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ON THE C^∞ INVARIANCE OF THE CANONICAL CLASSES OF CERTAIN ALGEBRAIC SURFACES

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1. The results announced in this article concern certain aspects of the diffeomorphism classification of algebraic surfaces, and in particular, the role of the canonical class. We establish our results by developing a general criterion under which the possibilities for Donaldson's polynomial invariants for smooth 4-manifolds [**2**] are severely limited. We then use these limitations to conclude that in many cases the canonical class of an algebraic surface is a diffeomorphism invariant up to a multiple. Two classes of surfaces satisfying our general criterion are complete intersections and simply connected elliptic surfaces with $p_g \equiv 0 \pmod 2$ (see Corollary 8). A third class of such surfaces are certain abelian branched coverings of $\mathbf{C}P^1 \times \mathbf{C}P^1$ which are surfaces of general type (see §4). These latter surfaces provide infinitely many examples of pairs of homeomorphic, nondiffeomorphic, simply connected surfaces of general type.

§2 gives a brief review of the part of Donaldson's theory needed for what we do here. The material described in §3 represents the work of the first and last authors and is some evidence for a general conjecture described in [**5**]. The material described in §4 represents the work of the second author and will be explained in detail in [**6**].

The authors owe much to Simon Donaldson. Not only were we inspired by his general theory, but also the earliest versions of Theorem 5 were worked out jointly with him. It is a pleasure to express our gratitude to him.

2. Donaldson's polynomial invariants. In this section M denotes a closed, smooth, oriented, simply connected 4-manifold; $q_M\colon H_2(M;\mathbf{Z}) \otimes H_2(M;\mathbf{Z}) \to \mathbf{Z}$ is its intersection form; and $b_2^{\pm}(M)$ is the rank of any maximal subspace of $H_2(M;\mathbf{Z})$ on which q_M is positive (negative) definite. We assume that $b_2^+(M) = 2p+1$ with $p \geq 1$. For such a manifold and for all n sufficiently large, Donaldson has defined invariants $\gamma_n(M) \in \mathrm{Sym}^{d(n)/2}(H_2(M)^*)$ where $d(n) = 8n-3(b_2^+(M)+1)$. More explicitly, for $n > 0$ let $P_n \to M$ be the unique principal SU(2)-bundle with $c_2(P_n) = n$. Let $\mathcal{X}$ be the space of equivalence classes of irreducible connections on P_n under the action of the gauge group of P_n. Let g be a generic riemannian metric on M, and let $\mathcal{M}_n \subset \mathcal{X}$ be the space of equivalence classes of connections on P_n anti-self-dual with respect to g. It is a smooth manifold (usually noncompact) of dimension $d(n)$. Over $M \times \mathcal{M}_n$ there is a universal U(2)-bundle ξ_n. We have $c_2(\xi_n) \in H^4(M \times \mathcal{M}_n;\mathbf{Z})$, and we use this class to define $\mu\colon H_2(M) \to H^2(\mathcal{M}_n)$ by $\mu(\alpha) = c_2(\xi_n)\backslash\alpha$.

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Donaldson has constructed a compactification $\overline{\mathcal{M}}_n$ of $\mathcal{M}_n$ by adding strata at infinity built out of the moduli spaces $\mathcal{M}_l$ for $l < n$. If n is sufficiently large then $\overline{\mathcal{M}}_n$ carries a top class $[\overline{\mathcal{M}}_n] \in H_{d(n)}(\overline{\mathcal{M}}_n; \mathbf{Z})$. Furthermore, the classes $\mu(\alpha)$ extend to classes which we also denote by $\mu(\alpha) \in H^2(\overline{\mathcal{M}}_n)$. We define $\gamma_n(M)$: $\mathrm{Sym}^{d(n)/2}(H_2(M)) \to \mathbf{Z}$ by

$$\gamma_n(M)(\alpha_1, \ldots, \alpha_{d(n)/2}) = \langle \mu(\alpha_1) \cup \cdots \cup \mu(\alpha_{d(n)/2}), [\overline{\mathcal{M}}_n] \rangle.$$

The formal properties of $\gamma_n(M)$ are as follows:

THEOREM 1 [2]. (a) *$\gamma_n(M)$ is independent of the choice of metric.*

(b) *$\gamma_n(M)$ is natural up to sign under orientation-preserving diffeomorphisms, i.e., if $f: M \to M'$ is an orientation-preserving diffeomorphism, then $f^*(\gamma_n(M')) = \pm\gamma_n(M)$.*

(c) *If M is an algebraic surface, then $\gamma_n(M) \neq 0$ for all n sufficiently large.*

3. Algebraic surfaces with large diffeomorphism groups. In this section S is an algebraic surface, and k_S is the first chern class of the canonical bundle K_S over S. Clearly, we can view $k_S \in \mathrm{Sym}^1(H_2(S)^*)$ and $q_S \in \mathrm{Sym}^2(H_2(S)^*)$. Let $O(q_S)$ (resp., $\mathrm{SO}(q_S)$) be the algebraic subgroup of automorphisms of $H_2(S)$ which preserve q_S (resp., and which have determinant 1). Let $\mathrm{Stab}^0(k_S) = \{\alpha \in \mathrm{SO}(q_S) \mid \alpha(k_S) = k_S\}$. Clearly, all these are algebraic group schemes defined over $\mathbf{Z}$. We denote by $\mathrm{SO}_{\mathbf{Z}}(q_S)$ and $\mathrm{SO}_{\mathbf{C}}(q_S)$ the groups of integral and complex points, and similarly for $O(q_S)$ and $\mathrm{Stab}^0(k_S)$. Let $\mathrm{Diff}_+(S)$ be the component group of the group of orientation-preserving diffeomorphisms. Let ψ: $\mathrm{Diff}_+(S) \to O_{\mathbf{Z}}(q_S)$ be the natural map. If π: $\mathcal{S} \to T$ is a family (i.e., if π is a smooth proper holomorphic map of connected complex spaces) with $\pi^{-1}(t_0) = S$, then we have the monodromy representation ρ: $\pi_1(T, t_0) \to \mathrm{Diff}_+(S)$. The image of $\psi \circ \rho$ preserves k_S.

DEFINITION 2. The family π: $\mathcal{S} \to T$ has *big monodromy* if $\mathrm{Im}(\psi \circ \rho) \cap \mathrm{Stab}^0_{\mathbf{Z}}(k_S)$ is Zariski dense in $\mathrm{Stab}^0_{\mathbf{C}}(k_S)$. The surface S has *big monodromy* if it sits as a fiber in a family with big monodromy.

REMARK 3. If $b_2^+(S) \geq 3$, then it is easy to see that $\mathrm{Stab}^0_{\mathbf{C}}(k_S)$ is a connected group and that $\mathrm{Stab}^0_{\mathbf{Z}}(k_S)$ is Zariski dense. Thus, if $\mathrm{Im}(\psi \circ \rho) \cap \mathrm{Stab}^0_{\mathbf{Z}}(k_S)$ is of finite index in $\mathrm{Stab}^0_{\mathbf{Z}}(k_S)$, then it is Zariski dense in $\mathrm{Stab}^0_{\mathbf{C}}(k_S)$.

The following proposition gives two classes of examples of this phenomenon.

PROPOSITION 4. *Suppose that S is a smooth complete intersection or a minimal, simply connected elliptic surface. Then S has a big monodromy group.*

The result follows from the results in [**3**], cf. also Beauville [**1**].

Now let us see the effect of big monodromy on Donaldson's polynomial invariants.

THEOREM 5. *Let S be a simply connected algebraic surface with $p_g > 0$ and with a big monodromy group. Then for all n sufficiently large, $\gamma_n(S)$ is a polynomial in q_S and k_S, i.e., $\gamma_n(S) \in \mathbf{C}[q_S, k_S] \subset \mathrm{Sym}^*(H_2(S)^*)$.*

The proof is a simple exercise in invariant theory. Applying this result, we conclude:

THEOREM 6. *Let S and S' be algebraic surfaces as in Theorem 5. Suppose that $p_g(S) = p_g(S') \equiv 0 \pmod 2$. Suppose that $f\colon S \to S'$ is an orientation-preserving diffeomorphism. Then $f^*k_{S'} = \lambda(f)k_S$ for some $\lambda(f) \in \mathbf{Q}$. If $S = S'$ or if $k_S^2 \neq 0$, then $\lambda(f) = \pm 1$.*

The key point is that since $p_g \equiv 0 \pmod 2$, we have $d(n)/2 \equiv 1 \pmod 2$. Thus, k_S divides $\gamma_n(S)$ and is its only linear factor up to multiples.

Here are two consequences of Theorem 6.

COROLLARY 7. *Let S and S' be minimal simply connected surfaces with $p_g(S) = p_g(S') \equiv 0 \pmod 2$. Suppose that $f\colon S \to S'$ is an orientation-preserving diffeomorphism.*

(a) *If each of S and S' is a complete intersection or an elliptic surface, then $f^*k_{S'} = \lambda(f)k_S$ for some $\lambda \in \mathbf{Q}$.*

(b) *If S and S' are of general type and have big monodromy groups, then the divisibilities of k_S and $k_{S'}$ in integral cohomology are equal.*

(Notice that in (b) the hypothesis that S is of general type implies that $k_S^2 \neq 0$.)

4. Homeomorphic, nondiffeomorphic surfaces of general type. Here we describe an infinite set of pairs of simply connected surfaces of general type which are homeomorphic but not diffeomorphic. We see that they are homeomorphic by invoking Freedman's result [**4**], and we see that they are not diffeomorphic by Corollary 7. The surfaces in question are all abelian coverings of $X = \mathbf{C}P^1 \times \mathbf{C}P^1$. Let $E \subset \mathbf{C}P^1 \times \mathbf{C}P^1$ be the divisor $\{\mathrm{pt}\} \times \mathbf{C}P^1 + \mathbf{C}P^1 \times \{\mathrm{pt}\}$. For any sequence of positive integers $\{x_1, \dots, x_k\}$ we construct, by induction on k, a branched covering

$$g(x_1, \dots, x_k)\colon X(x_1, \dots, x_k) \to X.$$

Given $g(x_1, \dots, x_{k-1})\colon X(x_1, \dots, x_{k-1}) \to X$, let

$$f(x_k)\colon X(x_1, \dots, x_k) \to X(x_1, \dots, x_{k-1})$$

be the cyclic 3-sheeted covering branched over a nonsingular divisor $B_{x_k} \subset X(x_1, \dots, x_{k-1})$ linearly equivalent to $g(x_1, \dots, x_{k-1})^*(3x_k \cdot E)$. We set $g(x_1, \dots, x_k) = g(x_1, \dots, x_{k-1}) \circ f(x_k)$. One establishes easily by induction on k that $X(x_1, \dots, x_k)$ is simply connected, that $p_g(X(x_1, \dots, x_k))$ is even, that the divisibility of $k_{X(x_1, \dots, x_k)}$ is $2(\sum_{i=1}^k x_i - 1)$, and that $X(x_1, \dots, x_k)$ and $X(y_1, \dots, y_l)$ are orientation-preserving homeomorphic provided that (i) $k \equiv l \pmod 2$, (ii) $\sum_{i=1}^k x_i = 3^{(l-k)/2} \sum_{j=1}^l y_j - 3^{(l-k)/2} + 1$, and (iii) $\sum_{i=1}^k x_i^2 = 3^{(l-k)} \sum_{j=1}^l y_j^2$.

From Proposition-Example 15 of [**6**], it follows that if $x_{i_0} \geq 2$ for some i_0, $1 \leq i_0 \leq k$, then $X(x_1, \dots, x_k)$ has a big monodromy group. We now have the following result.

THEOREM 8. *For any positive integers* $(z_1, \ldots, z_m)$ *with* $\sum_{i=1}^m z_i \equiv 0 \pmod 2$, *the surfaces* $X(1,1,1,1,6,z_1,\ldots,z_m)$ *and* $X(2,10,16,3z_1,\ldots,3z_m)$ *are homeomorphic but not diffeomorphic.*

Previously, examples of orientation-reversing homeomorphic, nondiffeomorphic surfaces were worked out by Donaldson and the second author.

REFERENCES

1. A. Beauville, *Le groupe de monodromie des familles universelles d'hypersurfaces et d'intersections complètes*, Complex Analysis and Algebraic Geometry (H. Grauert ed.), Lecture Notes in Math., vol. 1194, Springer-Verlag, Berlin and New York, 1986.

2. S. Donaldson, *Polynomial invariants for smooth* 4-*manifolds* (in preparation).

3. W. Ebeling, *An arithmetic characterization of the symmetric monodromy groups of singularities*, Invent. Math. **77** (1984), 85–99.

4. M. Freedman, *The topology of four-dimensional manifolds*, J. Differential Geom. **17** (1982), 357–454.

5. R. Friedman and J. Morgan, *Algebraic surfaces and* 4-*manifolds: Some conjectures and speculations*, Bull. Amer. Math. Soc. (N.S.) **18** (1988), 1–19.

6. B. Moishezon, *Analogs of Lefschetz theorems for linear systems with isolated singularities* (to appear).

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