Open access • Journal Article • DOI:10.1007/S00022-006-0042-4

On the intersection of Hermitian surfaces — Source link

Luca Giuzzi

Institutions: Instituto Politécnico Nacional

Published on: 01 Sep 2006 - Journal of Geometry (Birkhäuser-Verlag)

Topics: Intersection, Hermitian matrix and Variety (universal algebra)

ON THE INTERSECTION OF HERMITIAN SURFACES

L. GIUZZI

ABSTRACT. We provide a description of the configuration arising from intersection of two Hermitian surfaces in $\mathrm{PG}(3,\mathrm{q})$, provided that the linear system they generate contains at least a degenerate variety.

1. INTRODUCTION

In [2], the seven point–line configurations arising from the intersection of two Hermitian curves are described and a classification of Hermitian pencils yielding a given configuration is provided. In [3] it has been shown that these configurations are projectively unique and their full collineation group has been determined. Given two Hermitian varieties $\mathcal{H}_1$ and $\mathcal{H}_2$, their intersection $\mathcal{E}$ is just the base locus of the $\mathrm{GF}(\sqrt{\mathrm{q}})$–linear system they generate, namely

$$\Gamma(\mathcal{H}_1,\mathcal{H}_2)=\{\mathcal{H}_1+\lambda\mathcal{H}_2:\lambda\in\mathrm{GF}(\sqrt{\mathrm{q}})\}.$$

In this paper we determine the size of such an intersection depending on the number of degenerate varieties in Γ and describe the actual point–line configurations arising in the 3–dimensional case, provided that Γ contains at least a degenerate surface.

2. INTERSECTION NUMBERS

The set of all singular points of a Hermitian variety $\mathcal{H}$ is a subspace $\mathrm{rad}\,\mathcal{H}$, the *radical* of $\mathcal{H}$. We recall that the *rank* of a Hermitian variety in $\mathrm{PG}(\mathrm{n},\mathrm{q})$ is the number $r=n+1-\dim\mathrm{rad}\,\mathcal{H}$.

Let now $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hermitian hypersurfaces of $\mathrm{PG}(\mathrm{n},\mathrm{q})$ and denote by r_i the number of varieties of rank i in the $\mathrm{GF}(\sqrt{\mathrm{q}})$–pencil Γ they generate. We shall call the list $(r_1,\ldots,r_n)$ the *rank sequence* of Γ.

It has been observed in [2] that the cardinality of the base locus $\mathcal{E}$ of Γ depends only on its rank sequence. In fact the considerations provided in [2] about the 2–dimensional case may be generalised to arbitrary dimension n, as it has been done in [4].

Proposition 1. *The rank sequence $(r_1,\ldots,r_n)$ of a pencil Γ of Hermitian varieties in $\mathrm{PG}(\mathrm{n},\mathrm{q})$ satisfies the inequality*

$$\sum_{i=1}^{n}(n-i+1)r_i\leq n+1.$$

Since, see [1], the total number of points of the non–degenerate Hermitian hypersurface $\mathcal{H}$ of $\mathrm{PG}(\mathrm{n},\mathrm{q})$ is,

$$\mu(n,q)=[q^{(n+1)/2}+(-1)^n][q^{n/2}-(-1)^n)]/(q-1)],$$

it is possible to formulate the following proposition.

2000 *Mathematics Subject Classification.* Primary: 11E39, 51E20, Secondary: 05B25, 05B30.

Key words and phrases. Hermitian surfaces, intersection numbers, finite projective spaces, point–line configurations.

The present research was performed within the activity of G.N.S.A.G.A. of the Italian INDAM with the financial support of the Italian Ministry M.I.U.R., project “Strutture geometriche, combinatorica e loro applicazioni”, 2003–04.

Proposition 2. *Let $\mathcal{H}_1$, $\mathcal{H}_2$ be two non–degenerate Hermitian varieties in* $\mathrm{PG}(n,q)$, *and let* $(r_1,\dots,r_n)$ *be the rank sequence of the pencil* $\Gamma(\mathcal{H}_1,\mathcal{H}_2)$. *Then,*

$$|\mathcal{H}_1\cap\mathcal{H}_2| = \eta_n(\Gamma,q) = \frac{1}{\sqrt{q}(q-1)}\{(1-q^{n+1})+\sum_{i=1}^{n} r_i[(q\sqrt{q}\mu(i-1,q)+1)(q^{n+1-i}-1) - (q-1)\mu(n,q)]\} + (1+\frac{1}{\sqrt{q}})\mu(n,q).$$

Table 1 outlines the possible intersection sizes for any two non–degenerate Hermitian surfaces in $\mathrm{PG}(3,q)$. All cases are possible.

r_1	r_2	r_3	$\eta_3(\Gamma,q)$
0	0	0	$(q+1)^2$
0	0	1	$(q+\sqrt{q}+1)(q-\sqrt{q}+1)$
0	0	2	(q^2+1)
0	0	3	q^2-q+1
0	0	4	$(q-1)^2$
0	1	0	$q^2+q\sqrt{q}+q+1$
0	1	1	$q^2+q\sqrt{q}+1$
0	1	2	$(\sqrt{q}+1)(q\sqrt{q}-q+1)$
0	2	0	$(\sqrt{q}+1)(q\sqrt{q}+q-\sqrt{q}+1)$
1	0	0	$q\sqrt{q}+q+1$
1	0	1	$q\sqrt{q}+1$

TABLE 1. Possible intersection numbers for Hermitian surfaces: non–degenerate pencil.

In the rest of this paper we shall usually write just Γ for $\Gamma(\mathcal{H}_1,\mathcal{H}_2)$ when no ambiguity might arise and we shall denote by $\mathcal{E}$ the intersection $\mathcal{H}_1\cap\mathcal{H}_2$.

3. DESCRIPTION OF THE CONFIGURATIONS

3.1. **Pencils with a degenerate surface of rank** 1. The simplest case to consider is when the linear system Γ contains a degenerate surface $\mathcal{C}$ of rank 1, that is a plane repeated $q+1$ times.

In this case, the intersection is either a degenerate or non–degenerate Hermitian, according as $\mathcal{C}$ is secant or tangent to all the other surfaces in the pencil. It follows directly from Table 1 that the former case occurs when Γ contains also a surface of rank 3, a *Hermitian cone*, whereas the latter is possible only if all the surfaces in $\Gamma\setminus\{\mathcal{C}\}$ are non–degenerate.

3.2. **Pencils whose degenerate surfaces have all rank** 2. A Hermitian surface $\mathcal{P}$ of rank 2 is a set of $\sqrt{q}+1$ planes through a line, the radical of $\mathcal{P}$. In the following propositions we distinguish several cases.

Proposition 3. *Suppose that Γ contains exactly one degenerate surface $\mathcal{P}$ of rank 2. Then, either 1, 2 or $(\sqrt{q}+1)$ components of $\mathcal{P}$ are degenerate Hermitian curves.*

Proof. The radical of $\mathcal{P}$ meets $\mathcal{E}$ in either 1, $\sqrt{q}+1$ or $q+1$ points. Let $n=|\mathrm{rad}\,\mathcal{P}\cap\mathcal{E}|$ and denote by v_2 the number of components of $\mathcal{P}$ which meet $\mathcal{E}$ in a degenerate Hermitian curve.

(1) $n = 1$. Then,

$$q^2 + q\sqrt{q} + q + 1 = v_2(q\sqrt{q} + q) + (\sqrt{q} + 1 - v_2)q\sqrt{q} + 1;$$

hence, $v_2 = 1$.

(2) $n = \sqrt{q} + 1$. Then,

$$q^2 + q\sqrt{q} + q + 1 = v_2(q\sqrt{q} + q - \sqrt{q}) + \sqrt{q}(\sqrt{q} + 1 - v_2)(q - 1) + \sqrt{q} + 1;$$

hence, $v_2 = 2$.

(3) $n = q + 1$. Then,

$$q^2 + q\sqrt{q} + q + 1 = v_2(q\sqrt{q}) + q(\sqrt{q} + 1 - v_2)(\sqrt{q} - 1) + q + 1;$$

hence, $v_2 = \sqrt{q} + 1$.

□

Observe that in general, if $\mathcal{P}$ and $\mathcal{P}'$ are any two degenerate surfaces in Γ, then $\mathcal{R} = \operatorname{rad}\mathcal{P} \cap \operatorname{rad}\mathcal{P}' = \emptyset$ since, otherwise, any point $V \in \mathcal{R}$ would be singular for all the surfaces of Γ. Suppose now that there are two distinct Hermitian surfaces $\mathcal{P}$ and $\mathcal{P}'$ both of rank 2 in Γ; the previous observation proves that $\operatorname{rad}\mathcal{P}$ and $\operatorname{rad}\mathcal{P}'$ have to be mutually skew. Furthermore, both $\operatorname{rad}\mathcal{P}$ and $\operatorname{rad}\mathcal{P}'$ meet any non–degenerate surface in Γ in $(\sqrt{q} + 1)$ points. Thus, we obtain the following proposition.

Proposition 4. *The intersection of two non degenerate Hermitian surfaces $\mathcal{H}_1$, $\mathcal{H}_2$ spawning a pencil with $r_2 = 2$ is the union of all generators of $\mathcal{H}_1$ which pass through two skew $(\sqrt{q} + 1)$–secants.*

3.3. **Pencils whose degenerate surfaces have rank 2 and 3.** Given a Hermitian cone $\mathcal{C}$ of vertex V and a non–degenerate Hermitian variety $\mathcal{H}$, we denote by $\Gamma'(\mathcal{C}, \mathcal{H})$ the $\mathrm{GF}(\sqrt{\mathrm{q}})$–linear system of Hermitian curves generated by $\mathcal{C}' = \mathcal{C} \cap \pi$ and $\mathcal{H}' = \mathcal{H} \cap \pi$, where π is the polar plane of V with respect to $\mathcal{H}$.

Lemma 5. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two distinct Hermitian cones of vertices respectively V_1 and V_2. Assume that the pencil $\Gamma(\mathcal{C}_1, \mathcal{C}_2)$ contains at least a non–singular surface and that $V_1 \notin \mathcal{E}$. Then, V_2 belongs to the polar plane of V_1 with respect to any non–degenerate Hermitian surface in Γ.*

Proof. Fix a non–degenerate Hermitian surface $\mathcal{H} \in \Gamma$ and let π be the polar plane of V_1 with respect to $\mathcal{H}$. Since $V_1 \notin \mathcal{H}$, the plane π cuts a non–singular Hermitian curve on $\mathcal{H}$. Suppose $V_2 \notin \pi$; then the line V_1V_2 would meet $\mathcal{H}$ in $\sqrt{q}+1$ points. On the other hand, V_1V_2 meets $\mathcal{C}_1$ and $\mathcal{C}_2$ in either 1 or $q + 1$ points — a contradiction. It follows that $V_2 \in \pi$ and $|V_1V_2 \cap \mathcal{E}| \leq 1$. □

Lemma 6. *Take $\mathcal{C}$ to be a Hermitian cone of vertex V and $\mathcal{H}$ to be a non–degenerate Hermitian surface; let also π be the polar plane of V with respect to $\mathcal{H}$ and let $\Gamma = \Gamma(\mathcal{C}, \mathcal{H})$. Then,*

(1) *if $V \notin \mathcal{H}$,*

$$\eta_3(\Gamma, q) = q^2 + q\sqrt{q} + \sqrt{q} + 1 - \eta_2(\Gamma', q)\sqrt{q};$$

(2) *if $V \in \mathcal{H}$,*

$$\eta_3(\Gamma, q) = q^2 - q + |\pi \cap \mathcal{E}|.$$

Proof. Let $\mathcal{H}' = \mathcal{H} \cap \pi$ and $\mathcal{C}' = \mathcal{C} \cap \pi$. Take $h = \eta_2(\Gamma', q)$. Observe that any line through V tangent to $\mathcal{H}$ is of the form PV with $P \in \mathcal{H}'$. If $V \notin \mathcal{H}$, every line through V meets $\mathcal{H}$ in either 1 or $\sqrt{q} + 1$ points; on the other hand, exactly h generators of $\mathcal{C}$ are tangent to $\mathcal{H}$, whence it follows

$$\eta_3(\Gamma, q) = h + (q\sqrt{q} + 1 - h)(\sqrt{q} + 1).$$

Assume $V \in \mathcal{H}$. Hence, π is the tangent plane to $\mathcal{H}$ at V and $\mathcal{H}'$ consists of $\sqrt{q}+1$ lines through V. However, $\mathcal{C}'$ consists of either 1 line or a degenerate Hermitian curve. In the former case we would have $\eta_3(\Gamma,q)=q^2+q+1$, which is not possible. Hence, $\mathcal{C}'$ is the union of $\sqrt{q}+1$ lines through V and

$$\eta_3(\Gamma,q)=\sqrt{q}(q\sqrt{q}-\sqrt{q})+|\pi\cap\mathcal{E}|.$$

Observe that in this case, all the curves in the linear system Γ are degenerate. □

Using the cardinality formula of Proposition 2, together with Lemma 6, it is possible to reconstruct the rank sequence of $\Gamma(\mathcal{C},\mathcal{H})$ from the rank sequence of Γ'.

Lemma 7. *Assume Γ to contain at least one cone $\mathcal{C}$ of vertex $V\notin\mathcal{E}$, and let the rank sequence of Γ' be (r_1',r_2'). Then, the rank sequence of Γ is $(0,r_1',r_2'+1)$.*

In fact, it is possible to describe in an accurate way the actual configuration $\mathcal{E}\cap\pi$ in terms of the seven classes of [2]. We shall denote such classes as in [3].

Proposition 8. *Let Γ be a $\mathrm{GF}(\sqrt{\mathrm{q}})$-pencil of Hermitian surfaces with rank sequence $(0,r_1',r_2'+1)$. Then, the base configuration $\mathcal{E}'=\mathcal{E}\cap\pi$ is uniquely determined.*

Proof. The cardinality of $\mathcal{E}'$ is determined by Lemma 7. Observe that 5 of the 7 classes of [2] are uniquely determined by their rank sequence. However, both class III and IV correspond to the same rank sequence $(r_1',r_2')=(0,1)$. By Lemma 7, Γ has necessarily rank sequence $(0,0,2)$ and $|\mathcal{E}|=q^2+1$. Denote then by $\mathcal{C}_1$ and $\mathcal{C}_2$ the two distinct Hermitian cones of Γ and assume they have respectively vertices V_1 and V_2. There are two possibilities for $\mathcal{E}'=\mathcal{E}\cap\pi$:

(1) $\mathcal{E}'$ belongs to class III, that is $\mathcal{E}'$ consists of $\sqrt{q}-1$ sublines, all disjoint, and 2 more points;
(2) $\mathcal{E}'$ belongs to class IV, that is $\mathcal{E}'$ consists of $\sqrt{q}$ sublines, which meet all in a point P.

The former case occurs if $V_2\notin\mathcal{E}$. If $\mathcal{E}'$ belongs to class IV, then $V_2=P\in\mathcal{E}$. □

Proposition 9. *Let Γ be a non–degenerate linear system of Hermitian surfaces with $r_2(\Gamma)=r_3(\Gamma)=1$. Then, $\mathcal{E}$ is either the union of $\sqrt{q}+1$ non–degenerate Hermitian curves all with a point in common, or the union of $\sqrt{q}$ non–degenerate Hermitian curves and a degenerate Hermitian curve, all sharing a $(\sqrt{q}+1)$–secant.*

Proof. Let $\mathcal{P}$ and $\mathcal{C}$ be respectively the only surface of rank 2 and the only Hermitian cone in Γ. Let also $L=\operatorname{rad}\mathcal{P}$ and V be the vertex of $\mathcal{C}$. Observe that $l=|L\cap\mathcal{E}|\in\{1,\sqrt{q}+1\}$.

(1) $l=1$. Let M be the point of intersection of $\mathcal{C}$ and L; clearly $M\neq V$. If $V\in\mathcal{E}$, then there is a component π of $\mathcal{P}$ such that $V\in\pi$. However, in this case $\mathcal{C}\cap\pi=PM$. However, this cannot be a plane section of a non–degenerate Hermitian surface; hence, it follows that $V\notin\mathcal{E}$. This being the case, all the $\sqrt{q}+1$ sections cut on $\mathcal{C}$ by $\mathcal{P}$ are non–degenerate Hermitian curves having the point M in common.
(2) $l=\sqrt{q}+1$: Let v_1 be the number of the components of $\mathcal{P}$ which meet $\mathcal{C}$ in a degenerate Hermitian curve. Observe that $v_1\leq 1$ and equality occurs if and only if $V\in\mathcal{E}$. Since,

$$(q^2+q\sqrt{q}+1)=(\sqrt{q}+1)+(\sqrt{q}+1-v_1)(q\sqrt{q}-\sqrt{q})+v_1(q\sqrt{q}+q^2-\sqrt{q}),$$

we get $v_1=1$ and $V\in\mathcal{E}$.

□

Proposition 10. *Let Γ be a non–degenerate linear system of Hermitian surfaces with $r_2(\Gamma)=1$ and $r_3(\Gamma)=2$. Then, $\mathcal{E}$ is the union of $\sqrt{q}+1$ non degenerate Hermitian curves, all sharing a chord.*

Proof. Let $\mathcal{P}$ be the only Hermitian surface or rank 2 in Γ and denote by $\mathcal{C}_1$, $\mathcal{C}_2$ be the two Hermitian cones. As before, denote by V_1 and V_2 the vertices of respectively $\mathcal{C}_1$ and $\mathcal{C}_2$ and define $L=\operatorname{rad}\mathcal{P}$ and $l=|L\cap\mathcal{E}|$. Either $l=1$ or $l=\sqrt{q}+1$. If it were $l=1$, then we would obtain that $\mathcal{E}$ is the union of $\sqrt{q}+1$ non–degenerate Hermitian curves, all with a point in common. However, this is a contradiction because of Proposition 2. Assume then $l = \sqrt{q}+1$ and denote by v_1 the number of components of $\mathcal{P}$ meeting $\mathcal{E}$ in a degenerate Hermitian curve. Then,

$$(\sqrt{q}+1)(q\sqrt{q}-q+1) = (\sqrt{q}+1) + (\sqrt{q}+1-v_1)(q\sqrt{q}-\sqrt{q}) + v_1(q\sqrt{q}+q^2-\sqrt{q}).$$

This is possible only if $v_1 = 0$, that is, $\mathcal{E}$ is the union of $\sqrt{q}+1$ non–degenerate Hermitian curves, all sharing a chord. □

3.4. **Pencils whose degenerate surfaces have all rank** 3. All the pencils considered in this section contains at least a cone $\mathcal{C}$. We shall denote by s_1, s_2 and s_3 the number of generators of $\mathcal{C}$ meeting $\mathcal{E}$ in respectively $q+1$, $\sqrt{q}+1$ or 1 points.

Lemma 11. *Let Γ contain at least 3 distinct cones $\mathcal{C}_1 \ldots \mathcal{C}_3$ of vertices $V_1 \ldots V_3$. Then, either the vertices of all the cones in Γ are collinear or at most one of them is in $\mathcal{E}$.*

Proof. Suppose $V_1, V_2 \in \mathcal{E}$; then, $V_1V_2 \subseteq \mathcal{E}$. However, for V_1V_2 to be a subset of $\mathcal{E}$, it is necessary for it to be a generator of $\mathcal{C}_3$ also. It follows $V_3 \in V_1V_2$. □

Observe that when Γ contains at least two Hermitian cones, the number of lines in $\mathcal{E}$ is at most 1.

Proposition 12. *If Γ contains 4 Hermitian cones, then none of the vertices of such cones belongs to $\mathcal{E}$ and exactly $\sqrt{q}(q-\sqrt{q}-2)$ generators of any cone meet $\mathcal{E}$ in $(\sqrt{q}+1)$ points, the remaining $(\sqrt{q}+1)^2$ being tangent lines.*

Proof. By Proposition 2, $|\mathcal{E}| = (q-1)^2$. Suppose $V_1, V_2, V_3, V_4 \in \mathcal{E}$. Then,

$$(q-1)^2 = (q+1) + s_2\sqrt{q} + (q\sqrt{q} - s_2),$$

that is $q(q^2-\sqrt{q}-3) = s_2(\sqrt{q}-1)$, a contradiction since $(\sqrt{q}-1)$ does not divide $q^2-\sqrt{q}-3$. If $V_1 \in \mathcal{E}$ while $V_2, V_3, V_4 \notin \mathcal{E}$. Observe that a generator of $\mathcal{C}_1$ either meets a non–degenerate surface $\mathcal{H}$ is 1 or in $\sqrt{q}+1$ points. Then,

$$(q-1)^2 = s_2\sqrt{q}+1,$$

which gives $s_2 = q\sqrt{q} - 2\sqrt{q}$, that is $2\sqrt{q}+1$ generators of $\mathcal{C}$ meet $\mathcal{H}$ in V only and all these generators lie in the tangent plane to $\mathcal{H}$ at V. However, the number of generators of $\mathcal{C}$ on any plane is at most $\sqrt{q}+1$, which gives a contradiction. It follows that $\mathcal{E}$ does not contain the vertex of any cone in Γ and

$$(q-1)^2 = s_2(\sqrt{q}+1) + (q\sqrt{q}+1-s_2),$$

that is, $s_2 = \sqrt{q}(q-\sqrt{q}-2)$, which gives the result. □

Proposition 13. *Suppose Γ to contain exactly 3 cones $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$. Then, there are two possibilities:*

(1) *$V_1, V_2, V_3 \notin \mathcal{E}$: then $\sqrt{q}(q-\sqrt{q}-1)$ components of each cone are $(\sqrt{q}+1)$–secants to any non–degenerate Hermitian surface in Γ;*
(2) *$V_1 \in \mathcal{E}$ but $V_2, V_3 \notin \mathcal{E}$: then $\sqrt{q}(q-1)$ generators of the cone $\mathcal{C}_1$ meet $\mathcal{E}$ in $(\sqrt{q}+1)$ points, the remaining intersecting $\mathcal{E}$ in V_1 only; the number of generators of the cones $\mathcal{C}_2$ and $\mathcal{C}_3$ meeting $\mathcal{E}$ in $(\sqrt{q}+1)$ points is $q\sqrt{q}-q-\sqrt{q}$, the others meeting $\mathcal{E}$ in distinct points;*

Proof. The cardinality of $\mathcal{E}$ is q^2-q+1.

(1) Since $V_1 \notin \mathcal{E}$,

$$q^2-q+1 = s_2(\sqrt{q}+1) + (q\sqrt{q}-s_2+1).$$

Hence, there are $\sqrt{q}(q-\sqrt{q}-1)$ components of $\mathcal{C}_1$ are meeting $\mathcal{E}$ in $(\sqrt{q}+1)$ points, the remaining $q+\sqrt{q}+1$ being tangent to any surface.

(2) Since $V_1 \in \mathcal{E}$, each generator of $\mathcal{C}_1$ meets $\mathcal{E}$ in either 1 or $\sqrt{q}+1$ points. We get
$$q^2-q+1=1+t\sqrt{q}.$$
It follows that $\sqrt{q}(q-1)$ generators through V meet $\mathcal{E}$ in $\sqrt{q}+1$ points. Consider now another cone $\mathcal{C}_2 \in \Gamma$. Since $V_2 \notin \mathcal{E}$, we get
$$q^2-q+1=s_2(\sqrt{q}+1)+(q\sqrt{q}+1-s_2);$$
hence $s_2=q\sqrt{q}-q-\sqrt{q}$.

Suppose now $V_1, V_2 \in \mathcal{E}$. Then, the line V_1V_2 is a generator of any surface $\mathcal{H} \in \Gamma$ and we have
$$q^2-q+1=q+1+s_2\sqrt{q}.$$
It follows that $s_2=q\sqrt{q}-2\sqrt{q}$. This gives that there should be $2\sqrt{q}+1>\sqrt{q}+1$ generators through V_1 meeting $\mathcal{H}$ in V_1 only — a contradiction. □

Proposition 14. *Assume that the pencil Γ contains exactly two cones $\mathcal{C}_1$, $\mathcal{C}_2$ of respectively vertices V_1 and V_2. Then, one of the following possibilities holds:*

(1) *both $V_1, V_2 \in \mathcal{E}$; then, $\mathcal{E}$ contains the line V_1V_2; $q(\sqrt{q}-1)$ components of each cone meet $\mathcal{E}$ in $\sqrt{q}+1$ points.*
(2) *$V_1 \in \mathcal{E}$, while $V_2 \notin \mathcal{E}$; $\mathcal{E}$ does not contain any line; $q\sqrt{q}$ components of $\mathcal{C}_1$ and $q(\sqrt{q}-1)$ components of $\mathcal{C}_2$ meet $\mathcal{E}$ in $\sqrt{q}+1$ points.*
(3) *$V_1, V_2 \notin \mathcal{E}$ belong to $\mathcal{E}$; $q(\sqrt{q}-1)$ components of each cone meet $\mathcal{E}$ in $(q+1)$ points.*

Proof. Let V be the vertex of any cone $\mathcal{C}$ in Γ. Observe that for $V \notin \mathcal{E}$,
$$q^2+1=s_2(\sqrt{q}+1)+(q\sqrt{q}+1-s_2);$$
hence, $s_2=q\sqrt{q}-q$. On the other hand, if $V \in \mathcal{E}$
$$q^2+1=s_1q+s_2\sqrt{q}+1;$$
hence, $s_2=\sqrt{q}(q-s_1)$. The result now follows from $s_1 \leq 1$. □

Proposition 15. *Assume that the only degenerate surface in the pencil Γ is a cone $\mathcal{C}$. Then, either*

(1) *$V \notin \mathcal{E}$ and $\sqrt{q}(q-\sqrt{q}+1)$ components of $\mathcal{C}$ are meet $\mathcal{E}$ in $\sqrt{q}+1$ points, or*
(2) *$V \in \mathcal{E}$ and $\mathcal{E}$ contains at least a line.*

Proof.

(1) If $V \notin \mathcal{E}$, then $s_1=0$ and
$$q^2+q+1=s_2(\sqrt{q}+1)+(q\sqrt{q}+1-s_2);$$
hence, $s_2=\sqrt{q}(q-\sqrt{q}+1)$.
(2) If $V \in \mathcal{E}$,
$$q^2+q+1=s_1q+s_2\sqrt{q}+1;$$
hence, $s_2=\sqrt{q}(q+1-s_1)$. Since the total number of components of $\mathcal{C}$ is $q\sqrt{q}+1$, it follows that $s_2 \leq q\sqrt{q}+1$ and $s_1 \geq 1$.

□

Suppose that Γ contains exactly one Hermitian cone whose vertex belongs to $\mathcal{E}$, and let π be the tangent plane at P to a non–degenerate Hermitian surface in Γ. Clearly, all generators in $\mathcal{E}$ lie in π; furthermore $s_1 \in \{1,2,\sqrt{q}+1\}$. Consequently, $s_2 \in \{q\sqrt{q}, q\sqrt{q}-\sqrt{q}, q\sqrt{q}-q\}$. We observe also that if $s_1>1$, then all non–degenerate Hermitian surfaces in Γ share the same tangent plane at P.

References

[1] J.W.P. Hirschfeld, *Projective geometries over finite fields*, Oxford University Press, 1998
[2] B.C. Kestenband, Unital intersections in finite projective planes, *Geom. Dedicata* **11** (1981), no. 1, 107–117.
[3] L. Giuzzi, Collineation groups of the intersection of two classical unitals, *J. Combin Designs* **9** (2001); 445–459.
[4] L. Giuzzi, *Hermitian varieties over finite fields*, DPHIL Thesis, *University of Sussex.*

Dipartimento di Matematica, Facoltà di Ingegneria, Università degli studi di Brescia, via Valotti 9, 25133 Brescia, Italy.