $x_1 = x_2$
Similarly we have $x_1 = x_2$ if u_{2k+1} and u_{2k+2} are compatible of type b - sinks. ■

Lemma 2.5. *Let $\Gamma^*(H)$ be the core graph of the finitely generated subgroup H of the free group F on generators a, b. If $\Gamma^*(H)$ has more than two types of compatible branch points then the largest number of one type of compatible branch points is n, where $\#Br$ $(\Gamma^*(H) = 2n \geq 4$ or $x_i < n$ for all $i = 1, 2 \dots r$, where r is the number of typing of compatible branch points X_i.*

Proof. Suppose $\Gamma^*(H)$ has only three types of compatible branch points X_1, X_2 and X_3.

Let x_i be the number of compatible branch points of type X_i, $i = 1, 2, 3$. Let x_1 be the largest number of compatible branch points.

If $2n = 4$ then $x_1 = n$, otherwise we have $x_i < n$ for all i, $i = 1, 2, 3, 4$.

Now suppose $x_1 = k$ for $m = 2k < 2n$.

Similarly as in proof of Lemma 2.4 then $x_1 = k + 1$ for $m = 2k + 2$, therefore $x_1 = n$.

Thus by induction on the number of typing of compatible branch points we have $x_1 = n$ and $x_2 + x_3 + \dots + x_r = n$ or $x_i < n$ for all $i = 1, \dots r$.
■

Theorem 2.6. *Let $\Gamma^*(H)$ and $\Gamma^*(K)$ be the core graphs of the finitely generated subgroups H and K of the free group F on generators a, b.*

If $\Gamma^*(H) \cup \Gamma^*(K)$ has at least two types of compatible branch points then

$$\#Br\ (\text{core } \Gamma^*(H) \tilde{\times} \Gamma^*(K)) \leq \frac{\#Br(\Gamma^*(H)) \times \#Br(\Gamma^*(K))}{2}$$

Proof. Since the product of incompatible branch points is a vertex of degree at most 2 or a branch point is not in core of $\Gamma^*(H) \tilde{\times} \Gamma^*(K)$ so by Lemmas 2.3 and 2.4

$$\#Br\ (\text{core } \Gamma^*(H) \tilde{\times} \Gamma^*(K)) \leq 2x_1y_1 = \frac{\#Br(\Gamma^*(H)) \times \#Br(\Gamma^*(K))}{2}$$

or by Lemma 2.5

$$\#Br(\text{core } \Gamma^*(H) \tilde{\times} \Gamma^*(K)) < \frac{\#Br(\Gamma^*(H)) \times \#Br(\Gamma^*(K)}{2}$$

where x_i and y_i are the number of compatible branch points of types X_i and Y_i in $\Gamma^*(H)$ and $\Gamma^*(K)$ respectively, $\#X_i \leq \#Y_i$ and X_i and Y_i are compatible. ■

Theorem 2.7. *If H and K are finitely generated subgroups of a free group F on generators a, b of ranks m, n respectively and if all vertices in $\Gamma^*(H)$ and $\Gamma^*(K)$ are of degree 2 and 3 only, then the rank N of $H \cap K$ satisfies $N \leq mn - m - n + 2$.*

Proof. By Theorem 2.6 and Propositions 2.1 and 2.2 the result follows. ■

Corollary 2.8 [1] *If H and K are finitely generated subgroups of a free group F on generators a, b of rank m, n respectively. If H or K has a finite index in F then the rank N of $H \cap K$ satisfies $N \leq mn - n - m + 2$.*

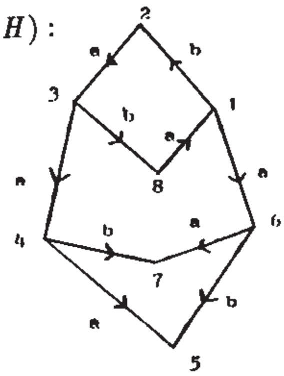
Proof. If H say, has a finite index in F, then $\Gamma^*(H) = \Gamma(H)$ and all vertices in $\Gamma^*(H)$ are of degree 4 only.

Therefore as in [12], we can reduce the degree of vertices in $\Gamma^*(H)$ into vertices of degree 3 only. Thus we have a new consistent graph π^* which has branch points of degree 3 only.

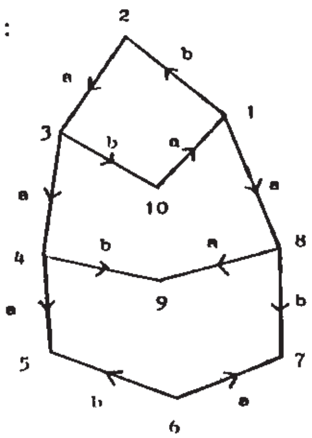
By Theorem 2.7 the result follows. ■

Example 1:

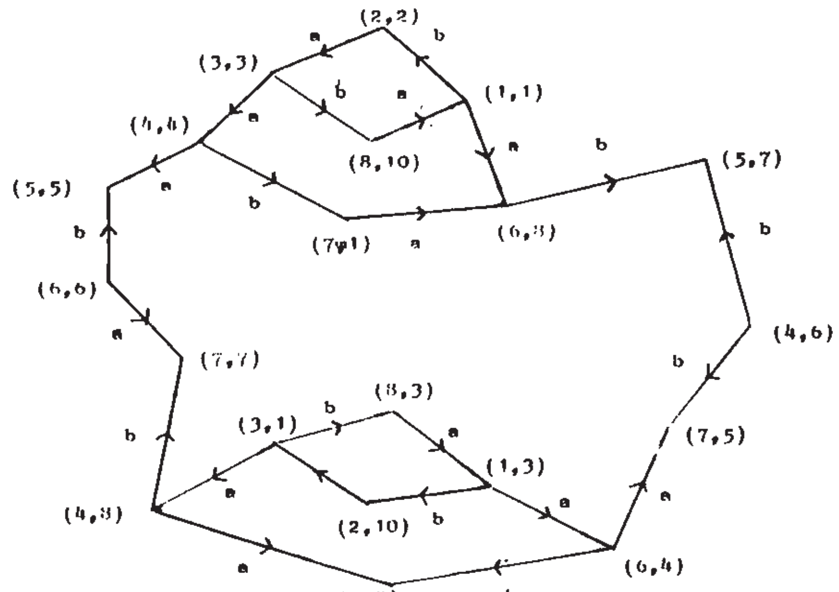
$\Gamma^*(H)$:



$\Gamma^*(K)$:

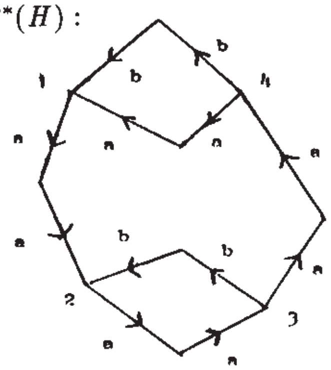


Core of $\Gamma^*(H) \tilde{\times} \Gamma^*(K)$:

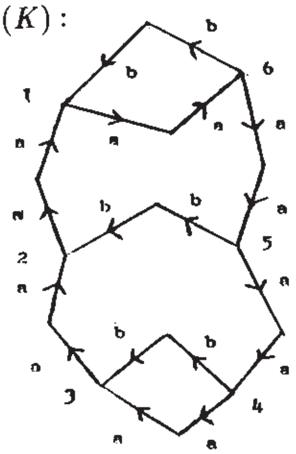


Example 2:

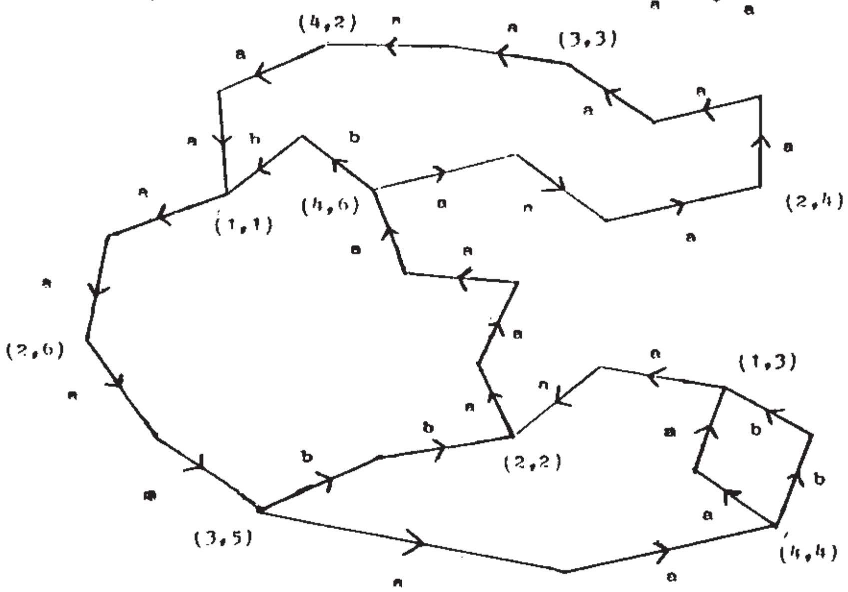
$\Gamma^*(H)$:



$\Gamma^*(K)$:

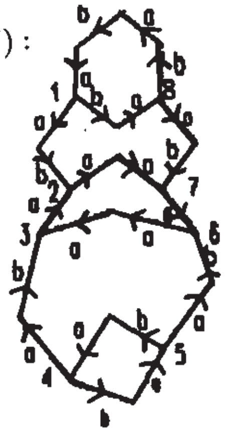


Core of $\Gamma^*(H) \mathbin{\tilde{\times}} \Gamma^*(K)$:

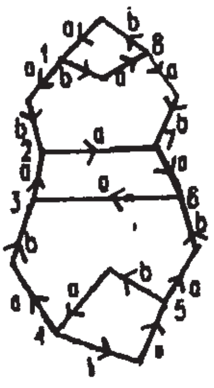


Example 3:

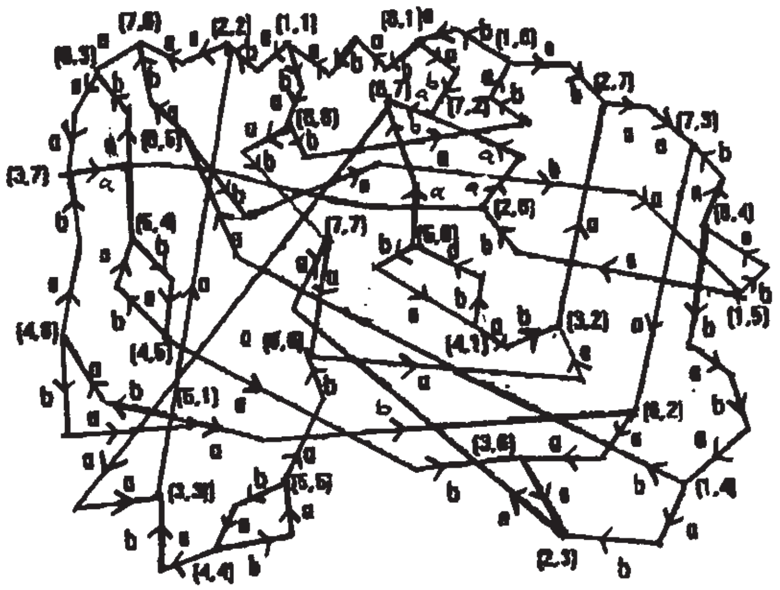
$\Gamma^*(H)$:



$\Gamma^*(K)$:



Core of $\Gamma^*(H) \tilde{\times} \Gamma^*(K)$:



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