On the intersection of free subgroups in free products of groups

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February 25, 2008

Dedicated to the memory of Prof. Charles Thomas

Abstract

Let $(G_i \mid i \in I)$ be a family of groups, let F be a free group, and let $G = F * \mathop{\ast}\limits_{i \in I} G_i$, the free product of F and all the G_i.

Let $\mathcal{F}$ denote the set of all finitely generated subgroups H of G which have the property that, for each $g \in G$ and each $i \in I$, $H \cap G_i^g = \{1\}$. By the Kurosh Subgroup Theorem, every element of $\mathcal{F}$ is a free group. For each free group H, the *reduced rank* of H, denoted $\bar{\mathrm{r}}(H)$, is defined as $\max\{\mathrm{rank}(H) - 1, 0\} \in \mathbb{N} \cup \{\infty\} \subseteq [0, \infty]$. To avoid the vacuous case, we make the additional assumption that $\mathcal{F}$ contains a non-cyclic group, and we define

$$\sigma \ := \ \sup\{\tfrac{\bar{\mathrm{r}}(H \cap K)}{\bar{\mathrm{r}}(H) \cdot \bar{\mathrm{r}}(K)} : H, K \in \mathcal{F} \text{ and } \bar{\mathrm{r}}(H) \cdot \bar{\mathrm{r}}(K) \neq 0\} \ \in \ [1, \infty].$$

We are interested in precise bounds for σ. In the special case where I is empty, Hanna Neumann proved that $\sigma \in [1, 2]$, and conjectured that $\sigma = 1$; fifty years later, this interval has not been reduced.

With the understanding that $\frac{\infty}{\infty - 2}$ is 1, we define

$$\theta \ := \ \max\{\tfrac{|L|}{|L|-2} : L \text{ is a subgroup of } G \text{ and } |L| \neq 2\} \ \in \ [1, 3].$$

Generalizing Hanna Neumann's theorem, we prove that $\sigma \in [\theta, 2\,\theta]$, and, moreover, $\sigma = 2\,\theta$ whenever G has 2-torsion. Since σ is finite, $\mathcal{F}$ is closed under finite intersections. Generalizing Hanna Neumann's conjecture, we conjecture that $\sigma = \theta$ whenever G does not have 2-torsion.

2000 *Mathematics Subject Classification.* Primary: 20E06; Secondary: 20F06, 20F32.

Key words. Intersection of free subgroups, free product of groups, Bass–Serre tree, group-action on a tree.

1 Outline

Let us first record the conventions and notation that we shall be using.

Throughout the article, let G be a group. Except where otherwise specified, our G-actions will be on the left.

1.1 Definitions. To indicate disjoint unions, we shall use the symbols $\vee$, $\bigvee$ in place of $\cup$, $\bigcup$.

We let $\mathbb{N}$ denote the set of finite cardinals, $\{0, 1, 2, \ldots\}$. For each set S, we define $|S| \in \mathbb{N} \vee \{\infty\} \subseteq [0, \infty]$ to be the cardinal of S if S is finite, and to be ∞ if S is infinite.

For any $n \in \{1, 2, 3, \ldots\} \vee \{\infty\}$, we let C_n denote a multiplicative cyclic group of order n. For any $n \in \mathbb{N}$, we let Sym_n denote the group of permutations of $\{1, 2, \ldots, n\}$, and we let Alt_n denote the subgroup of even permutations.

Let a, b be elements of G, and let S be a subset of G. We shall denote the inverse of a by $\overline{a}$. Also, $b^a := \overline{a}ba$, $\overline{S} := \{\overline{c} \mid c \in S\}$, and $S^a = \{c^a \mid c \in S\}$.

The *rank* of G is defined as

$$\mathrm{rank}(G) := \min\{|S| : S \text{ is a generating set of } G\} \in \mathbb{N} \vee \{\infty\} \subseteq [0, \infty].$$

If G is a free group, the *reduced rank* of G is defined as

$$\bar{\mathrm{r}}(G) := \max\{\mathrm{rank}(G) - 1, 0\} \in \mathbb{N} \vee \{\infty\} \subseteq [0, \infty];$$

thus, $\bar{\mathrm{r}}(G) = b_1^{(2)}(G)$, the first L^2-Betti number of G; see, for example, [17, Example 7.19].

Define $\alpha_3(G) := \inf\{|L| : L \text{ is a subgroup of } G \text{ and } |L| \geq 3\}$; it is understood that the infimum of the empty set is ∞. By the Sylow Theorems, $\alpha_3(G)$ is ∞ or 4 or an odd prime.

Let θ denote the bijective, strictly decreasing (or orientation-reversing) function $\theta\colon [3, \infty] \to [1, 3]$, $x \mapsto \frac{x}{x-2}$. Let $\theta\alpha_3(G) := \theta(\alpha_3(G)) = \frac{\alpha_3(G)}{\alpha_3(G)-2}$; thus,

$$\theta\alpha_3(G) \in \{\tfrac{3}{1}, \tfrac{4}{2}, \tfrac{5}{3}, \tfrac{7}{5}, \tfrac{11}{9}, \ldots, \tfrac{\infty}{\infty-2}\} = \{1, \ldots, \tfrac{11}{9}, \tfrac{7}{5}, \tfrac{5}{3}, 2, 3\} \subseteq [1, 3].$$

For example: $\theta\alpha_3(G) = 3$ if G has a subgroup of order 3; $\theta\alpha_3(G) = \frac{7}{5}$ if G has a subgroup of order 7 but none of order 3, 4, or 5; and $\theta\alpha_3(G) = 1$ if every finite subgroup of G has order at most 2. It is easy to see that if $|G| \geq 3$, then $\theta\alpha_3(G) = \max\{\frac{|L|}{|L|-2} : L \text{ is a subgroup of } G \text{ and } |L| \neq 2\}$.

Finally, define $\beta_2(G) := \begin{cases} 2 & \text{if } G \text{ has a subgroup of order two,} \\ 1 & \text{otherwise.} \end{cases}$

One could define $\beta_2(G)$ as $\sup\{|L| : L \text{ is a subgroup of } G \text{ and } |L| \leq 2\}$ to mirror the definition of $\alpha_3(G)$. □

Our main interest in this article is the following.

1.2 Notation. Let $(G_i \mid i \in I)$ be a family of groups, let F be a free group, and let $G = F * \mathop{\ast}\limits_{i\in I} G_i$, the free product of F and all the G_i.

For each $j \in I$, we write $G_{\neg j} := F * \mathop{\ast}\limits_{i\in I-\{j\}} G_i$, which gives $G = G_j * G_{\neg j}$.

Let $\mathcal{F}$ denote the set of all finitely generated subgroups H of G which have the property that, for each $g \in G$ and each $i \in I$, $H \cap G_i^g = \{1\}$. It follows from Kurosh's classic Subgroup Theorem [9, Theorem I.7.8] that every element of $\mathcal{F}$ is a free group; see, for example, [9, Theorem I.7.7].

To avoid the vacuous case, we assume that some element of $\mathcal{F}$ has rank at least two. We then define

$$(1.2.1)\quad \sigma(\mathcal{F}) \quad = \quad \sup\{\tfrac{\bar{\mathrm{r}}(H\cap K)}{\bar{\mathrm{r}}(H)\cdot \bar{\mathrm{r}}(K)} \mid H, K \in \mathcal{F},\ \bar{\mathrm{r}}(H)\cdot \bar{\mathrm{r}}(K) \neq 0\} \quad \in \quad [1, \infty];$$

notice that $\sigma(\mathcal{F}) \geq 1$ since $\mathcal{F}$ contains some free group H of rank two, and, then, for $K = H$, we have $\frac{\bar{\mathrm{r}}(H\cap K)}{\bar{\mathrm{r}}(H)\cdot \bar{\mathrm{r}}(K)} = \frac{1}{1\cdot 1}$. □

1.3 Observations. Suppose that Notation 1.2 holds.

We are interested in bounds for $\sigma(\mathcal{F})$.

1.3.1 Remarks. Consider the case where I is empty.

Here, G is a free group, $\mathcal{F}$ is the set of all finitely generated (free) subgroups of G, and $\beta_2(G) = \theta\alpha_3(G) = 1$.

Let us write $\sigma = \sigma(\mathcal{F})$.

In 1954, in [12], A. G. Howson proved that $\sigma \in [1, 5]$, and, hence, the intersection of any two finitely generated subgroups of a free group is again finitely generated, that is, $\mathcal{F}$ is closed under finite intersections. In 1956, in [19], Hanna Neumann proved that $\sigma \in [1, 3]$; then, in 1957, in [20], she proved that $\sigma \in [1, 2]$ and she conjectured that $\sigma = 1$. Fifty years later, the interval has not been reduced any further, although the conjecture has received much attention; see, for example, [3], [8], [10], [13], [11], [25], [27], [28].

We now return to the general case.

1.3.2 Remarks. Let us write $\sigma = \sigma(\mathcal{F})$, $\beta = \beta_2(G)$ and $\theta = \theta\alpha_3(G)$.

We conjecture that $\sigma = \beta\cdot\theta$.

In Theorem 6.5, we prove that $\sigma \in [\beta\cdot\theta,\ 2\cdot\theta]$.

In the case where G has 2-torsion, that is, $\beta = 2$, then $\sigma = \beta\cdot\theta = 2\cdot\theta$, and this case of the conjecture is true.

In the case where G is 2-torsion free, that is $\beta = 1$, then $\sigma \in [\theta,\ 2\cdot\theta]$; this generalizes Hanna Neumann's Theorem. Here, our conjecture reduces to $\sigma = \theta$, which generalizes Hanna Neumann's Conjecture.

Since $2\cdot\theta$ is finite, $\mathcal{F}$ is closed under finite intersections. This generalizes Howson's Theorem. An even more general statement can be deduced from the proof of [26, Theorem 2.13(1)]; see Remarks 6.6(iv), below. See also [14, Theorem 2] for the case where F is trivial.

T. Soma [24] studied the intersection of subgroups of surface groups and the intersection of subgroups of free products of two groups. As observed in [4], a peripheral consequence of Soma's results is that if F is trivial and I has two elements then $\sigma \le 18$. Later, in [14], [15], it was shown that $\sigma \le 6$, and that this cannot be reduced; see Remarks 2.6.2, below.

1.3.3 Remarks. The condition that some element of $\mathcal{F}$ has rank at least two implies the following.

For each $j \in I$, $|G_{\neg j}| \ge 2$.

Moreover, if, for some $j \in I$, $|G_{\neg j}| = 2$, then there exists a unique $j' \in I - \{j\}$ such that $|G_{j'}| = 2$ and, here, $|G_{\neg j'}| \ge 3$.

1.3.4 Remark. The condition that some element of $\mathcal{F}$ has rank at least two is equivalent to the condition that exactly one of the following holds.

(i). All the G_i are trivial and $\operatorname{rank}(F) \ge 2$.

(ii). There exists some $i_0 \in I$ such that $|G_{i_0}| \ge 2$ and $|G_{\neg i_0}| \ge 3$.

1.3.5 Remarks. By the Kurosh Subgroup Theorem, again, each finite subgroup of G lies in a conjugate of some G_i; see, for example, [9, Proposition I.7.11]. Hence, if I is nonempty, then

$$\alpha_3(G) = \min\{\alpha_3(G_i) \mid i \in I\} \quad \text{and} \quad \theta\alpha_3(G) = \max\{\theta\alpha_3(G_i) \mid i \in I\};$$

we can arrange for I to be nonempty by adding a trivial group to the family.

1.3.6 Remark. In the case where each G_i is a torsion group, $\mathcal{F}$ is the set of *all* finitely generated free subgroups of G. $\square$

The organization of the paper is as follows.

In Section 2, we use Euler characteristics and Bass–Serre theory, see [1], [22], [9], to show that $\sigma(\mathcal{F}) \ge \beta_2(G)\cdot\theta\alpha_3(G)$.

Let A and B be finite subsets of G with at least two elements each. By a *single-quotient* subset of $A \times B$, we mean any subset C with the property that $\big|\{a\overline{b} \mid (a,b) \in C\}\big| = 1$. Sections 3, 4, and 5 are devoted to proving Corollary 3.5(ii) which says that, if $\mathcal{C}$ is a set of pairwise-disjoint, single-quotient subsets of $A \times B$, then $\sum\limits_{C\in\mathcal{C}} (|C| - 2) \;\le\; \theta\alpha_3(G)\cdot(|A| - 2)\cdot(|B| - 2)$.

In Section 6, we use the latter result and Bass–Serre theory to show that $\sigma(\mathcal{F}) \le 2\cdot\theta\alpha_3(G)$. As in the extension of Hanna Neumann's theorem by W. D. Neumann [21], we find that all the results remain valid if, in the definition of $\sigma(\mathcal{F})$ in (1.2.1), we replace $\bar{\mathrm{r}}(H \cap K)$ with $\sum\limits_{s\in S} \bar{\mathrm{r}}(H^s \cap K)$ for any set S of (H,K)-double coset representatives in G; see Theorem 6.3, below.

2 Lower bounds

In this section, in Proposition 2.9, we prove that, if Notation 1.2 holds, then $\sigma(\mathcal{F}) \ge \beta_2(G)\cdot\theta\alpha_3(G)$.

The following is standard; see, for example, [9, Definition IV.1.10].

2.1 Review. Suppose that G is (isomorphic to) the fundamental group of a finite graph of finite groups, $\pi(G(-), Y, Y_0)$.

We write VY and EY for the vertex-set and edge-set of Y, respectively.

The *Euler characteristic* of G is defined as

$$\chi(G) = \Big(\sum_{v\in VY} \tfrac{1}{|G(v)|}\Big) - \Big(\sum_{e\in EY} \tfrac{1}{|G(e)|}\Big).$$

By Bass–Serre Theory, if L is any subgroup of G of finite index, then L is also the fundamental group of some finite graph of finite groups, and $\chi(L) = (G:L){\cdot}\chi(G)$.

There exists a normal subgroup H of G of finite index such that, for each $v \in VY$, the composite $G(v) \hookrightarrow G \twoheadrightarrow G/H$ is injective. Moreover, any such subgroup H is a finitely generated free group, and $\chi(H) = 1 - \operatorname{rank}(H)$. Thus, if $\chi(G) < 0$, then $0 > (G:H){\cdot}\chi(G) = \chi(H) = -\bar{\mathrm{r}}(H)$. □

For the purposes of this section, we introduce the following.

2.2 Notation. If G contains a free subgroup of rank 2, we let $\sigma(G)$ denote the value given by $\sigma(\mathfrak{F})$ in (1.2.1) when $\mathfrak{F}$ is taken to be the set of all finitely generated free subgroups of G. □

2.3 Proposition. *Suppose that G is the fundamental group of a finite graph of finite groups and that $\chi(G) < 0$. If H and K are free normal subgroups of G of finite index such that $HK = G$, then $\bar{\mathrm{r}}(H \cap K) = \frac{-1}{\chi(G)}{\cdot}\,\bar{\mathrm{r}}(H){\cdot}\,\bar{\mathrm{r}}(K) > 0$, and, hence, $\sigma(G) \geq \frac{-1}{\chi(G)}$.*

Proof. Notice that $(G:K) = (HK:K) = (H:H\cap K)$, since $H \cap K$ is the kernel of the induced map $H \twoheadrightarrow HK/K$. Hence,

$$\begin{aligned}\chi(H){\cdot}\chi(K) &= (G:H){\cdot}\chi(G){\cdot}(G:K){\cdot}\chi(G) = (G:H){\cdot}\chi(G){\cdot}(H:H\cap K){\cdot}\chi(G)\\ &= (G:H\cap K){\cdot}\chi(G){\cdot}\chi(G) = \chi(H\cap K){\cdot}\chi(G).\end{aligned}$$

Since $\chi(G) < 0$, we have

$$(-\,\bar{\mathrm{r}}(H)){\cdot}(-\,\bar{\mathrm{r}}(K)) = \chi(H){\cdot}\chi(K) = (-\,\bar{\mathrm{r}}(H\cap K)){\cdot}\chi(G) > 0,$$

and the result follows. The hypothesis that H is a normal subgroup can be omitted. □

We now consider four concrete examples which will be used in the proof of Proposition 2.9.

2.4 Example. Let $G = C_2 * C_2 * C_2$.

Then $\chi(G) = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} - 1 - 1 = \frac{-1}{2}$.

We have a presentation $G = \langle x, y, z \mid x^2 = y^2 = z^2 = 1\rangle$.

In Sym_2, consider $x' = y' = z' = (1,2)$. There is an induced homomorphism $G \to \mathrm{Sym}_2$ which sends w to w' for each $w \in \{x, y, z\}$. Let H be the kernel of this homomorphism. As in Review 2.1, H is a free normal subgroup of G of finite index. Notice that H contains xy and xz.

In Sym_4, consider $x'' = (1,2)$, $y'' = (3,4)$, $z'' = (1,2)(3,4)$. There is an induced homomorphism $G \to \mathrm{Sym}_4$ which sends w to w'' for each $w \in \{x,y,z\}$. Let K be the kernel of this homomorphism. As in Review 2.1, K is a free normal subgroup of G of finite index. Notice that K contains xyz.

Then HK contains xy, xz and xyz. It follows that $HK = G$. By Proposition 2.3, $\sigma(G) \geq \frac{-1}{\chi(G)} = 2$. This was also shown in [14, Theorem 3]. □

In the three remaining examples, we shall tacitly use analogous constructions of free normal subgroups of G of finite index, H and K.

2.5 Example. Let $G = C_2 * V$ where $V = C_2 \times C_2$.

Then $\chi(G) = \frac{1}{2} + \frac{1}{4} - 1 = \frac{-1}{4}$.

We have a presentation $G = \langle x, y, z \mid x^2 = y^2 = z^2 = (yz)^2 = 1\rangle$.

In Sym_4, consider $x' = (1,2)(3,4)$, $y' = (1,2)$, and $z' = (3,4)$. The resulting kernel H contains xyz, $(xy)^2$ and $(xz)^2$.

In Sym_4, consider $x'' = (1,3)$, $y'' = (1,2)$, and $z'' = (3,4)$. Here,

$$x''y'' = (1,2,3) \quad \text{and} \quad x''z'' = (1,3,4).$$

The resulting kernel K contains $(xy)^3$ and $(xz)^3$.

Then HK contains xyz, $(xy)^2$, $(xz)^2$, $(xy)^3$ and $(xz)^3$. It follows that $HK = G$. By Proposition 2.3, $\sigma(G) \geq \frac{-1}{\chi(G)} = 4$. □

2.6 Example. Let p be 4 or an odd prime, and let $G = C_2 * C_p$.

Then $\chi(G) = \frac{1}{2} + \frac{1}{p} - 1 = \frac{2-p}{2p}$.

We have a presentation $G = \langle x, y \mid x^2 = y^p = 1\rangle$.

Let $q = \begin{cases} 2 & \text{if } p = 4, \\ p & \text{if } p \text{ is an odd prime.} \end{cases}$

In Sym_{q+2}, consider $x' = \begin{cases} (1,3)(2,4) & \text{if } p = 4, \\ (p+1, p+2) & \text{if } p \text{ is an odd prime,} \end{cases}$

$$y' = (1, 2, \ldots, p-1, p).$$

Then $x'y' = y'x'$. The resulting kernel H contains $(xy)^{2p}$ and $x(xy)^q$.

In Sym_{2p}, consider $x'' = (1, p+1)(2,3)$,

$$y'' = (1, 2, \ldots, p)(p+1, p+2, \ldots, 2p).$$

Then $x''y'' = (1, 3, 4, \ldots, p, p+1, \ldots, 2p-1, 2p)$. The resulting kernel K contains $(xy)^{2p-1}$.

Then, HK contains $(xy)^{2p}$, $x(xy)^q$ and $(xy)^{2p-1}$. It follows that $HK = G$. By Proposition 2.3, $\sigma(G) \geq \frac{-1}{\chi(G)} = \frac{2p}{p-2}$. For $p = 3$, this was also shown in [15, Theorem 1].

2.6.1 Remarks. For $p \geq 4$, the foregoing K has rather large rank.

For $p = 4$, an alternative K can be constructed by taking, in Sym_4, $x'' = (1, 2)$, $y'' = (1, 2, 3, 4)$. Then, $x''y'' = (2, 3, 4)$ and K contains $(xy)^3$. Here, 3 is coprime to $2p$.

For $p \geq 5$, an alternative K can be constructed by taking, in Sym_{p+1}, $x''\colon t \mapsto -\frac{1}{t}$, $y''\colon t \mapsto t + 1$, where we identify $\{1, \ldots, p + 1\}$ with the projective line over the field with p elements, $\mathbb{F}_p \vee \{\infty\}$. Then, $x''y''\colon t \mapsto -\frac{1}{t+1}$ and K contains $(xy)^3$. Here, 3 is coprime to $2p$.

2.6.2 Remarks. For $p = 3$, there are interesting examples related to the action of the arithmetic group $\mathrm{PSL}_2(\mathbb{Z}) \simeq C_2 * C_3$ by Möbius transformations on the upper half-plane $\mathfrak{h}$, the set of complex numbers with positive imaginary part.

Let $n \in \mathbb{N}$ with $n \geq 2$. Let $\overline{\Gamma}(n)$ denote the kernel of the mod-n map $\mathrm{PSL}_2(\mathbb{Z}) \twoheadrightarrow \mathrm{PSL}_2(\mathbb{Z}_n)$. Then $\overline{\Gamma}(n)$ acts freely on $\mathfrak{h}$, and the quotient space $\overline{\Gamma}(n)\backslash\mathfrak{h}$ is a punctured Riemann surface with fundamental group $\overline{\Gamma}(n)$. By [23, (1.6.4)] (a reference kindly provided by Chris Cummins), the topological genus of $\overline{\Gamma}(n)\backslash\mathfrak{h}$ equals

$$1 + \frac{n-6}{12n}\left|\mathrm{PSL}_2(\mathbb{Z}_n)\right|.$$

If we supplement $\mathfrak{h}$ with the projective rational line, $\mathbb{Q} \vee \{\infty\}$, then we can think of the punctures as cusps or C_∞-points. Then $\mathrm{PSL}_2(\mathbb{Z}_n)$ acts faithfully on the set of cusps of $\overline{\Gamma}(n)\backslash\mathfrak{h}$.

The following facts are well known.

(1) $\mathrm{PSL}_2(\mathbb{Z}_2) = \langle x, y \mid x^2 = y^3 = (xy)^2 = 1\rangle = \mathrm{Sym}_3$, of order 6.

(2) $\bar{\mathrm{r}}(\overline{\Gamma}(2)) = 1$ and $\overline{\Gamma}(2)$ is free of rank two.

(3) $\overline{\Gamma}(2)\backslash\mathfrak{h}$ is a sphere with three cusps, and $\mathrm{PSL}_2(\mathbb{Z}_2) \simeq \mathrm{Sym}_3$ acts naturally on the set of cusps.

(4) $\mathrm{PSL}_2(\mathbb{Z}_3) = \langle x, y \mid x^2 = y^3 = (xy)^3 = 1\rangle = \mathrm{Alt}_4$, of order 12.

(5) $\bar{\mathrm{r}}(\overline{\Gamma}(3)) = 2$ and $\overline{\Gamma}(3)$ is free of rank three.

(6) $\overline{\Gamma}(3)\backslash\mathfrak{h}$ is a sphere with four cusps, like a tetrahedron, and $\mathrm{PSL}_2(\mathbb{Z}_3) \simeq \mathrm{Alt}_4$ acts naturally on the set of cusps.

(7) $\mathrm{PSL}_2(\mathbb{Z}_6) \simeq \mathrm{PSL}_2(\mathbb{Z}_2) \times \mathrm{PSL}_2(\mathbb{Z}_3) \simeq \mathrm{Sym}_3 \times \mathrm{Alt}_4$, of order 72.

(8) $\bar{\mathrm{r}}(\overline{\Gamma}(6)) = 12$ and $\overline{\Gamma}(6)$ is free of rank 13, and $\overline{\Gamma}(6) = \overline{\Gamma}(2) \cap \overline{\Gamma}(3)$.

(9) $\overline{\Gamma}(6)\backslash\mathfrak{h}$ is a torus with twelve cusps, and $\mathrm{PSL}_2(\mathbb{Z}_6) \simeq \mathrm{Sym}_3 \times \mathrm{Alt}_4$ acts faithfully on the set of cusps. □

2.7 Example. Let p be an odd prime, and let $G = C_p * C_p$.

Then $\chi(G) = \frac{1}{p} + \frac{1}{p} - 1 = \frac{2-p}{p}$.

We have a presentation $G = \langle x, y \mid x^p = y^p = 1\rangle$.

In Sym_p, consider $x' = y' = x'' = (1, 2, \ldots, p-1, p)$, and $y'' = (p, p-1, \ldots, 2, 1)$. The resulting kernels H and K contain $x\overline{y}$ and xy, respectively; recall that the overline indicates the inverse. Now HK contains $y\overline{x}$, xy and y^p. It follows that $HK = G$. By Proposition 2.3, $\sigma(G) \geq \frac{-1}{\chi(G)} = \frac{p}{p-2}$. □

2.8 Remark. Let us record triples $(\bar{\mathrm{r}}(H), \bar{\mathrm{r}}(K), \bar{\mathrm{r}}(H \cap K))$ obtained in the above examples.

(i). In $C_2 * C_2 * C_2$, $(\bar{\mathrm{r}}(H), \bar{\mathrm{r}}(K), \bar{\mathrm{r}}(H \cap K)) = (1, 2, 4)$.
(ii). In $C_2 * C_3$, $(\bar{\mathrm{r}}(H), \bar{\mathrm{r}}(K), \bar{\mathrm{r}}(H \cap K)) = (1, 2, 12)$.
(iii). In $C_2 * V$ and $C_2 * C_4$, $(\bar{\mathrm{r}}(H), \bar{\mathrm{r}}(K), \bar{\mathrm{r}}(H \cap K)) = (1, 6, 24)$.
(iv). In $C_2 * C_p$, $p \geq 5$, p prime,
$(\bar{\mathrm{r}}(H), \bar{\mathrm{r}}(K), \bar{\mathrm{r}}(H \cap K)) = (p-2, \frac{1}{4}(p^2-1)(p-2), \frac{1}{4}(2p)(p^2-1)(p-2))$.
(v). In $C_p * C_p$, p odd, $(\bar{\mathrm{r}}(H), \bar{\mathrm{r}}(K), \bar{\mathrm{r}}(H \cap K)) = (p-2, p-2, p(p-2))$. $\square$

We now have a candidate for a sharp lower bound.

2.9 Proposition. *If* Notation 1.2 *holds, then* $\sigma(\mathcal{F}) \geq \beta_2(G) \cdot \theta\alpha_3(G)$.

Proof. Let $p = \alpha_3(G)$.

Thus p is ∞, 4, or an odd prime, and $\theta\alpha_3(G) = \theta(p) = \frac{p}{p-2}$.

We consider two cases, with two subcases each.

Case 1. $\beta_2(G) = 2$, that is, G has an element of order two.

Here, there exists $j \in I$ such that G_j has a subgroup which we can identify with C_2. By Remarks 1.3.3, we may assume that $|G_{\neg j}| \geq 3$. Let a, b and c be three distinct elements of $G_{\neg j}$.

Subcase 1.1. $p = \infty$.

We have $C_2^a * C_2^b * C_2^c \leq G_j^a * G_j^b * G_j^c \leq G$, and, hence, $C_2 * C_2 * C_2$ embeds in G in such a way that the finitely generated free subgroups of $C_2 * C_2 * C_2$ are carried to $\mathcal{F}$.

By Example 2.4, $\sigma(\mathcal{F}) \geq 2 = 2 \cdot \theta(\infty) = \beta_2(G) \cdot \theta\alpha_3(G)$.

Subcase 1.2. p is 4 or an odd prime.

Here, there exists $i \in I$ such that G_i has a subgroup P of order p. Then $C_2^a * P^b \leq G_j^a * G_i^b \leq G$, and, hence, $C_2 * P$ embeds in G in such a way that the finitely generated free subgroups of $C_2 * P$ are carried to $\mathcal{F}$.

By Examples 2.5 and 2.6, $\sigma(\mathcal{F}) \geq \frac{2p}{p-2} = 2 \cdot \theta(p) = \beta_2(G) \cdot \theta\alpha_3(G)$.

Case 2. $\beta_2(G) = 1$, that is, G has no element of order two.

Subcase 2.1. $p = \infty$.

In Notation 1.2, we saw that $\sigma(\mathcal{F}) \geq 1 = 1 \cdot \theta(\infty) = \beta_2(G) \cdot \theta\alpha_3(G)$.

Subcase 2.2. p is 4 or an odd prime.

Notice that $p \neq 4$ since $\beta_2(G) \neq 2$.

Here, there exists $j \in I$ such that G_j has a subgroup which we can identify with C_p.

By Remarks 1.3.3, $|G_{\neg j}| \geq 2$. Let a and b be two distinct elements of $G_{\neg j}$. Then $C_p^a * C_p^b \leq G_j^a * G_j^b \leq G$, and, hence, $C_p * C_p$ embeds in G in such a way that the finitely generated free subgroups of $C_p * C_p$ are carried to $\mathcal{F}$.

By Example 2.7, $\sigma(\mathcal{F}) \geq \frac{p}{p-2} = 1 \cdot \theta(p) = \beta_2(G) \cdot \theta\alpha_3(G)$. $\square$

2.10 Exercise. Use the foregoing proof to show that $\beta_2(G)\cdot\theta\alpha_3(G)$ equals

$$\text{(2.10.1)} \qquad \max\{\tfrac{(L:H)}{\bar{\mathrm{r}}(H)} \mid H \in \mathcal{F},\ \bar{\mathrm{r}}(H) \geq 1,\ H \leq L \leq G,\ (L:H) < \infty\};$$

here, $\frac{-1}{\chi(L)} = \frac{(L:H)}{\bar{\mathrm{r}}(H)}$. □

3 Single-quotient subsets

In this section, and in the next two sections, G is an arbitrary group. Our main objective is to prove, in Corollary 3.5(ii), that, if A and B are finite subsets of G with at least two elements each, and $\mathcal{C}$ is a set of pairwise-disjoint, single-quotient subsets of $A \times B$, then $\sum\limits_{C\in\mathcal{C}} (|C| - 2) \ \leq \ \theta\alpha_3(G)\cdot(|A| - 2)\cdot(|B| - 2)$. We recall that $\theta\alpha_3(G)$ was described in Definitions 1.1, and we now recall what we mean by a 'single-quotient' subset of $A \times B$.

3.1 Definitions. Let A and B be finite subsets of G.

A subset C of $A \times B$ is said to be a *single-product* subset of $A \times B$ if $|\{ab \mid (a, b) \in C\}| = 1$. Similarly, C is said to be a *single-quotient* subset if $\big|\{a\bar{b} \mid (a, b) \in C\}\big| = 1$.

For $x \in G$, we let $\mathrm{rep}(x, A \times B) \ := \ \{(a, b) \in A \times B \mid ab = x\} \ \subseteq \ A \times B$.

For each positive integer i, we let

$$\begin{array}{rcccl} A{\cdot_i}B & := & \{x \in G : |\mathrm{rep}(x, A \times B)| \geq i\} & \subseteq & G, \\ A{\cdot_{[=i]}}B & := & \{x \in G : |\mathrm{rep}(x, A \times B)| = i\} & \subseteq & G. \end{array}$$

Thus, an element of $A{\cdot_i}B$, resp. $A{\cdot_{[=i]}}B$, is an element of G which has at least, resp. exactly, i distinct representations of the form ab with $(a, b) \in A \times B$.

We shall be interested in $A{\cdot_1}B = AB$, $A{\cdot_2}B$, and $A{\cdot_{[=1]}}B = AB - A{\cdot_2}B$. □

The following result will be used frequently.

3.2 Lemma. *For any finite subsets A, B of G, the following hold.*

(i). *If $|B| = 2$, then $|AB| + |A{\cdot_2}B| \ = \ 2\,|A| + 2\,|B| - 4$.*

(ii). *If $|B| \geq 2$, then $|AB| + |A{\cdot_2}B| \ \geq \ 2\,|A|$.*

Proof. Suppose that b_1 and b_2 are two distinct elements of B, and let $B' = \{b_1, b_2\}$.

Then $B \supseteq B'$, $AB \supseteq AB' = Ab_1 \cup Ab_2$ and $A{\cdot_2}B \supseteq A{\cdot_2}B' = Ab_1 \cap Ab_2$. Hence,

$$\begin{aligned} &|AB| + |A{\cdot_2}B| \geq |AB'| + |A{\cdot_2}B'| = |Ab_1 \cup Ab_2| + |Ab_1 \cap Ab_2| = |Ab_1| + |Ab_2| \\ &= 2\,|A| = 2\,|A| + 2\,|B'| - 4. \end{aligned}$$

This proves (ii), and the case $B = B'$ proves (i). □

We call the next result *the key inequality.* Recall from Definitions 1.1 that $\alpha_3(G)$ is ∞ or 4 or an odd prime, and that $\theta\alpha_3(G) = \frac{\alpha_3(G)}{\alpha_3(G)-2} \in [1,3]$.

3.3 Theorem (= Theorem 5.10). *For any finite subsets A, B of G, if $|A| \ge 2$ and $|B| \ge 2$, then* $|AB| + |A{\cdot_2}B| \;\ge\; \min\{2\,|A| + 2\,|B| - 4,\ 2{\cdot}\alpha_3(G)\}$.

Proof. We postpone the lengthy proof to the next two sections; see Theorem 5.10. □

3.4 Corollary. *For any finite subsets A, B of G, if $|A| \ge 2$ and $|B| \ge 2$, then* $|A|\,|B| - |AB| - |A{\cdot_2}B| \;\le\; \theta\alpha_3(G){\cdot}(|A|-2){\cdot}(|B|-2)$.

Proof. By symmetry, we may assume that $|A| \ge |B|$.

Let $p = \alpha_3(G)$. Recall, from Definitions 1.1, that

(3.4.1) the function $\theta\colon [3,\infty] \to [1,3]$, $x \mapsto \frac{x}{x-2}$, is strictly decreasing,

and $\theta\alpha_3(G) = \theta(p) = \frac{p}{p-2} \in [1,3]$.

We claim that at least one of the following holds.

(1). $|AB| + |A{\cdot_2}B| \;\ge\; 2\,|A| + 2\,|B| - 4$.

(2). $|A| \;\ge\; p$.

(3). $|A| \;<\; p$ and $\infty \;>\; |AB| + |A{\cdot_2}B| \;\ge\; 2p$.

To see this, notice that if (1) and (2) fail, then (3) holds, by Theorem 3.3 (= Theorem 5.10).

We now have three (overlapping) cases.

Case 1. $|AB| + |A{\cdot_2}B| \;\ge\; 2\,|A| + 2\,|B| - 4$.

$$\begin{aligned} \text{Here, } |A|\,|B| - |AB| - |A{\cdot_2}B| &\le |A|\,|B| - 2\,|A| - 2\,|B| + 4 \\ &= \theta(\infty){\cdot}(|A|-2){\cdot}(|B|-2) \\ &\le \theta(p){\cdot}(|A|-2){\cdot}(|B|-2) \text{ by (3.4.1).} \end{aligned}$$

Case 2. $|A| \;\ge\; p \;\ge\; 3$.

$$\begin{aligned} \text{Here, } |A|\,|B| - |AB| - |A{\cdot_2}B| &\le |A|\,|B| - 2\,|A| \text{ by Lemma 3.2(ii)} \\ &= \theta(|A|){\cdot}(|A|-2){\cdot}(|B|-2) \\ &\le \theta(p){\cdot}(|A|-2){\cdot}(|B|-2) \text{ by (3.4.1).} \end{aligned}$$

Case 3. $|AB| + |A{\cdot_2}B| \;\ge\; 2p$ and $\infty \;>\; p \;>\; |A| \;\ge\; |B| \;\ge\; 2$.

Here,

$$\begin{aligned} (p-2)(|A|\,|B| - |AB| - |A{\cdot_2}B|) &\le (p-2)(|A|\,|B| - 2p) \\ &\le (p-2)(|A|\,|B| - 2p) + 2(p-|A|)(p-|B|) \\ &= p\,|A|\,|B| - 2p^2 - 2\,|A|\,|B| + 4p + 2p^2 - 2p\,|B| - 2p\,|A| + 2\,|A|\,|B| \\ &= p\,|A|\,|B| + 4p - 2p\,|B| - 2p\,|A| \\ &= (p-2){\cdot}\,\theta(p){\cdot}(|A|-2){\cdot}(|B|-2). \end{aligned}$$

The desired result holds in all cases. □

Part (ii) of the following is the result that we shall apply in Section 6.

3.5 Corollary. *Let A and B be finite subsets of a group G such that $|A| \geq 2$ and $|B| \geq 2$.*

(i). *If $\mathcal{E}$ is a set of pairwise-disjoint, single-product subsets of $A \times B$, then* $\sum\limits_{E \in \mathcal{E}} (|E| - 2) \;\leq\; \theta\alpha_3(G)\cdot(|A| - 2)\cdot(|B| - 2)$.

(ii). *If $\mathcal{C}$ is a set of pairwise-disjoint, single-quotient subsets of $A \times B$, then* $\sum\limits_{C \in \mathcal{C}} (|C| - 2) \;\leq\; \theta\alpha_3(G)\cdot(|A| - 2)\cdot(|B| - 2)$.

Proof. (i). If there exists some $E_0 \in \mathcal{E}$ such that $|E_0| \leq 1$, then we may replace $\mathcal{E}$ with $\mathcal{E} - \{E_0\}$. This respects the hypotheses and increases $\sum\limits_{E \in \mathcal{E}} (|E| - 2)$ by $2 - |E_0|$. By repeating this procedure as often as necessary, we may assume that, for each $E \in \mathcal{E}$, $|E| \geq 2$, and, hence, there exists a unique $x_E \in A{\cdot_2}B$ such that $\operatorname{rep}(x_E, A \times B) \supseteq E$.

If there exist some $E' \neq E'' \in \mathcal{E}$ such that $x_{E'} = x_{E''}$, then the disjoint union $E' \vee E''$ is again a single-product subset of $A \times B$, and we may replace $\mathcal{E}$ with

$$\mathcal{E} - \{E', E''\} \cup \{E' \vee E''\}.$$

This respects the hypotheses and increases $\sum\limits_{E \in \mathcal{E}} (|E| - 2)$ by 2. By repeating this procedure as often as necessary, we may assume that the map $\mathcal{E} \to A{\cdot_2}B$, $E \mapsto x_E$, is injective. Thus,

$$\begin{aligned}
\sum_{E \in \mathcal{E}} (|E| - 2) &\leq \sum_{E \in \mathcal{E}} (|\operatorname{rep}(x_E, A \times B)| - 2) \\
&\leq \sum_{x \in A\cdot_2 B} (|\operatorname{rep}(x, A \times B)| - 2) \\
&= \Big|\bigvee_{x \in A\cdot_2 B} \operatorname{rep}(x, A \times B)\Big| - 2\,|A{\cdot_2}B| \\
&= |A \times B| - \big|A{\cdot_{[=1]}}B\big| - 2\,|A{\cdot_2}B| \\
&= |A|\,|B| - |AB| - |A{\cdot_2}B| \\
&\leq \theta\alpha_3(G)\cdot(|A| - 2)\cdot(|B| - 2) \text{ by Corollary 3.4.}
\end{aligned}$$

(ii). The bijection $A \times B \to A \times \overline{B}$, $(a, b) \mapsto (a, \bar{b})$, carries single-quotient subsets of $A \times B$ to single-product subsets of $A \times \overline{B}$. Hence, by (i), we see that

$$\sum_{C \in \mathcal{C}} (|C| - 2) \;\leq\; \theta\alpha_3(G)\cdot(|A| - 2)\cdot(\big|\overline{B}\big| - 2),$$

and the result follows. □

3.6 Examples. (i) Suppose that G has an element g whose order is at least 3. Let $A = B = \{1, g, g^2\}$, and let $\mathcal{C} = \{\{(1, 1), (g, g), (g^2, g^2)\}\}$. Here, Corollary 3.5(ii) asserts that $(3 - 2) \leq \theta\alpha_3(G)\cdot(3 - 2)(3 - 2)$.

(ii). Suppose that G has a finite, nontrivial subgroup L. Let $A = B = L$, and let $\mathcal{C} = \{\{(xy, y) \mid y \in L\} \mid x \in L\}$. Here, Corollary 3.5(ii) asserts that $|L|\cdot(|L| - 2) \;\leq\; \theta\alpha_3(G)\cdot(|L| - 2)\cdot(|L| - 2)$. □

4 Blocks and the Kemperman transform

4.1 Remarks. To put the key inequality, Theorem 5.10/3.3, into historical perspective, we record the following.

Kemperman's Theorem. *If A and B are finite, nonempty subsets of a group G, then there exists a subgroup L of G such that*

$$|A| + |B| - |AB| \;\le\; |L| \;\le\; |AB|\,. \tag{4.1.1}$$

Moreover, if $A{\cdot_2}B \neq AB$, then $|L|$ can be taken to be 1. □

This is a consequence of Theorems 5 and 3 of J. H. B. Kemperman's 1956 paper [16]; it is a curious coincidence that 1956 also saw the publication of Hanna Neumann's paper [19]. In the case where G has prime order, (4.1.1) is the famous Cauchy-Davenport Theorem, discovered by A. Cauchy [5] in 1813 and by H. Davenport [7] in 1935.

We will be using (a variant of) the marvellous 'Kemperman transform' which was introduced unnamed in the proofs of Theorems 5 and 3 of [16]; see Definition 4.8, below. Kemperman pointed out that this transform is closely related to the type of reasoning that H. B. Mann [18] had employed to prove the Landau–Schur–Khintchine $\alpha + \beta$-conjecture. □

In this section, we introduce concepts that will be used in the proof in the next section.

4.2 Definitions. For each $n \in \mathbb{N}$, we let $\mathcal{S}_n$ denote the set of pairs (A, B) such that A and B are finite subsets of G with $|A| \ge n$ and $|B| \ge n$. We shall be interested in $\mathcal{S}_2 \subseteq \mathcal{S}_0$.

For $(A, B) \in \mathcal{S}_0$, we define $\Omega(A, B) := |AB| + |A{\cdot_2}B| - 2\,|A| - 2\,|B| \in \mathbb{Z}$.

By a *block* (in G) we mean a subset of G of the form cPd where c and d are elements of G, and P is a subgroup of G whose order is either 4 or an odd prime. We remark that $|cPd| = |P| \ge \alpha_3(G)$. By replacing the triple (c, P, d) with the triple $(cd, P^d, 1)$, we can arrange that $d = 1$.

If C is a finite subset of G, we let blocks(C) denote the number of subsets of C which are blocks in G.

An element (A, B) of $\mathcal{S}_2$ is said to be *sound* if (at least) one of the following holds: $|AB| + |A{\cdot_2}B| \;\ge\; 2\,|A| + 2\,|B| - 4$ (equivalently, $\Omega(A, B) \ge -4$), or blocks$(A{\cdot_2}B) \ge 1$, or blocks$(AB) \ge 2$. □

In the next section, we shall show that every element of $\mathcal{S}_2$ is sound.

4.3 Examples. (i). Suppose that G has an element g whose order is at least 3, and take $A = B = \{1, g\}$.

Then $AB = \{1, g, g^2\}$ and $A{\cdot_2}B = \{g\}$.

Here, $\Omega(A, B) = |AB| + |A{\cdot_2}B| - 2\,|A| - 2\,|B| = 3 + 1 - 4 - 4 = -4$.

Also, blocks$(A{\cdot_2}B) = 0$ and blocks$(AB) \le 1$.

(ii). Suppose that G has a subgroup P of order 4 or an odd prime, and take $A = B = P$.

Then $AB = A{\cdot_2}B = P$.

Here, $\mathrm{blocks}(AB) = \mathrm{blocks}(A{\cdot_2}B) = 1$.

Also, $\Omega(A, B) = |AB| + |A{\cdot_2}B| - 2\,|A| - 2\,|B| = -2\,|P| < -4$.

(iii). We do not know of an example where $\mathrm{blocks}(AB) \geq 2$ but $\Omega(A, B) < -4$ and $\mathrm{blocks}(A{\cdot_2}B) = 0$.

Added February 20, 2008: David Grynkiewicz has proved that any such example would have to be nonabelian. □

4.4 Lemma. *Let* $(A, B) \in \mathcal{S}_2$. *If* (A, B) *is sound, then*

$$(4.4.1) \qquad |AB| + |A{\cdot_2}B| \quad \geq \quad \min\{2\,|A| + 2\,|B| - 4,\; 2{\cdot}\alpha_3(G)\}.$$

Proof. From Definitions 4.2, we have three possibilities.

Case 1. $|AB| + |A{\cdot_2}B| \;\geq\; 2\,|A| + 2\,|B| - 4$.

Here, (4.4.1) holds.

Case 2. $\mathrm{blocks}(A{\cdot_2}B) \geq 1$.

Here, $|A{\cdot_2}B| \geq \alpha_3(G)$. Hence, $|AB| + |A{\cdot_2}B| \;\geq\; 2{\cdot}\,|A{\cdot_2}B| \;\geq\; 2{\cdot}\alpha_3(G)$, and, hence, (4.4.1) holds.

Case 3. $\mathrm{blocks}(AB) \geq 2$.

We subdivide this case into two subcases.

Subcase 3.1. $|A{\cdot_2}B| \geq 2$.

We have $AB \supseteq c_1P_1 \cup c_2P_2$ where c_1P_1 and c_2P_2 are two different blocks in G.

We claim that $|c_1P_1 \cap c_2P_2| \leq 2$. Suppose that d is an element of $c_1P_1 \cap c_2P_2$. Then $dP_1 = c_1P_1$ and $dP_2 = c_2P_2$. Hence, $dP_1 \neq dP_2$, and, hence, $P_1 \neq P_2$, and, hence, $|P_1 \cap P_2| \leq 2$, by the conditions on the orders. Now, $c_1P_1 \cap c_2P_2 = dP_1 \cap dP_2 = d(P_1 \cap P_2)$, and the claim is proved.

Thus $|AB| \geq |c_1P_1| + |c_2P_2| - |c_1P_1 \cap c_2P_2| \geq \alpha_3(G) + \alpha_3(G) - 2$.

Since $|A{\cdot_2}B| \geq 2$, we see that $|AB| + |A{\cdot_2}B| \geq 2{\cdot}\alpha_3(G)$, and (4.4.1) holds.

Subcase 3.2. $|A{\cdot_2}B| \leq 1$.

If $|B| = 2$, then (4.4.1) holds by Lemma 3.2(i). Thus, we may assume that $|B| \geq 3$. Here, $(|A| - 2){\cdot}(|B| - 2 - |A{\cdot_2}B|) \geq (|A| - 2){\cdot}(3 - 2 - 1) = 0$, and it follows that

$$(4.4.2) \qquad |A|\,{\cdot}(|B| - |A{\cdot_2}B|) + 2\,|A{\cdot_2}B| \geq 2\,|A| + 2\,|B| - 4.$$

Since $A{\cdot_{[=1]}}B = \bigvee_{a\in A} (aB \cap A{\cdot_{[=1]}}B) = \bigvee_{a\in A} (aB - A{\cdot_2}B)$, we see that

$$\begin{aligned} \left|A{\cdot_{[=1]}}B\right| &= \Big|\bigvee_{a\in A} (aB - A{\cdot_2}B)\Big| = \sum_{a\in A} |aB - A{\cdot_2}B| \\ &\geq \sum_{a\in A} (|aB| - |A{\cdot_2}B|) = \sum_{a\in A} (|B| - |A{\cdot_2}B|) = |A|\,{\cdot}(|B| - |A{\cdot_2}B|). \end{aligned}$$

Now,

$$\begin{aligned}&|AB| + |A{\cdot_2}B| = \left|A{\cdot_{[=1]}}B\right| + 2\,|A{\cdot_2}B|\\ &\geq |A| \cdot (|B| - |A{\cdot_2}B|) + 2\,|A{\cdot_2}B| \ \text{ by the foregoing}\\ &\geq 2\,|A| + 2\,|B| - 4 \text{ by (4.4.2)},\end{aligned}$$

and, hence, (4.4.1) holds.

Thus, (4.4.1) holds in all cases. □

4.5 Definitions. We endow $\mathcal{S}_0$ with a partial order by assigning four indicators to each $(A, B) \in \mathcal{S}_0$.

The *first indicator* of (A, B) is $|AB| \in \mathbb{N} \subset \mathbb{Z}$.

The *second indicator* of (A, B) is $\Omega(A, B) = |AB| + |A{\cdot_2}B| - 2\,|A| - 2\,|B| \in \mathbb{Z}$.

The *third indicator* of (A, B) is $|B| \in \mathbb{N} \subset \mathbb{Z}$.

The *fourth indicator* of (A, B) is $|A| \in \mathbb{N} \subset \mathbb{Z}$.

We say that the *indicator sequence* of (A, B) is $(|AB|, \Omega(A, B), |B|, |A|)$.

Considered lexicographically, the indicator sequence gives a partial order, denoted $\succcurlyeq$, on $\mathcal{S}_0$. Thus, if (A', B') is an element of $\mathcal{S}_0$, we write $(A, B) \succ (A', B')$ if and only if

$$(|AB|, \Omega(A, B), |B|, |A|) \ > \ (|A'B'|, \Omega(A', B'), |B'|, |A'|)$$

in the lexicographic ordering of $\mathbb{Z}^4$. □

4.6 Lemma. *There are no infinite, strictly descending chains in $(\mathcal{S}_2, \succcurlyeq)$.*

Proof. Recall that the indicator sequence of (A, B) is $(|AB|, \Omega(A, B), |B|, |A|)$. In any infinite descending chain in $(\mathcal{S}_2, \succcurlyeq)$, the first indicator eventually becomes constant. Once the first indicator is constant, the other three indicators can take only finitely many values, and, hence, eventually become constant also.

This is also true in $\mathcal{S}_1$, but not in $\mathcal{S}_0$. □

4.7 Notation. Let us think of $\{A, B, \cdot_1, \cdot_2, \Omega\}$ as a set of five functions with domain $\mathcal{S}_0$, where A and B denote the projections onto the first and second coordinates, respectively, of elements of $\mathcal{S}_0$.

Let (A_1, B_1), (A_2, B_2) be elements of $\mathcal{S}_0$.

We define a map $\delta = \delta^{((A_2,B_2),(A_1,B_1))} \colon \{A, B, \cdot_1, \cdot_2, \Omega\} \to \mathbb{Z}$ with the following values:

$$\begin{aligned}&\delta(A) := |A_2| - |A_1|; \quad \delta(B) := |B_2| - |B_1|;\\ &\delta(\cdot_1) := |A_2B_2| - |A_1B_1|; \quad \delta(\cdot_2) := |(A_2){\cdot_2}(B_2)| - |(A_1){\cdot_2}(B_1)|;\\ &\delta(\Omega) := \Omega(A_2, B_2) - \Omega(A_1, B_1) = \delta(\cdot_1) + \delta(\cdot_2) - 2\delta(A) - 2\delta(B).\end{aligned}$$

In applications, A_1 will always be denoted A, with little risk of confusion. □

4.8 Definition. Let $(A,B) \in \mathcal{S}_0$ and let $x \in G$.

Set $(A^+, B^-) = (A \cup Ax, B \cap \overline{x}B)$ and $(A^-, B^+) = (A \cap A\overline{x}, B \cup xB)$. Clearly,

$$A^+B^- \subseteq AB \quad \text{and} \quad A^-B^+ \subseteq AB. \tag{4.8.1}$$

With Notation 4.7, let $\delta^+ = \delta^{((A^+,B^-),(A,B))}$ and $\delta^- = \delta^{((A^-,B^+),(A,B))}$.

We define the (revised) *Kemperman transform of* (A,B) *with respect to* x to be

$$(A',B') := \begin{cases} (A^-,B^+) & \text{if } \delta^-(\Omega) < 0, \\ (A^+,B^-) & \text{if } \delta^-(\Omega) \geq 0 \text{ and } \delta^+(\Omega) < 0, \\ (A^+,B^-) & \text{if } \delta^-(\Omega) \geq 0 \text{ and } \delta^+(\Omega) \geq 0 \text{ and } \delta^+(B) < 0, \\ (A^-,B^+) & \text{if } \delta^-(\Omega) \geq 0 \text{ and } \delta^+(\Omega) \geq 0 \text{ and } \delta^+(B) \geq 0. \end{cases} \tag{4.8.2}$$

Thus (A',B') is a well-defined element of $\mathcal{S}_0$.

We now make a sequence of remarks about this construction.

We call the bijection $G \times G \to G \times G$, $(a,b) \mapsto (\overline{b}, \overline{a})$, the *dual* map. Any statement about $G \times G$ can be "dualized" in a natural way.

4.8.3 Remark. $\delta^+(A) + \delta^-(A) = \delta^+(B) + \delta^-(B) = 0.$

Proof. Notice that $|A - A\overline{x}| = |(Ax - A)\overline{x})| = |Ax - A|$. Now,

$$\begin{aligned} \delta^+(A) + \delta^-(A) \ &= \ (\left|A^+\right| - |A|) + (\left|A^-\right| - |A|) \\ = \ (|A \cup Ax| - |A|) + (|A \cap A\overline{x}| - |A|) \ &= \ |Ax - A| - |A - A\overline{x}| = 0. \end{aligned}$$

Dualizing, we see that $\delta^+(B) + \delta^-(B) = 0$. □

4.8.4 Remark. $\delta^+(\cdot_1) \ = \ -|AB - A^+B^-| \ \leq \ 0$ and $\delta^-(\cdot_1) \ = \ -|AB - A^-B^+| \ \leq \ 0.$

Proof. This is clear from (4.8.1). □

4.8.5 Remark. $\max\{0, \delta^-(\cdot_2)\} \quad \leq \quad \left|(A^-)\cdot_2(B^+) \ \cap \ A\cdot_{[=1]}B\right|.$

Proof.
$$\begin{aligned} \delta^-(\cdot_2) \ &= \ \left|(A^-)\cdot_2(B^+)\right| - |A\cdot_2 B| \\ &= \ \left|(A^-)\cdot_2(B^+) \ \cap \ AB\right| - |A\cdot_2 B| \ \text{ since } A^-B^+ \subseteq AB \\ &= \ \left|(A^-)\cdot_2(B^+) \ \cap \ A\cdot_{[=1]}B\right| + \left|(A^-)\cdot_2(B^+) \ \cap \ A\cdot_2 B\right| - |A\cdot_2 B| \\ &\leq \ \left|(A^-)\cdot_2(B^+) \ \cap \ A\cdot_{[=1]}B\right|. \end{aligned}$$
□

4.8.6 Remark. $A^+B^- \ \cap\ (A^-)\cdot_2(B^+) \ \cap\ A\cdot_{[=1]}B \quad = \quad \emptyset.$

Proof. Suppose that

$$c \in A^+B^- \ \cap\ (A^-)\cdot_2(B^+) \ \cap\ A\cdot_{[=1]}B, \tag{4.8.7}$$

and let (a, b) denote the unique element of $\operatorname{rep}(c, A \times B)$.

By (4.8.7), the equation $c = a'b'$ has at least two solutions (a', b') with (a', b') in $(A^-) \times (B^+) = (A \cap A\overline{x}) \times (B \cup xB)$.

Type 1. $b' \in B$.
Here, $(a', b') \in \operatorname{rep}(c, A \times B) = \{(a, b)\}$. Hence, $a = a' \in A \cap A\overline{x}$. Observe that if $b \in xB$ then $(ax, \overline{x}b) \in \operatorname{rep}(ab, A \times B) = \{(a, b)\}$, which is a contradiction; hence, here, $b \in B - xB$.

Type 2. $b' \in xB - B$.
Here, $(a'x, \overline{x}b') \in \operatorname{rep}(c, A \times B) = \{(a, b)\}$. Hence, $a = a'x \in Ax$. Moreover, $b = \overline{x}b' \in B - \overline{x}B = B - B^-$. Here, $(a', b') = (a\overline{x}, xb)$.

In summary, the equation $c = a'b'$ has *exactly* two solutions $(a', b') \in A^- \times B^+$, one of each type, namely, (a, b) and $(a\overline{x}, xb)$.

It follows that $(a, b) \in (Ax \cap A \cap A\overline{x}) \times (B - (\overline{x}B \cup xB))$.

By (4.8.7), there exists some $(a'', b'') \in A^+ \times B^- = (A \cup Ax) \times (B \cap \overline{x}B)$ such that $a''b'' = c$.

Case 1. $a'' \in A$.
Here, $(a'', b'') \in \operatorname{rep}(c, A \times B) = \{(a, b)\}$. Hence, $b = b'' \in B^- \subseteq \overline{x}B$. This contradicts the fact that $b \in B - \overline{x}B$.

Case 2. $a'' \in Ax - A$.
Here, $(a''\overline{x}, xb'') \in \operatorname{rep}(c, A \times B) = \{(a, b)\}$. Hence, $a = a''\overline{x} \in A - A\overline{x}$. This contradicts the fact that $a \in A\overline{x}$.

This completes the proof of Remark 4.8.6. □

On dualizing Remark 4.8.6, we get the following.

4.8.8 Remark. $A^-B^+ \ \cap\ (A^+)\cdot_2(B^-) \ \cap\ A\cdot_{[=1]}B \quad = \quad \emptyset.$ □

4.8.9 Remark. $\delta^+(\Omega) + \delta^-(\Omega) \leq 0.$

Proof. Here,

$$\begin{aligned} \delta^-(\cdot_2) &\leq \left|(A^-)\cdot_2(B^+) \ \cap\ A\cdot_{[=1]}B\right| \text{ by Remark 4.8.5} \\ &\leq \left|AB - A^+B^-\right| \text{ by Remark 4.8.6} \\ &= -\delta^+(\cdot_1) \text{ by Remark 4.8.4.} \end{aligned} \tag{4.8.10}$$

By dualizing, we see that

$$\delta^+(\cdot_2) \;\le\; -\delta^-(\cdot_1). \tag{4.8.11}$$

By combining Remark 4.8.3 with (4.8.10) and (4.8.11), we obtain

$$\begin{aligned}\delta^+(\cdot_1) + \delta^+(\cdot_2) &- 2\delta^+(A) - 2\delta^+(B)\\ &+ \delta^-(\cdot_1) + \delta^-(\cdot_2) - 2\delta^-(A) - 2\delta^-(B)) \quad \le \quad 0,\end{aligned}$$

and Remark 4.8.9 is proved. □

4.8.12 Remark. The following holds:

$$(A', B') = \begin{cases} (A^-, B^+) & \text{if } \delta^-(\Omega) < 0,\\ (A^+, B^-) & \text{if } \delta^-(\Omega) \ge 0 \text{ and } \delta^+(\Omega) < 0,\\ (A^+, B^-) & \text{if } \delta^-(\Omega) = \delta^+(\Omega) = 0 \text{ and } \delta^+(B) < 0,\\ (A^-, B^+) & \text{if } \delta^-(\Omega) = \delta^+(\Omega) = 0 \text{ and } \delta^+(B) = 0.\end{cases}$$

Of course, if $(A^+, B^-) = (A^-, B^+)$, then $(A^+, B^-) = (A^-, B^+) = (A, B)$.

Proof. The description of (A', B') follows from (4.8.2), and Remark 4.8.9, and the fact that $\delta^+(B) \le 0$; recall that $\delta^+(B) = |B^-| - |B|$. □

This completes the desired description of the Kemperman transform. □

5 Proof of the key inequality

This section is structured as the proof of the key inequality. Recall Definitions 4.2. We fix, throughout the proof, an element (A, B) of $\mathcal{S}_2$ and we show that (A, B) is sound by progressively finding various assumptions that we are free to make.

5.1 Assumptions. Let (A, B) be an element of $\mathcal{S}_2$. We want to show that (A, B) is sound.

By Lemma 4.6 and transfinite induction, we have the following (transfinite) induction hypothesis: we assume, without loss of generality, that in $(\mathcal{S}_2, \succcurlyeq)$, every element which is strictly smaller than (A, B) is sound. □

5.2 Lemma. *With* Assumptions 5.1*, if* $|A| < |B|$*, then* (A, B) *is sound.*

Proof. Recall that the indicator sequence of (A, B) is $(|AB|, \Omega(A, B), |B|, |A|)$. In passing from (A, B) to its dual, $(\overline{B}, \overline{A})$, the first two indicators stay the same, while the third indicator decreases by $|B| - |A|$. By the induction hypothesis, Assumptions 5.1, $(\overline{B}, \overline{A})$ is sound. Dualizing, we see that (A, B) is sound. □

Also, by Lemma 3.2(i), (A, B) is sound if $|B| = 2$.

5.3 Assumptions. We assume, without loss of generality, that $|A| \geq |B| \geq 3$. □

5.4 Lemma. *With* Assumptions 5.1 *and* 5.3*, the following hold.*

(i). *If, for some $a \in A$, $\left|aB \;\cap\; A{\cdot_{[=1]}}B\right| \geq 2$, then (A, B) is sound.*

(ii). *If, for some $b \in B$, $\left|Ab \;\cap\; A{\cdot_{[=1]}}B\right| \geq 2$, then (A, B) is sound.*

Proof. For (i), set $(A', B') = (A - \{a\}, B)$; for (ii), set $(A', B') = (A, B - \{b\})$.

In both cases, $(A', B') \in \mathcal{S}_2$, by Assumptions 5.3.

It is easy to see that, for (i), $A'B' = AB - (aB \cap A{\cdot_{[=1]}}B)$, while, for (ii), $A'B' = AB - (Ab \cap A{\cdot_{[=1]}}B)$.

In both cases, $A'{\cdot_2}B' \subseteq A{\cdot_2}B$.

Thus, in both cases, $|A'| + |B'| = |A| + |B| - 1$, $|A'B'| \leq |AB| - 2$, and $|A'{\cdot_2}B'| \leq |A{\cdot_2}B|$. Now the two cases are handled together.

Recall that the indicator sequence of (A, B) is $(|AB|, \Omega(A, B), |B|, |A|)$. In passing from (A, B) to (A', B'), the first indicator decreases, by at least 2. By the induction hypothesis, Assumptions 5.1, (A', B') is sound. By Definitions 4.2, there are three possibilities.

Case 1. $|A'B'| + |A'{\cdot_2}B'| \;\geq\; 2|A'| + 2|B'| - 4$.

Here,
$|AB| + |A{\cdot_2}B| \;\geq\; 2 + |A'B'| + |A'{\cdot_2}B'| \;\geq\; 2 + 2|A'| + 2|B'| - 4 \;=\; 2|A| + 2|B| - 4$.
Thus, (A, B) is sound.

Case 2. $\text{blocks}(A'{\cdot_2}B') \geq 1$.

Since $A{\cdot_2}B \supseteq A'{\cdot_2}B'$, we see that $\text{blocks}(A{\cdot_2}B) \geq \text{blocks}(A'{\cdot_2}B') \geq 1$, and (A, B) is sound.

Case 3. $\text{blocks}(A'B') \geq 2$.

Since $AB \supseteq A'B'$, we see that $\text{blocks}(AB) \geq \text{blocks}(A'B') \geq 2$, and (A, B) is sound.

Hence, (i) and (ii) hold. □

5.5 Assumptions. We assume, without loss of generality, that the following hold.

(i). For each $a \in A$, $\left|aB \;\cap\; A{\cdot_{[=1]}}B\right| \;\leq\; 1$.

(ii). For each $b \in B$, $\left|Ab \;\cap\; A{\cdot_{[=1]}}B\right| \;\leq\; 1$. □

The proofs of Lemmas 5.7 and 5.8, which are modelled on the proofs of Theorem 5 and Theorem 3 of [16], respectively, have a large common part which we now describe.

5.6 Hypotheses. With Assumptions 5.1, 5.3, and 5.5, let x be an element of G such that $A \neq Ax$ and let (A', B') be the Kemperman transform of (A, B) with respect to x, with notation as in Definition 4.8.

Since $Ax \neq A$, we see that $x \neq 1$, and that $A^- \subset A \subset A^+$.

5.6.1 Consequence. *If* $1, x \in B$, *and*

$$2\left|A^{+}\right| + \left|(A^{-})\cdot_2(B^{+}) \ \cap \ A\cdot_{[=1]}B\right| \quad \geq \quad 2\left|A\right| + 2\left|B\right| - 2,$$

then (A, B) *is sound.*

Proof. Observe that $AB \cap A^{+}B^{-} \ \supseteq \ (A1 \cup Ax) \cap (A^{+}1) \ = \ A^{+}$. Hence, by Remark 4.8.6,

$$|AB| \quad \geq \quad \left|A^{+}\right| + \left|(A^{-})\cdot_2(B^{+}) \ \cap \ A\cdot_{[=1]}B\right|.$$

Since $A\cdot_2 B \ \supseteq \ A^{+} - (A\cdot_{[=1]}B)$, it follows from Assumptions 5.5(ii) that

$$|A\cdot_2 B| \geq \left|A^{+}\right| - 2.$$

Hence,

$$|AB| + |A\cdot_2 B| \ \geq \ 2\left|A^{+}\right| + \left|(A^{-})\cdot_2(B^{+}) \ \cap \ A\cdot_{[=1]}B\right| - 2 \ \geq \ 2\left|A\right| + 2\left|B\right| - 4,$$

and (A, B) is sound. □

5.6.2 Consequence. *If*

$$2\left|A^{+}\right| + \left|(A^{-})\cdot_2(B^{+}) \ \cap \ A\cdot_{[=1]}B\right| \quad \leq \quad 2\left|A\right| + 2\left|B\right| - 3,$$

then $(A', B') \in \mathcal{S}_2$.

Proof. By Remark 4.8.5, the hypothesis implies that

$$2\left|A^{+}\right| + 0 \quad \leq \quad 2\left|A\right| + 2\left|B\right| - 3, \text{ and,} \tag{5.6.3}$$

$$2\left|A^{+}\right| + \delta^{-}(\cdot_2) \quad \leq \quad 2\left|A\right| + 2\left|B\right| - 3. \tag{5.6.4}$$

Case 1. $(A', B') = (A^{-}, B^{+})$.

Using (5.6.3) and Assumptions 5.3, we see that

$$2\left|A^{-}\right| \ = \ 2(2\left|A\right| - \left|A^{+}\right|) \ \geq \ 2\left|A\right| - 2\left|B\right| + 3 \ \geq \ 0 + 3.$$

Thus, $|A^{-}| \geq \frac{3}{2}$, and, hence, $(A', B') \in \mathcal{S}_2$.

Case 2. $(A', B') = (A^{+}, B^{-})$.

It follows from Remark 4.8.12 that $\delta^{-}(\Omega) \geq 0$. Hence

$$\begin{aligned}
0 \quad &\leq \quad \delta^{-}(\Omega) \\
&= \quad \delta^{-}(\cdot_1) + \delta^{-}(\cdot_2) - 2\delta^{-}(A) - 2\delta^{-}(B) \\
&\leq \quad 0 + \delta^{-}(\cdot_2) - 2\delta^{-}(A) - 2\delta^{-}(B) \quad \text{by Remark 4.8.4} \\
&= \quad \delta^{-}(\cdot_2) + 2\delta^{+}(A) + 2\delta^{+}(B) \quad \text{by Remark 4.8.3} \\
&= \quad \delta^{-}(\cdot_2) + 2\left|A^{+}\right| - 2\left|A\right| + 2\left|B^{-}\right| - 2\left|B\right| \\
&\leq \quad -3 + 2\left|B^{-}\right| \quad \text{by (5.6.4).}
\end{aligned}$$

Here, $|B^{-}| \geq \frac{3}{2}$, and, hence, $(A', B') \in \mathcal{S}_2$.

In all cases then, $(A', B') \in \mathcal{S}_2$ and Consequence 5.6.2 is proved. □

5.6.5 Consequences. *The following hold:* $A'B' \subseteq AB$; $\Omega(A',B') \le \Omega(A,B)$; $(A,B) \succ (A',B')$; *and, if* $(A',B') \in \mathcal{S}_2$, *then* (A',B') *is sound.*

Proof. The first assertion follows from (4.8.1).

With Notation 4.7, let $\delta' = \delta^{((A',B'),(A,B))}$. It follows from Remark 4.8.12 that $\delta'(\Omega) \le 0$, and, hence, $\Omega(A',B') \le \Omega(A,B)$.

Recall that the indicator sequence of (A,B) is $(|AB|, \Omega(A,B), |B|, |A|)$. We now discuss how the four indicators change in passing from (A,B) to (A',B'). We have just seen that the first two indicators do not increase.

If the second indicator does not change, then Remark 4.8.12 shows that $\delta'(B) \le 0$ and, hence, the third indicator does not increase.

If the second and third indicators do not change, then Remark 4.8.12 shows that $A' = A^- \subset A$, and, hence, the fourth indicator decreases by at least 1.

Hence, $(A,B) \succ (A',B')$.

If $(A',B') \in \mathcal{S}_2$ then, by the induction hypothesis, Assumptions 5.1, (A',B') is sound, and we have proved Consequences 5.6.5. $\square$

This completes the list of consequences. $\square$

A substantial part of the proof of the following result is similar to the proof of Theorem 5 in [16].

5.7 Lemma. *With* Assumptions 5.1, 5.3, *and* 5.5, *if* $\mathrm{blocks}(A \cdot_2 B) = \mathrm{blocks}(AB)$, *then* (A,B) *is sound.*

Proof. Consider the possibility that, for all b_1, b_2 in B, we have $Ab_1 = Ab_2$. Let

$$L := \langle b_1 \bar{b}_2 \mid b_1, b_2 \in B \rangle \le G.$$

Here, $AL = A$. Consider any $(a,b) \in A \times B$. Then, $AB \supseteq Ab = ALb \supseteq aLb$, and L is finite. Also, $L \supseteq \langle B\bar{b} \rangle \supseteq B\bar{b}$, and, $|L| \ge \left|B\bar{b}\right| = |B| \ge 3$, by Assumptions 5.3. By the Sylow theorems, L contains a subgroup which has order 4 or an odd prime. Thus, $\mathrm{blocks}(AB) \ge 1$. Hence, $\mathrm{blocks}(A \cdot_2 B) = \mathrm{blocks}(AB) \ge 1$, and (A,B) is sound.

It remains to consider the case where $\mathrm{blocks}(AB) = 0$ and, here, by the foregoing, there exist b_1 and b_2 in B such that $Ab_1 \ne Ab_2$.

Without loss of generality, we may replace B with $B\bar{b}_1$. On setting $x = b_2\bar{b}_1$, we have $\{1, x\} \subseteq B$ and $A \ne Ax$, and, hence, $1 \ne x$. Let (A',B') be the Kemperman transform of (A,B) with respect to x, as in Definition 4.8. Now Hypotheses 5.6 apply.

By Consequences 5.6.1 and 5.6.2, we may assume that $(A',B') \in \mathcal{S}_2$.

By Consequences 5.6.5, $A'B' \subseteq AB$, $\Omega(A',B') \le \Omega(A,B)$ and (A',B') is sound. Since $A' \cdot_2 B' \subseteq A'B' \subseteq AB$, we see that

$$\mathrm{blocks}(A' \cdot_2 B') \;\le\; \mathrm{blocks}(A'B') \;\le\; \mathrm{blocks}(AB) \;=\; 0.$$

By soundness, $\Omega(A',B') \ge -4$. Hence, $\Omega(A,B) \ge \Omega(A',B') \ge -4$, and, hence, (A,B) is sound. $\square$

A substantial part of the proof of the following result is similar to the proof of Theorem 3 in [16].

5.8 Lemma. *With* Assumptions 5.1, 5.3, *and* 5.5*, if* $\mathrm{blocks}(A{\cdot_2}B) \neq \mathrm{blocks}(AB)$*, then* (A,B) *is sound.*

Proof. Here, there exists some block C which is contained in AB but is not contained in $A{\cdot_2}B$. Hence, $C \cap A{\cdot_{[=1]}}B$ is nonempty. Let (a,b) be an element of $A \times B$ such that $ab \in C \cap A{\cdot_{[=1]}}B$.

By replacing (A,B,a,b,C) with $(\overline{a}A, B\overline{b}, 1, 1, \overline{a}C\overline{b})$, we may assume that $(a,b) = (1,1)$. In particular, $1 \in A{\cdot_{[=1]}}B \cap C$. By Assumptions 5.5(ii) and (i), $A - \{1\}$ and $B - \{1\}$ are subsets of $A{\cdot_2}B$.

Consider first the case where $A - \{1\}$ and $B - \{1\}$ are disjoint. Then

$$|AB| + |A{\cdot_2}B| \geq 2\,|A{\cdot_2}B| \geq 2(|A-\{1\}| + |B-\{1\}|) = 2\,|A| + 2\,|B| - 4,$$

and (A,B) is sound. Therefore, we may assume that $A - \{1\}$ and $B - \{1\}$ are not disjoint and, hence, there exists some $x \in (A \cap B) - \{1\}$.

Since $1 \in A{\cdot_{[=1]}}B$, we see that $1 \in A1 - Ax$. In particular, $A \neq Ax$ and $1 \neq x$. Let (A', B') be the Kemperman transform of (A,B) with respect to x, as in Definition 4.8. Now Hypotheses 5.6 apply.

By Consequences 5.6.1 and 5.6.2, we may assume that $(A', B') \in \mathcal{S}_2$.

By Consequences 5.6.5, $A'B' \subseteq AB$, $\Omega(A', B') \leq \Omega(A,B)$ and (A', B') is sound. By Definitions 4.2, there are three possibilities.

Case 1. $\Omega(A', B') \geq -4$.

Here, $\Omega(A,B) \geq \Omega(A',B') \geq -4$, and (A,B) is sound.

Case 2. $\mathrm{blocks}(A'{\cdot_2}B') \geq 1$.

Here, $A'{\cdot_2}B'$ contains some block, D.

We claim that $C \neq D$. Since $1 \in C$ and $D \subseteq A'{\cdot_2}B'$, it suffices to show that $1 \notin A'{\cdot_2}B'$.

Notice that $1 = 1{\cdot}1 \in (A^+){\cdot}(B^-)$.

By Remark 4.8.6, $1 \notin (A^-){\cdot_2}(B^+) \cap A{\cdot_{[=1]}}B$. Since $1 \in A{\cdot_{[=1]}}B$, we see that $1 \notin (A^-){\cdot_2}(B^+)$.

Similarly, $1 = 1{\cdot}1 \in (A^-){\cdot}(B^+)$ and, by Remark 4.8.8, $1 \notin (A^+){\cdot_2}(B^-)$.

Hence, $1 \notin (A'){\cdot_2}(B')$, as desired.

Thus, C and D are two different blocks which are contained in AB.

Hence, $\mathrm{blocks}(AB) \geq 2$ and (A,B) is sound.

Case 3. $\mathrm{blocks}(A'B') \geq 2$.

Since $AB \supseteq A'B'$, we see that $\mathrm{blocks}(AB) \geq \mathrm{blocks}(A'B') \geq 2$, and (A,B) is sound. □

By Lemmas 5.7 and 5.8, the induction argument is complete, and we have proved the following.

5.9 Theorem. *Every element (A,B) of $\mathcal{S}_2$ is sound.*

By Lemma 4.4, we have the key inequality.

5.10 Theorem. *Let A and B be finite subsets of a group G. If $|A| \geq 2$ and $|B| \geq 2$, then $|AB| + |A{\cdot_2}B| \;\geq\; \min\{2\,|A| + 2\,|B| - 4,\, 2{\cdot}\,\alpha_3(G)\}$.* □

The proof of Corollary 3.5(ii) is now complete.

6 Upper bounds

In this section, we use the viewpoint of Mihalis Sykiotis [26, Proof of Theorem 2.13(1)] together with Corollary 3.5(ii) to rewrite and generalize results of [14] and [15].

The following is well known and easy to prove.

6.1 Lemma. *Let H and K be subgroups of a group G, and let S be a set of (H,K)-double coset representatives in G. Then the map*

$$\bigvee_{s\in S} ((H^s \cap K)\backslash G) \quad\to\quad (H\backslash G)\times(K\backslash G), \qquad (H^s\cap K)g \mapsto (Hsg, Kg),$$

is bijective. The inverse map is given by $(Hx, Ky) \mapsto (H^s \cap K)ky$ for the unique $s \in S$ such that $Hx\overline{y}K = HsK$, and any $k \in K$ such that $Hx\overline{y} = Hsk$; here $(H^s \cap K)k$ is unique. □

It is convenient to recall the following.

6.2 Review. Suppose that H is a group and that T is an H-free H-tree, that is, H acts freely on T.

Then, with respect to any basepoint, the fundamental group of the quotient graph $H\backslash T$ is isomorphic to H; see, for example, [9, Corollary I.4.2]. In particular, H is a free group.

The *core* of $H\backslash T$, denoted $\mathrm{core}(H\backslash T)$, is the subgraph of $H\backslash T$ consisting of all those vertices and edges which lie in cyclically reduced closed paths in $H\backslash T$.

Let $X = \mathrm{core}(H\backslash T)$. We write VX and EX for the vertex-set and edge-set of X, respectively. Every vertex of X has valence at least two.

If H is trivial, then $H\backslash T$ is the tree T, and X is empty.

Now suppose that H is nontrivial.

Then $H\backslash T$ is not a tree, and X is nonempty and its fundamental group is isomorphic to H. Moreover, H is finitely generated if and only if X is finite.

Suppose further that H is finitely generated, or, equivalently, that X is finite.

For each $v \in VX$, let $\deg_X(v)$ denote the valence of v in X. Then

$$\begin{aligned}\sum_{v\in VX}(\deg_X(v)-2) &= \Big(\sum_{v\in VX}\deg_X(v)\Big) - \Big(\sum_{v\in VX}2\Big) = \Big(\sum_{e\in EX}2\Big) - \Big(\sum_{v\in VX}2\Big)\\ &= 2{\cdot}\,|EX| - 2{\cdot}\,|VX| = -2{\cdot}\chi(X) = -2{\cdot}\chi(H) = 2{\cdot}\,\bar{\mathrm{r}}(H).\end{aligned}$$

Thus $\bar{\mathrm{r}}(H) = \frac{1}{2}\sum\limits_{v\in VX}(\deg_X(v)-2)$. □

We now come to our main upper-bound result. Recall from Definitions 1.1 that $\alpha_3(G)$ is ∞ or 4 or an odd prime, and that $\theta\alpha_3(G) = \frac{\alpha_3(G)}{\alpha_3(G)-2} \in [1,3]$.

6.3 Theorem. *Suppose that* Notation 1.2 *holds. Let H and K be elements of $\mathcal{F}$, and let S be a set of (H,K)-double coset representatives in G. Then*
$$\sum_{s\in S} \bar{\mathrm{r}}(H^s \cap K) \quad \leq \quad 2{\cdot}\theta\alpha_3(G){\cdot}\,\bar{\mathrm{r}}(H){\cdot}\,\bar{\mathrm{r}}(K).$$

Proof. Clearly, we may assume that H and K are nontrivial.

Let $\{x_j \mid j \in J\}$ be a free generating set of F.

We view G as the fundamental group of the following graph of groups.

Let $V = \{v_i \mid i \in I \vee \{0\}\}$, a set indexed by the disjoint union $I \vee \{0\}$.

Let $E = \{e_i \mid i \in I \vee J\}$, a set indexed by the disjoint union $I \vee J$.

Let $Z = (Z, V, E, \overline{\iota}, \overline{\tau})$ denote the (oriented) graph with vertex set V, edge set E, and incidence relations such that, for each $i \in I$ and $j \in J$, we have $\overline{\iota}(e_i) = \overline{\iota}(e_j) = \overline{\tau}(e_j) = v_0$, and $\overline{\tau}(e_i) = v_i$.

Let $Z_0 = Z - \{e_j \mid j \in J\}$, the unique maximal subtree of Z.

Let $(G(-), Z)$ be the unique graph of groups such that $G(v_0) = \{1\}$, and for each $i \in I$, $G(v_i) = G_i$, and, for each $i \in I \vee J$, $G(e_i) = \{1\}$.

In a natural way, the fundamental group $\pi(G(-), Z, Z_0)$ can be identified with the free product $F * \mathop{\ast}\limits_{i\in I} G_i = G$.

Let $T = T(G(-), Z, Z_0)$ be the Bass–Serre tree for $(G(-), Z, Z_0)$. Thus $T = (T, VT, ET, \iota, \tau)$ is the G-graph described as follows.

The vertex set is $VT = \bigvee\limits_{i\in I\vee\{0\}} Gv_i$, where, for each $i \in I \vee \{0\}$, the stabilizer G_{v_i} is $G(v_i)$.

The edge set is $ET = \bigvee\limits_{i\in I\vee J} Ge_i$, where, for each $i \in I \vee J$, the stabilizer G_{e_i} is $G(e_i) = \{1\}$.

The incidence relations are such that, for each $g \in G$, $i \in I$, and $j \in J$, we have $\iota(ge_i) = \iota(ge_j) = gv_0$, $\tau(ge_j) = gx_jv_0$, and, $\tau(ge_i) = gv_i$.

By Bass–Serre theory, T is a G-tree; see, for example, [9, Theorem I.7.6].

Here, G acts freely on the edge set ET, and H and K act freely on all of T.

We now use the argument in the proof of [26, Theorem 2.13(1)]; see also [8, p.380].

We identify $G\backslash T = Z$.

The pullback of the two graph maps $H\backslash T \to Z$ and $K\backslash T \to Z$ will be denoted $(H\backslash T) \times_Z (K\backslash T)$. As a set, $(H\backslash T) \times_Z (K\backslash T)$ is a subset of $(H\backslash T) \times (K\backslash T)$; moreover, $(H\backslash T) \times_Z (K\backslash T)$ has a natural graph structure.

We consider the map
$$\Phi\colon \bigvee_{s\in S} ((H^s \cap K)\backslash T) \quad \to \quad (H\backslash T) \times_Z (K\backslash T), \qquad (H^s \cap K)t \mapsto (Hst, Kt).$$
Here, Φ is a graph map. By Lemma 6.1, Φ is bijective on the edge sets, and on the sets of vertices that map to v_0 in Z, since G acts freely on $ET \vee Gv_0$. In particular, Φ is surjective.

Let us write

$$X = \operatorname{core}(H\backslash T),\ Y = \operatorname{core}(K\backslash T) \text{ and } W = \bigvee_{s\in S} \operatorname{core}((H^s \cap K)\backslash T).$$

Since Φ carries cores to cores, Φ induces a graph map $\phi\colon W \to X \times_Z Y$. Here, ϕ is injective on the edge sets, and on the sets of vertices which map to v_0 in Z.

By Review 6.2, X and Y are finite and

$$\bar{\mathrm{r}}(H) = \tfrac{1}{2} \sum_{x\in VX} (\deg_X(x) - 2), \quad \bar{\mathrm{r}}(K) = \tfrac{1}{2} \sum_{y\in VY} (\deg_Y(y) - 2).$$

Since ϕ embeds EW in the finite set $EX \times_{EZ} EY$, we see that W is finite, and, by Review 6.2,

$$\sum_{s\in S} \bar{\mathrm{r}}(H^s \cap K) = \tfrac{1}{2} \sum_{w\in VW} (\deg_W(w) - 2).$$

At this stage, we leave the proof of [26, Theorem 2.13(1)] and switch to the proof of [14, Theorem 2].

Notice that the result we want to prove can be reformulated as

$$\tfrac{1}{2}\cdot \sum_{w\in VW} (\deg_W(w)-2) \le 2\cdot\theta\alpha_3(G)\cdot(\tfrac{1}{2}\cdot \sum_{x\in VX} (\deg_X(x)-2))\cdot(\tfrac{1}{2}\cdot \sum_{y\in VY} (\deg_Y(y)-2)),$$

that is,

$$\sum_{w\in VW} (\deg_W(w) - 2) \quad \le \quad \theta\alpha_3(G) \cdot \sum_{(x,y)\in VX\times VY} ((\deg_X(x) - 2)\cdot(\deg_Y(y) - 2)).$$

Consider any $(x, y) \in VX \times_{VZ} VY$, and let $\phi^{-1}(x, y)$ denote the preimage in VW of (x, y) under the map $\phi : VW \to VX \times_{VZ} VY$. To prove the desired result, it then suffices to show that

$$\text{(6.3.1)} \qquad \sum_{w\in\phi^{-1}(x,y)} (\deg_W(w) - 2) \quad \le \quad \theta\alpha_3(G)\cdot(\deg_X(x) - 2)\cdot(\deg_Y(y) - 2).$$

Let z denote the common image of x and y in Z. Thus, there exists a unique $i \in I \vee \{0\}$ such that $z = v_i$.

Case 1. $i = 0$.

We have seen that the graph map $\phi\colon W \to X \times_Z Y$ is injective on the sets of vertices mapping to v_0 in Z. Thus, here, $\phi^{-1}(x, y)$ consists of a single element, w_0, say. Since (6.3.1) is clear when all the w have valence 2, we may assume that $\deg_W(w_0) \ge 3$. Recall that $\iota_W^{-1}\{w_0\}$, resp. $\tau_W^{-1}\{w_0\}$, denotes the set of edges of W whose initial, resp. terminal, vertex is w_0. Then

$$\left|\iota_W^{-1}\{w_0\}\right| + \left|\tau_W^{-1}\{w_0\}\right| = \deg_W(w_0).$$

It is not difficult to show that the induced map $\iota_W^{-1}\{w_0\} \to EZ$ is injective, and, hence, $\iota_W^{-1}\{w_0\} \to \iota_X^{-1}\{x\}$ is injective, and, hence, $\left|\iota_W^{-1}\{w_0\}\right| \le \left|\iota_X^{-1}\{x\}\right|$. Similarly, $\left|\tau_W^{-1}\{w_0\}\right| \le \left|\tau_X^{-1}\{x\}\right|$. Thus $\deg_W(w_0) \le \deg_X(x)$.

Similarly, $\deg_Y(y) \ge \deg_W(w_0) \ge 3$.

Now we have

$$\begin{aligned}\sum_{w\in\phi^{-1}(x,y)} (\deg_W(w)-2) \;&=\; \deg_W(w_0)-2 \;\le\; \deg_X(x)-2\\ &\le\; 1{\cdot}(\deg_X(x)-2){\cdot}(3-2) \le\; \theta\alpha_3(G){\cdot}(\deg_X(x)-2){\cdot}(\deg_Y(y)-2),\end{aligned}$$

as desired.

Case 2. $i \in I$.

Here, there exist g_x, $g_y \in G$ such that $x = Hg_xv_i$ and $y = Kg_yv_i$.

Notice that $\deg_X(x) = \left|\tau_X^{-1}\{x\}\right|$, and that

$$\tau_X^{-1}\{x\} \subseteq \{H\}g_xG_ie_i := \{Hg_xae_i \mid a \in G_i\}.$$

Hence, there exists a subset A of G_i such that $\tau_X^{-1}\{x\} = \{H\}g_xAe_i$. Moreover, A is unique, since G acts freely on ET (on the left) and G_i acts freely on $H\backslash G$ on the right. Hence, $|A| = \deg_X(x) \ge 2$.

Similarly, there exists a unique subset B of G_i such that $\tau_Y^{-1}\{y\} = \{K\}g_yBe_i$, and $|B| = \deg_Y(y) \ge 2$.

The embedding $\phi\colon EW \to EX \times_{EZ} EY$, gives an embedding

$$\phi\colon \bigvee_{w\in\phi^{-1}(x,y)} \tau_W^{-1}\{w\} \quad\hookrightarrow\quad \tau_X^{-1}\{x\} \times \tau_Y^{-1}\{y\} \;=\; \{H\}g_xAe_i \times \{K\}g_yBe_i,$$

which, when composed with the embedding

$$\{H\}g_xAe_i \times \{K\}g_yBe_i \quad\hookrightarrow\quad A \times B, \quad (Hg_xae_i, Kg_ybe_i) \mapsto (a,b),$$

gives an embedding

$$\psi\colon \bigvee_{w\in\phi^{-1}(x,y)} \tau_W^{-1}\{w\} \quad\hookrightarrow\quad A\times B, \quad e \mapsto \psi(e).$$

Let $w \in \phi^{-1}(x,y)$.

We claim that $\psi(\tau_W^{-1}\{w\})$ is a single-quotient subset of $A \times B$, as in Definitions 3.1. Let e, f be elements of $\tau_W^{-1}\{w\}$.

There exist $s_w \in S$ and $g_w \in G$ such that $w = (H^{s_w} \cap K)g_wv_i$. Also, there exists a unique subset C_w of G_i such that $\tau_W^{-1}\{w\} = (H^{s_w} \cap K)g_wC_we_i$, and, here, $|C_w| = \left|\tau_W^{-1}\{w\}\right| = \deg_W(w)$. There exist c_e, c_f in C_w such that

$$e = (H^{s_w} \cap K)g_wc_ee_i, \quad f = (H^{s_w}\cap K)g_wc_fe_i.$$

Let $(a_e, b_e) = \psi(e)$, $(a_f, b_f) = \psi(f)$. This means that, on applying the map $\phi\colon EW \to EX \times_{EZ} EY$, we have

$$(Hs_wg_wc_ee_i, Kg_wc_ee_i) = \phi(e) = (Hg_xa_ee_i, Kg_yb_ee_i),$$
$$(Hs_wg_wc_fe_i, Kg_wc_fe_i) = \phi(f) = (Hg_xa_fe_i, Kg_yb_fe_i).$$

Since G acts freely on ET, we have

$$(Hs_w g_w c_e, Kg_w c_e) = (Hg_x a_e, Kg_y b_e), \quad (Hs_w g_w c_f, Kg_w c_f) = (Hg_x a_f, Kg_y b_f).$$

Hence $Hg_x a_e \overline{c}_e = Hs_w g_w = Hg_x a_f \overline{c}_f$ and $Kg_y b_e \overline{c}_e = Kg_w = Kg_y b_f \overline{c}_f$. Since G_i acts freely on the right on both $H\backslash G$ and $K\backslash G$, we see that $a_e\overline{c}_e = a_f\overline{c}_f$ and $b_e\overline{c}_e = b_f\overline{c}_f$. Hence, $a_e\overline{b}_e = a_f\overline{b}_f$.

This completes the proof that $\psi(\tau_W^{-1}\{w\})$ is a single-quotient subset of $A\times B$.

Now

$$\begin{aligned}
\sum_{w\in\phi^{-1}(x,y)} (\deg_W(w) - 2) &= \sum_{w\in\phi^{-1}(x,y)} (\left|\tau_W^{-1}\{w\}\right| - 2) \\
&= \sum_{w\in\phi^{-1}(x,y)} (\left|\psi(\tau_W^{-1}\{w\})\right| - 2) \\
&\leq \theta\alpha_3(G)\cdot(|A|-2)\cdot(|B|-2) \quad \text{by Corollary 3.5(ii)} \\
&= \theta\alpha_3(G)\cdot(\deg_X(x)-2)\cdot(\deg_Y(y)-2). \qquad \square
\end{aligned}$$

For emphasis, we mention the extreme cases.

6.4 Corollary. *Suppose that* Notation 1.2 *holds. Let H and K be elements of $\mathcal{F}$, and let S be a set of (H,K)-double coset representatives in G. Then the following hold.*

(i). $\sum\limits_{s\in S} \bar{\mathrm{r}}(H^s\cap K) \;\leq\; 6\cdot \bar{\mathrm{r}}(H)\cdot \bar{\mathrm{r}}(K).$

(ii). *If G is torsion-free, or, more generally, every finite subgroup of G has order at most two, then* $\sum\limits_{s\in S} \bar{\mathrm{r}}(H^s\cap K) \;\leq\; 2\cdot \bar{\mathrm{r}}(H)\cdot \bar{\mathrm{r}}(K).$ $\square$

We remark that Corollary 6.4(i) generalizes [14, Theorem 2], while Corollary 6.4(ii) generalizes [15, Theorem 2].

By combining Proposition 2.9 and Theorem 6.3, we get our main result.

6.5 Theorem. *If* Notation 1.2 *holds, then $\mathcal{F}$ is closed under taking finite intersections. Moreover, $\sigma(\mathcal{F}) \in [\beta_2(G)\cdot\theta\alpha_3(G),\ 2\cdot\theta\alpha_3(G)]$, that is,*

$$\begin{cases} \sigma(\mathcal{F}) = \beta_2(G)\cdot\theta\alpha_3(G) = 2\cdot\theta\alpha_3(G) \text{ if } G \text{ has 2-torsion; and,} \\ \sigma(\mathcal{F}) \in [\theta\alpha_3(G),\ 2\cdot\theta\alpha_3(G)] \text{ if } G \text{ is 2-torsion free.} \end{cases} \qquad \square$$

We conclude by mentioning a more general problem.

6.6 Remarks. Suppose that G is a group and that T is a G-tree.

Let $\mathcal{F}$ denote the set of those finitely generated (free) subgroups H of G which have the property that, via the restriction of the G-action, H acts freely on T.

Let $\sigma(\mathcal{F})$ be defined as in (1.2.1).

(i). B. Baumslag [2] showed that if the G-stabilizers of the elements of ET are all trivial, and the G-stabilizers of the elements of VT are all Howson, then G itself is Howson; equivalently, the free product of a family of Howson groups is Howson. Recall that G is said to be *Howson* if the set of finitely generated subgroups of G is closed under finite intersections.

(ii). It follows from Theorem 6.5 that, if the G-stabilizers of the elements of ET are all trivial, then $\mathcal{F}$ is closed under finite intersections. (The proof of Baumslag's result given in [14, Theorem 1] shows this under the additional hypothesis that $G\backslash T$ is a tree.) Here we conjectured that $\sigma(\mathcal{F}) = \beta_2(G){\cdot}\theta\alpha_3(G)$, and Theorem 6.5 implies that $\sigma(\mathcal{F}) \in [\beta_2(G){\cdot}\theta\alpha_3(G),\ 2{\cdot}\theta\alpha_3(G)]$.

(iii). D. E. Cohen [6, Theorem 7], generalizing Baumslag's result, showed that if the G-stabilizers of the elements of ET are all finite, and the G-stabilizers of the elements of VT are all Howson, then G itself is Howson.

(iv). The proof of Cohen's result given by Sykiotis in [26, Corollary 2.14] shows that if the G-stabilizers of the elements of ET are all finite, then $\mathcal{F}$ is closed under finite intersections. (We recalled almost all of Sykiotis' argument in the above proof of Theorem 6.3.) Here we conjecture that $\sigma(\mathcal{F})$ is (again) given by the value in (2.10.1), but our techniques shed no light on this case. □

Acknowledgments

The research of the first-named author was jointly funded by the MEC (Spain) and the EFRD (EU) through Projects BFM2003-06613 and MTM2006-13544.

The research of the second-named author was partially funded by the NSF (USA) through grants DMS 00-99612 and DMS 04-00746.

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