

On the Inverse of Monoidal Transformation

By
Shigeo NAKANO

Introduction

When we have a complex analytic manifold X and a complex analytic submanifold M of codimension $r \geq 2$ of X , we can form the monoidal transform \tilde{X} of X with centre M . (By a manifold, we shall understand a paracompact connected one through this paper.) \tilde{X} is a complex analytic manifold with the same dimension n as X , there exists a holomorphic mapping π from \tilde{X} onto X , and π is an analytic homeomorphism between $X - S$ and $X - M$, where $S = \pi^{-1}(M)$. (More properly, we should say (\tilde{X}, π) is the monoidal transform of X .) S is an analytic submanifold of \tilde{X} of codimension 1, and is in a peculiar position in \tilde{X} : The restriction of π to S makes $\pi: S \rightarrow M$ an analytic fibre bundle with projective $(r-1)$ -space as the standard fibre. (More specifically, S is the normal bundle of M in X , with the zero cross section deleted and "divided" by the group \mathbb{C}^* operating as multiplication by constants on each fibre.) If we denote the fibre $\pi^{-1}(a)$ by $L_a (a \in M)$, then we have $[S]_{L_a} = [e]^{-1}$, where $[S]$ and $[e]$ denote the complex line bundles defined by the divisor S of \tilde{X} and the hyperplane e of $\mathbb{P}^{r-1} = L_a$ respectively, and $[S]_{L_a}$ denotes the restriction of $[S]$ to L_a .

Now the inverse problem of the monoidal transformation is the following: Suppose we have a complex analytic manifold \tilde{X} and a submanifold S of \tilde{X} of codimension 1. Let S have a structure of a holomorphic fibre bundle over an analytic manifold M^m with projective $(r-1)$ -space as a standard fibre ($m+r=n$). Then under what condi-

tion can we find a complex analytic manifold X^n and a holomorphic map $\pi: \tilde{X} \rightarrow X$ so that X contains M as a submanifold and \tilde{X} is the monoidal transform of X with centre M ? It is clear that the above condition $[S]_{L_a} = [e]^{-1}$ is necessary. It was first shown by K. Kodaira that this condition is sufficient in case M is a single point and \tilde{X} is a projective algebraic manifold (Kodaira [4]). In case $M =$ a point, we also have results by H. Grauert [2] and K. Kodaira [5] etc. For M of higher dimension, P. A. Griffiths [3] gave a sufficient condition that the inverse problem can be solved affirmatively. B. G. Mořezon [7] gave a necessary and sufficient condition in case X is compact and M has as many independent meromorphic functions as complex dimension of M . A. Lascu [6] treats this problem in abstract algebraic geometry. In this paper, we shall give a necessary and sufficient condition for the existence of X , namely, we shall prove the

Main Theorem. *Let \tilde{X} be a complex analytic manifold of complex dimension $n \geq 3$ and S an analytic submanifold of \tilde{X} of codimension 1. Suppose that S has a structure of an analytic fibre bundle over an analytic manifold M with a projective $(r-1)$ -space as the standard fibre and that $r > 1$. Denote by L_a the fibre over $a \in M$ in the bundle $S \rightarrow M$, and by $[e]$ the complex line bundle over $L_a \cong \mathbb{P}^{r-1}$ determined by the hyperplane. Then, in order that there exists an n -dimensional analytic manifold X containing M and a holomorphic map $\pi: \tilde{X} \rightarrow X$ in such a way that (\tilde{X}, π) is the monoidal transform of X with centre M and $S = \pi^{-1}(M)$, it is necessary and sufficient that the following conditions are satisfied:*

- (α) *for any $a \in M$, $[S]_{L_a} = [e]^{-1}$,*
- (β) *each L_a has a neighbourhood V (in \tilde{X}) such that $[\mathcal{K}]_V = (S)_V^k$, where \mathcal{K} is the canonical bundle of V and k is a non-negative integer.*

To prove the theorem, we make use of a certain type of cohomology vanishing theorem and this theorem will be proved making use of a variant of a method due to A. Andreotti and E. Vesentini [1].

§1. A Cohomology Vanishing Theorem

1.1. We shall consider the following situation: V is an n -dimensional complex analytic manifold and there exist a real valued C^∞ function Ψ and a complex line bundle \mathcal{B} on V , with the following properties:

(a) Ψ is plurisubharmonic, i. e. for any point x of V , we have

$$\left(\frac{\partial^2 \Psi}{\partial x^j \partial \bar{x}^k}\right) \geq 0 \quad (\text{positive semi-definite}),$$

where (x^j) is a system of analytic local coordinates around x .

(b) For any real value A , the set $\{x | \Psi(x) \leq A\}$ is empty or compact.

(c) \mathcal{B} is negative, this means the following: \mathcal{B} is defined by a system of transition functions $\{e_{\alpha\beta}\}$ with respect to an open covering $\{U_\alpha\}$ of V , and we have a system of positive valued C^∞ functions $\{a_\alpha\}$, a_α being defined in U_α and having the properties

$$a_\alpha / a_\beta = |e_{\alpha\beta}|^2 \quad \text{in } U_\alpha \cap U_\beta,$$

$$\left(\frac{\partial^2 \log a_\alpha}{\partial x^j \partial \bar{x}^k}\right) < 0 \quad \text{at each point of } U_\alpha.$$

(d) The canonical bundle of V is equal to \mathcal{B}^k ($k \geq 0$).

The theorem we want to establish in this section is the following

Theorem 1. *In the above situation, we have $H^q(V, \mathcal{O}(\mathcal{B}^{-\varepsilon})) = 0$ for $\varepsilon = 1, 2, \dots, q = 1, \dots, n-1$.*

The proof by the method of Carleman estimate will be given in the following.

1.2. Let us first establish

Proposition 1. *The system of functions $\{a_\alpha\}$ can be so chosen that the Kähler metric on V given by*

$$(1.1) \quad ds^2 = \sum \frac{\partial^2 \log a_\alpha^{-1}}{\partial x^j \partial \bar{x}^k} dx^j d\bar{x}^k$$

is complete.

Proof. Take a real valued C^∞ function μ of a real variable τ such that $\mu(\tau) \geq 0, \mu'(\tau) \geq 0, \mu''(\tau) \geq 0$ and $\int_0^\infty \sqrt{\mu''(\tau)} d\tau = +\infty$, and replace given a_α by $a'_\alpha = e^{(-\mu(\Psi))} a_\alpha$. The new system $\{a'_\alpha\}$ satisfies the condition (c) too. Define a Kähler metric on V by (1.1) with a'_α instead of a_α on the right hand side.

If V itself is compact, there is nothing to prove. Assume V is not compact, and take a differentiable arc $\gamma: x = x(t) (0 \leq t < 1)$ in V which is not contained in any compact subset of V . Set $f(t) = \Psi(x(t))$, then $f(t_\nu) \rightarrow \infty$ for some sequence $\{t_\nu\}$ of values of t . We have:

$$\begin{aligned} \left(\frac{ds}{dt}\right)^2 &= \sum_{j,k} \left\{ \mu'(f(t)) \frac{\partial^2 \Psi}{\partial x^j \partial \bar{x}^k} + \mu''(f(t)) \frac{\partial \Psi}{\partial x^j} \frac{\partial \Psi}{\partial \bar{x}^k} \right\} \frac{dx^j}{dt} \frac{d\bar{x}^k}{dt} \\ &+ \sum \frac{\partial^2 \log a_\alpha^{-1}}{\partial x^j \partial \bar{x}^k} \frac{dx^j}{dt} \frac{d\bar{x}^k}{dt} \geq \mu''(f(t)) \left| \sum_j \frac{\partial \Psi}{\partial x^j} \frac{dx^j}{dt} \right|^2 \\ &\geq \frac{1}{4} \mu''(f(t)) \left| \frac{df}{dt} \right|^2. \quad (x^j = x^j(x(t))) \end{aligned}$$

Then

$$\int_{t=0}^1 ds \geq \frac{1}{2} \int_0^1 \sqrt{\mu''(f(t))} \left| \frac{df}{dt} \right| dt = +\infty.$$

Therefore the proposition holds if we take a'_α instead of the original a_α .

From now, we shall assume that $\{a_\alpha\}$ is chosen so that (1.1) is a complete Kähler metric, and we shall fix this metric through this section.

1.3. We consider a Hermitian metric $\{b_\alpha\}$ on the fibres of \mathcal{B} , that is to say, b_α is a positive valued C^∞ function on U_α and we have

$$b_\alpha/b_\beta = |e_{\alpha\beta}|^2 \quad \text{in } U_\alpha \cap U_\beta.$$

The system $\{a_\alpha\}$ in 1.1, 1.2 is an example of such metric.

We denote by $\mathcal{D}^{p,q}(\mathcal{B})$ the \mathcal{C} -vector space of all \mathcal{B} -valued differential forms of type (p, q) with compact supports. (a \mathcal{B} -valued differential form φ is a system $\{\varphi_\alpha\}$ of differential forms, φ_α being defined on U_α and satisfying $\varphi_\alpha = e_{\alpha\beta} \varphi_\beta$ in $U_\alpha \cap U_\beta$.) The metric $\{b_\alpha\}$ on the fibres

defines a Hermitian inner product in $\mathcal{D}^{p,q}(\mathcal{B})$

$$(1.2) \quad (\varphi, \psi) = \int_V \frac{1}{b_\alpha} \varphi_{\alpha \wedge} * \bar{\psi}_\alpha \quad \varphi, \psi \in \mathcal{D}^{p,q}(\mathcal{B}),$$

and makes it a pre-Hilbert space.

With respect to this inner product, the adjoint of the operator $\bar{\partial}$: $\mathcal{D}^{p,q}(\mathcal{B}) \rightarrow \mathcal{D}^{p,q+1}(\mathcal{B})$, is given by the operator ϑ :

$$(1.3) \quad (\vartheta\varphi)_\alpha = - * D' * \varphi_\alpha,$$

where

$$(1.4) \quad D'\varphi_\alpha = \partial\varphi_\alpha + \rho_\alpha \wedge \varphi_\alpha,$$

$$(1.5) \quad \rho_\alpha = -\partial \log b_\alpha.$$

D' : $\mathcal{D}^{p,q}(\mathcal{B}) \rightarrow \mathcal{D}^{p+1,q}(\mathcal{B})$ is the (1,0)-part of the exterior differentiation defined in terms of the connection $\{\rho_\alpha\}$ on \mathcal{B} . As usual we define the Laplace-Beltrami operator \square by

$$(1.6) \quad \square = \bar{\partial}\vartheta + \vartheta\bar{\partial}.$$

We define another norm $n(\varphi)$ in $\mathcal{D}^{p,q}(\mathcal{B})$:

$$(1.8) \quad n(\varphi)^2 = (\varphi, \varphi) + (\bar{\partial}\varphi, \bar{\partial}\varphi) + (\vartheta\varphi, \vartheta\varphi).$$

The completions of $\mathcal{D}^{p,q}(\mathcal{B})$ with respect to the norms $(\varphi, \varphi)^{1/2}$ and $n(\varphi)$ are denoted by $\mathcal{L}^{p,q}(\mathcal{B})$ and $W^{p,q}(\mathcal{B})$ respectively.

We shall recall the following propositions:

Proposition 2. $W^{p,q}(\mathcal{B})$ can be considered as the set of elements φ of $\mathcal{L}^{p,q}(\mathcal{B})$ for which $\bar{\partial}\varphi \in \mathcal{L}^{p,q+1}(\mathcal{B})$ and $\vartheta\varphi \in \mathcal{L}^{p,q-1}(\mathcal{B})$. Here $\bar{\partial}$ and ϑ are taken in the sense of distributions.

For the proof we refer the reader to Andreotti-Vesentini [1]. (Proposition 5 of p. 93 in the paper. One can also find detailed exposition in Vesentini [8].) We shall recall that we make use of the completeness of the metric in the proof. We shall further recall that

$$(1.9) \quad (\bar{\partial}\varphi, \bar{\partial}\varphi) + (\vartheta\varphi, \vartheta\varphi) \leq \sigma(\square\varphi, \square\varphi) + \frac{1}{\sigma}(\varphi, \varphi)$$

for any $\sigma > 0$ and for any \mathcal{B} -valued C^∞ form φ such that each term in

the formula is defined. This holds because the metric is complete. (Formula (19) of Andreotti-Vesentini [1], p. 93.) The following proposition is an easy consequence of (1.9).

Proposition 3. *If $\varphi \in \mathcal{L}^{p,q}(\mathcal{B})$ is of class C^∞ and $\square\varphi \in \mathcal{L}^{p,q}(\mathcal{B})$, then φ belongs to $W^{p,q}(\mathcal{B})$. In particular, if $\varphi \in \mathcal{L}^{p,q}(\mathcal{B})$ and $\square\varphi=0$, then $\bar{\partial}\varphi=0$ and $\vartheta\varphi=0$.*

This is the content of Cor. 6 and Prop. 7 of Andreotti-Vesentini [1] (p. 93), the contents of Theorem 1 (p. 89) and Theorem 21 (p. 94) of the paper are given in the following Propositions:

Proposition 4. *If there exists a positive constant C such that*

$$(1.10) \quad (\bar{\partial}\varphi, \bar{\partial}\varphi) + (\vartheta\varphi, \vartheta\varphi) \geq C(\varphi, \varphi)$$

for all $\varphi \in \mathcal{D}^{p,q}(\mathcal{B})$, then for any $\alpha \in \mathcal{L}^{p,q}(\mathcal{B})$, there exists one and only one element x of $W^{p,q}(\mathcal{B})$ satisfying $\square x = \alpha$.

Here $\square x = \alpha$ is to be understood in the sense of weak solution, i. e., for any $\varphi \in \mathcal{D}^{p,q}(\mathcal{B})$ we have $(\bar{\partial}x, \bar{\partial}\varphi) + (\vartheta x, \vartheta\varphi) = (\alpha, \varphi)$. But since \square is elliptic, if α is C^∞ then x is also C^∞ and the equality $\square x = \alpha$ holds in strong sense.

Proposition 5. *With the above notations, if α is C^∞ with $\bar{\partial}\alpha=0$, then $\beta = \vartheta x$ is the unique C^∞ form in $\mathcal{L}^{p,q-1}(\mathcal{B})$ which satisfies $\bar{\partial}\beta = \alpha$ and $\vartheta\beta = 0$.*

We define Green's operator $G: \mathcal{L}^{p,q} \rightarrow W^{p,q}$ by $x = G\alpha$. It is also seen that

$$(G\alpha, G\alpha) \leq C^{-2}(\alpha, \alpha).$$

1.4. We have introduced the connection $\{\rho_\alpha\}$ combined with the metric $\{b_\alpha\}$. This enables one to define the covariant differentiation $D: \mathcal{D}^{p,q}(\mathcal{B}) \rightarrow \mathcal{D}^{p+1,q}(\mathcal{B}) \oplus \mathcal{D}^{p,q+1}(\mathcal{B})$ by $D = D' + \bar{\partial}$, D' being given in (1.4). It is straightforward to verify the relation

$$(1.11) \quad (D^2\varphi)_\alpha = \theta \wedge \varphi_\alpha,$$

where

$$(1.12) \quad \theta = \bar{\partial}\rho_\alpha$$

is a scalar form on V of type (1.1). We set

$$(1.13) \quad \chi = \sqrt{-1}\theta = \sqrt{-1}\bar{\partial}\bar{\partial}\log b_\alpha$$

and call this the curvature form of the connection $\{\rho_\alpha\}$.

Since the metric of the base manifold V is Kähler, we have the relations

$$(1.14) \quad L\delta' - \delta'L = \sqrt{-1}\bar{\partial}, \quad A\bar{\partial} - \bar{\partial}A = -\sqrt{-1}\delta'.$$

(See, for example, A. Weil [9], p. 44. These formulas are proven for scalar forms there, but they can be applied to \mathcal{B} -valued forms too, because the operations are defined for those forms and formulas are of local character.) If we take the adjoint of the first formula of (1.14), we have

$$(1.15) \quad D'A - AD' = -\sqrt{-1}\vartheta.$$

As (1.11) and (1.12) show, we have $D^2 = D'\bar{\partial} + \bar{\partial}D'$. Hence, denoting by $e(\chi)$ the exterior multiplication by χ , we have

$$\begin{aligned} e(\chi)A - Ae(\chi) &= \sqrt{-1}\{(D'\bar{\partial} + \bar{\partial}D')A - A(D'\bar{\partial} + \bar{\partial}D')\}, \\ &= \sqrt{-1}\{D'(\bar{\partial}A - A\bar{\partial}) + D'A\bar{\partial} + \bar{\partial}(D'A - AD') + \bar{\partial}AD'\} \\ &\quad - (AD' - D'A)\bar{\partial} - D'A\bar{\partial} - (A\bar{\partial} - \bar{\partial}A)D' - \bar{\partial}AD'\} \\ &= (\bar{\partial}\vartheta + \vartheta\bar{\partial}) - (D'\delta' + \delta'D'), \end{aligned}$$

that is to say,

$$(1.16) \quad e(\chi)A - Ae(\chi) = \square - *^{-1}\square*,$$

a formula first due to Calabi and Vesentini.

1.5. Now we turn to the proof of Theorem 1. It is enough to show $H_K^{n-q}(V, \mathcal{O}^q(\mathcal{B}^\varepsilon)) = 0$ ($1 \leq q \leq n-1$), because of Serre duality theorem. Here H_K means the cohomology group with compact supports. By condition (d), this is equivalent to saying $H_K^{n-q}(V, \mathcal{O}(\mathcal{B}^{k+\varepsilon})) = 0$.

Set $l = k + \varepsilon (> 0)$ and apply the considerations in 1.3 and 1.4 to

\mathcal{B}' instead of original \mathcal{B} . We take b_α to be equal to a'_α , where $\{a_\alpha\}$ is the system introduced in 1.2. (The meaning of $\{b_\alpha\}$ in (1.2) has now changed according to the change $\mathcal{B} \rightarrow \mathcal{B}'$, but $\{a_\alpha\}$ keeps the original meaning for \mathcal{B} .)

$$\begin{aligned} \rho_\alpha &= \partial \log b_\alpha = l(\partial \log a_\alpha) = l\rho_\alpha^{(1)} \\ \chi &= \sqrt{-1} \partial \rho_\alpha = l(\sqrt{-1} \partial \rho_\alpha^{(1)}) = l\chi^{(1)}. \end{aligned}$$

If we write $\chi^{(1)}$ as $\chi^{(1)} = \sqrt{-1} \sum \chi_{j\bar{k}} dx^j \wedge d\bar{x}^k$, we see $\chi_{j\bar{k}} = \frac{\partial^2 \log a_\alpha}{\partial x^j \partial \bar{x}^k} = -g_{j\bar{k}}$ according to the choice of the Kähler metric on V . Hence $e(\chi^{(1)}) = -L$, $e(\chi) = (-l)L$ and (1.16) takes the form

$$(1.18) \quad \square - *^{-1} \square * = l(AL - LA).$$

If $\varphi \in \mathcal{D}^{0, n-q}(\mathcal{B}')$, then $(AL - LA)\varphi = q\varphi$, and

$$(\bar{\partial}\varphi, \bar{\partial}\varphi) + (\vartheta\varphi, \vartheta\varphi) = (\square\varphi, \varphi) = lq(\varphi, \varphi) + (*^{-1}\square*\varphi, \varphi) \geq lq(\varphi, \varphi)$$

because $(*^{-1}\square*\varphi, \varphi) = (\square*\varphi, *\varphi) \geq 0$. This shows that formula (1.10) holds if we choose $C = lq$.

Intermediate Proposition. For $\varphi \in \mathcal{D}^{0, n-q}(\mathcal{B}')$ with $\bar{\partial}\varphi = 0$, we can find $\psi \in \mathcal{L}^{0, n-q-1}(\mathcal{B}')$ which is C^∞ and satisfies $\bar{\partial}\psi = 0$, $\vartheta\psi = 0$ and $(\psi, \psi) \leq \frac{4}{C}(\varphi, \varphi)$, where $C = lq$. We have the same result for any choice of Hermitian metric on the fibre, provided (1.10) holds with the same value of C .

Proof. By Propositions 4 and 5, we see that $\psi = \vartheta G\varphi$ satisfies the requirement except $(\psi, \psi) \leq \frac{4}{C}(\varphi, \varphi)$. Now since $x = G\varphi$ is in $W^{0,q}(\mathcal{B}')$, (1.10) holds for x too. Hence

$$(\bar{\partial}x, \bar{\partial}x) + (\vartheta x, \vartheta x) \geq C(x, x).$$

Apply (1.9) to x with $\sigma = \frac{C}{2}$, then we have

$$\begin{aligned} (\bar{\partial}x, \bar{\partial}x) + (\vartheta x, \vartheta x) &\leq \frac{2}{C}(\varphi, \varphi) + \frac{C}{2}(x, x) \\ &\leq \frac{2}{C}(\varphi, \varphi) + \frac{1}{2}\{(\bar{\partial}x, \bar{\partial}x) + (\vartheta x, \vartheta x)\}, \end{aligned}$$

whence

$$(\psi, \psi) \leq (\bar{\partial}x, \bar{\partial}x) + (\partial x, \partial x) \leq \frac{4}{C}(\varphi, \varphi).$$

It remains to show that we can take ψ to lie in $\mathcal{D}^{0, n-1}(\mathcal{B}^l)$, because this result means that the \mathcal{B}^c -valued Dolbeault cohomology groups with compact support vanish for type $(0, q)$, $q=1, 2, \dots, n-1$.

1.6. In order to achieve the last point, we proceed as follows: We take a real valued C^∞ function λ of a real variable t , satisfying the conditions

$$\begin{cases} \lambda(t), \lambda'(t), \lambda''(t) \geq 0 \\ \lambda(t) = \lambda'(t) = \lambda''(t) = 0 & \text{for } t > 0 \\ \lambda'(t) > 0 & \text{for } t > 0 \\ \lambda''(t) > 0 & \text{for } 0 < t < 1 \\ \lambda''(t) = 0 & \text{for } 1 \leq t, \end{cases}$$

and introduce new metrics on the fibres of \mathcal{B}^l ;

$$b_{\alpha, \nu} = e^{-\nu\lambda(\Psi-A)} a_\alpha^l \quad (\nu = 1, 2, 3, \dots).$$

Here A is a constant to be determined later. Then the meaning of the inner product changes according as ν changes. We shall use $(\ , \)_\nu$, $\mathcal{L}_\nu^{p, q}$, $W_\nu^{p, q}$, ϑ_ν , $\square_\nu \dots$, for symbols corresponding to the metric $\{b_{\alpha, \nu}\}$, while for original metric $a_\alpha^l = b_{\alpha, 0}$, we shall omit the suffix 0.

The change of the curvature form χ is to be noted. We readily see $\chi_\nu = \chi - \nu\omega$ where $\omega = \sqrt{-1} \partial \bar{\partial} \lambda(\Psi - A)$. Hence (1.16), (1.18) take the form

$$\square_\nu - *^{-1} \square_\nu * = l(\lambda L - L\lambda) + \nu(\lambda e(\omega) - e(\omega)\lambda).$$

If we show

$$(1.19) \quad ((\lambda e(\omega) - e(\omega)\lambda)\varphi, \varphi) \geq 0 \quad \text{for } \varphi \in \mathcal{D}^{0, n-q}(\mathcal{B}^l),$$

we shall have

$$(1.20) \quad (\square_\nu \varphi, \varphi)_\nu \geq lq(\varphi, \varphi) \quad \varphi \in \mathcal{D}^{0, n-q}(\mathcal{B}^l).$$

As for (1.19), we have, in a small neighbourhood V of a given point,

$$\begin{aligned} \bar{\partial}\bar{\partial}\lambda(\psi - A) &= \bar{\partial}\sum_k \lambda'(\psi - A) \frac{\partial\psi}{\partial\bar{x}^k} d\bar{x}^k \\ &= \sum_{j,k} \left\{ \lambda''(\psi - A) \frac{\partial\psi}{\partial x^j} \frac{\partial\psi}{\partial\bar{x}^k} + \lambda'(\psi - A) \frac{\partial^2\psi}{\partial x^j \partial\bar{x}^k} dx^j \wedge d\bar{x}^k \right\}. \end{aligned}$$

The quantities in { } make up a positive semi-definite matrix. Hence if we take an orthonormal basis {θ_j} of linear differential forms on U, we shall have the expressions

$$\begin{aligned} L &= \sqrt{-1} \sum e(\theta_j) e(\bar{\theta}_j), \\ e(\omega) &= \sqrt{-1} \sum w_{j\bar{k}} e(\theta_j) e(\bar{\theta}_k), \end{aligned}$$

where (w_{j \bar{k}}) ≥ 0. We can find a matrix (t_{ij}) such that w_{j \bar{k}} = ∑ t_{ij} t_{ik}. The adjoint of e(θ_j) is denoted by i(θ_j) and is given by -*e(θ_j)*. Because {θ_j} is an orthonormal basis, it can be verified that if A = (a₁, ..., a_r), B = (b₁, ..., b_s) are set of indices, and if θ_{A∧B} denotes θ_{a₁} ∧ ... ∧ θ_{a_r} ∧ θ_{b₁} ∧ ... ∧ θ_{b_s}, we have

$$(1.21) \quad \begin{cases} i(\theta_j)(\theta_{j\wedge A\wedge\bar{\theta}_B}) = \theta_{A\wedge\bar{\theta}_B} \\ i(\theta_j)(\theta_{A\wedge\bar{\theta}_B}) = 0 \end{cases} \quad \text{for } j \notin A,$$

and similarly for i(θ_k). From this we conclude the relations

$$(1.22) \quad \begin{cases} e(\theta_j)i(\theta_m) + i(\theta_m)e(\theta_j) = \delta_{jm} \\ e(\bar{\theta}_k)i(\bar{\theta}_m) + i(\bar{\theta}_m)e(\bar{\theta}_k) = \delta_{km} \\ e(\theta_j)i(\bar{\theta}_k) + i(\bar{\theta}_k)e(\theta_j) = 0 \\ e(\bar{\theta}_j)i(\bar{\theta}_k) + i(\bar{\theta}_k)e(\bar{\theta}_j) = 0. \end{cases}$$

Now Ae(ω) - e(ω)A = ∑_{j,k,m} w_{j \bar{k}} {i(θ_m)i(θ_m)e(θ_j)e(θ_k) - e(θ_j)e(θ_k)i(θ_m)i(θ_m)}

can be calculated according to (1.22) and gives

$$Ae(\omega) - e(\omega)A = \sum w_{j\bar{k}} \{i(\bar{\theta}_j)e(\bar{\theta}_k) - e(\theta_j)i(\theta_k)\}.$$

Hence if φ is a (0, n - q)-form with support contained in U, we have

$$\begin{aligned} ((Ae(\omega) - e(\omega)A)\varphi, \varphi) &= (\sum_{i,j,k} \bar{t}_{ij} i(\theta_j) t_{ik} e(\bar{\theta}_k) \varphi, \varphi) \\ &= \sum_i (\sum_k t_{ik} e(\bar{\theta}_k) \varphi, \sum_j t_{ij} e(\bar{\theta}_j) \varphi) \geq 0. \end{aligned}$$

This is enough to establish (1.19).

So we have shown (1.20) with the constant lq which is independent

of φ and ν .

1.7. Now consider $\varphi \in \mathcal{D}^{0, n-q}(\mathcal{B}^l)$ with $\bar{\partial}\varphi = 0$. Set $K = \text{supp } \varphi$, $A = \max_{x \in K} \Psi(x)$. We take $b_{\alpha, \nu}$ with this value of A , and apply the Intermediate Proposition to this case. Then there exists $\psi_\nu \in \mathcal{L}_\nu^{0, n-q-1}(\mathcal{B}^l)$ such that $\partial_\nu \psi_\nu = 0$, $\bar{\partial}\psi_\nu = \varphi$ and $(\psi_\nu, \psi_\nu)_\nu \leq \frac{4}{lq}(\varphi, \varphi)_\nu$. Because $b_{\alpha, \nu} \leq a'_\alpha$ and equality holds on the support of φ ,

$$(1.23) \quad \frac{4}{lq}(\varphi, \varphi) = \frac{4}{lq}(\varphi, \varphi)_\nu \geq (\psi_\nu, \psi_\nu)_\nu \geq (\psi_\nu, \psi_\nu),$$

and $\{\psi_\nu\}_{\nu=1,2,\dots}$ form a bounded set in $\mathcal{L}^{0, n-q-1}(\mathcal{B}^l)$. Hence, if we choose a suitable subsequence if necessary, ψ_ν converges to an element $\psi \in \mathcal{L}^{0, n-q-1}(\mathcal{B}^l)$ weakly. We have $(\psi, \psi) \leq \frac{4}{lq}(\varphi, \varphi)$ and $\bar{\partial}\psi = \varphi$ in the sense of distribution.

Set $M = \{x \in V \mid \Psi(x) > A + 1\}$, then from (1.23) it follows that

$$(1.24) \quad \int_M a_\alpha^{-l} e^{\nu\lambda(\Psi-A)} (\psi_\nu)_\alpha \wedge * \overline{(\psi_\nu)_\alpha} \leq (\psi_\nu, \psi_\nu)_\nu \leq \frac{4}{lq}(\varphi, \varphi).$$

On the other hand, there exists a positive constant c such that $\lambda(\Psi - A) \geq c$ on M . Hence the left hand side of (1.24) is not less than $e^{\nu c} \int_M a_\alpha^{-l} (\psi_\nu)_\alpha \wedge * \overline{(\psi_\nu)_\alpha}$. This shows that

$$\int_M a_\alpha^{-l} (\psi_\nu)_\alpha \wedge * \overline{(\psi_\nu)_\alpha} \rightarrow 0 \quad \text{for } \nu \rightarrow \infty.$$

If we take an element $u \in \mathcal{D}^{0, n-q-1}(\mathcal{B}^l)$ with support contained in M , then it follows from the above that $(\psi, u) = \lim_{\nu \rightarrow \infty} (\psi_\nu, u) = 0$. Hence the support of ψ as a distribution is contained in the compact set $V - M$.

Finally, we can apply to ψ the regularization process as given in Andreotti-Vesentini [1], Lemma 12 p. 97, and obtain a form $\psi_0 \in \mathcal{D}^{0, n-q-1}(\mathcal{B}^l)$ with $\bar{\partial}\psi_0 = \varphi$. This completes the proof of Theorem 1.

§2. Construction of a Plurisubharmonic Function

2.1. We consider an n -dimensional complex analytic manifold \tilde{X} and a submanifold S of codimension 1, and assume that the condition (α)

in the Main Theorem is satisfied. S is an analytic bundle of projective spaces \mathbf{P}^{r-1} over an analytic manifold M^m with $m+r=n$. Take a point a of M and a small coordinate neighbourhood D of a . D can be considered as the domain $\{(\zeta^1, \dots, \zeta^m) \in \mathbf{C}^m \mid \sum |\zeta^j|^2 < 1\}$ in \mathbf{C}^m , the point a corresponding to the origin and the part $\pi^{-1}(D)$ of S which lies over D has the form $\pi^{-1}(D) \cong D \times \mathbf{P}^{r-1}$. We take a set of homogeneous coordinates $(\gamma^1, \dots, \gamma^r)$ on \mathbf{P}^{r-1} . Then \mathbf{P} is covered by coordinate neighbourhoods $\{U_\alpha\}$; $U_\alpha = \{\gamma^\alpha \neq 0\}$, and the set of inhomogeneous coordinates $\{\xi_\alpha^\gamma = \gamma^\gamma / \gamma^\alpha \mid \gamma = 1, \dots, \hat{\alpha}, \dots, r\}$ is a set of local coordinates in U_α . (When we make use of the notation $\xi_{\alpha\alpha}^\alpha$, it is the constant 1.) Because $[S]_b = [e]^{-1}$ for any $b \in D$, $[S]_{\pi^{-1}(D)} = q^*[e]^{-1}$, where q is the canonical projection $D \times \mathbf{P}^{r-1} \rightarrow \mathbf{P}^{r-1}$ and $[e]$ is the complex line bundle on \mathbf{P}^{r-1} determined by a hyperplane.

$[e]$ on \mathbf{P}^{r-1} can be defined with respect to the open covering $\{U_\alpha\}$ by transition functions

$$(2.1) \quad \varepsilon_{\alpha\beta} = \gamma^\beta / \gamma^\alpha = \xi_{\alpha\beta}^\beta \quad \text{in } U_\alpha \cap U_\beta.$$

As the metric on the fibre, we take

$$(2.2) \quad a_\alpha^{(0)} = \sum_\gamma |\gamma^\gamma|^2 / |\gamma^\alpha|^2 = \sum_\gamma |\xi_\alpha^\gamma|^2.$$

The curvature form of this metric is the Kähler form associated to the standard Hodge metric of \mathbf{P}^{r-1} .

On $D \times \mathbf{P}$, the bundle $q^*[e]$, which we shall now denote by $[e]$, is defined by (2.1) with respect to $\{D \times U_\alpha\}$. We set

$$(2.3) \quad \phi(\xi) = \sum |\xi^j|^2$$

and take

$$(2.4) \quad a_\alpha = e^{\phi(\xi)} a_\alpha^{(0)}$$

as the metric on the fibres. The curvature form of this metric is again positive definite.

2.2. Local coordinates on $\pi^{-1}(D)$ can be considered as restrictions of local coordinates on \tilde{X} . More precisely, we can choose a finite open covering $\{U'_\lambda\}_{\lambda \in A}$ of \mathbf{P} which refines $\{U_\alpha\}$, open sets V'_λ on \tilde{X} such

that $V'_\lambda \cap S = D \times U'_\lambda$, and systems of local coordinates $(z'_\lambda, \dots, z''_\lambda, x^1_\lambda, \dots, \hat{x}^{\tau(\lambda)}_\lambda, \dots, x^r_\lambda, y_\lambda)$ on V'_λ in the following manner:

$$(2.5) \quad \tau \text{ is a map } A \rightarrow \{1, \dots, r\} \text{ such that } U'_\lambda \subset U_{\tau(\lambda)}.$$

$$(2.6) \quad z'_\lambda | S = \zeta^j, \quad x^\gamma_\lambda | S = \xi^\gamma_{\tau(\lambda)}$$

$$(2.7) \quad y_\lambda = 0 \text{ is the local equation for } S \text{ in } V'_\lambda.$$

(We set $x^{\tau(\lambda)}_\lambda \equiv 1$. Hence (2.6) holds for $\gamma = \tau(\lambda)$ too.)

Furthermore we may assume that the transition function $e_{\lambda\mu} = y_\lambda / y_\mu$ for $[S]$ satisfies

$$(2.8) \quad e_{\lambda\mu} | S = \varepsilon_{\lambda\mu}^{-1},$$

where we write $\varepsilon_{\lambda\mu}$ instead of $\varepsilon_{\tau(\lambda), \tau(\mu)}$. Also we set $a_\lambda = a_{\tau(\lambda)}$.

For the moment, we fix the index j and write z_λ instead of z'_λ . Then $z_\lambda - z_\mu$ is a holomorphic function on $V'_\lambda \cap V'_\mu$ and is zero on S . We set

$$(2.9) \quad f_{\lambda\mu} = y_\lambda^{-1}(z_\lambda - z_\mu),$$

then we have

$$f_{\lambda\nu} = f_{\lambda\mu} + e_{\lambda\mu}^{-1} f_{\mu\nu} \quad \text{in } V'_\lambda \cap V'_\mu \cap V'_\nu.$$

In other words $\{f_{\lambda\mu}\}$ is a 1-cocycle with values in $\mathcal{O}([S]^{-1})$, with respect to the open covering $\mathfrak{B} = \{V'_\lambda\}$ of $V' = \bigcup_\lambda V'_\lambda$. The restrictions $\varphi_{\lambda\mu} = f_{\lambda\mu} | S$ define an element of $Z^1(\mathfrak{U}, \mathcal{O}([e]))$, where $\mathfrak{U} = \{D \times U'_\lambda\}$. Since $H^1(\mathbf{P}^{r-1}, \mathcal{O}([e])) = 0$, we can find holomorphic functions φ_λ on $D \times U'_\lambda$ such that

$$\varphi_\lambda - \varepsilon_{\lambda\mu} \varphi_\mu = \varphi_{\lambda\mu} \quad \text{on } D \times (U'_\lambda \cap U'_\mu).$$

We extend φ_λ to U'_λ and denote it by f_λ , then $z'_\lambda = z_\lambda - y_\lambda f_\lambda$ has the property

$$(2.10) \quad z'_\lambda - z'_\mu \equiv 0 \pmod{y_\lambda^2} \quad \text{in } V'_\lambda \cap V'_\mu.$$

We set $f'_{\lambda\mu} = y_\lambda^{-2}(z'_\lambda - z'_\mu)$ and proceed as before. This time we make use of $H^1(\mathbf{P}^{r-1}, \mathcal{O}([e]^2)) = 0$, and arrive at $z''_\lambda = z'_\lambda - y_\lambda^2 f'_\lambda$ with $z''_\lambda - z''_\mu \equiv 0 \pmod{y_\lambda^3}$. Since $H^1(\mathbf{P}^{r-1}, \mathcal{O}([e]^k)) = 0$ for any $k > 0$, we can thus proceed as far as we want. Similar procedure can be applied to x^α_λ for

any fixed α to obtain an approximate holomorphic section of $[S]^{-1}$. Hence,

Proposition 6. *Given a positive integer l , we may take local coordinates $(z_\lambda, x_\lambda, y_\lambda)$ in such a way that the following hold in addition to (2.6) and (2.7).*

$$(2.11) \quad \begin{cases} z_\lambda^j - z_\mu^j = (y_\lambda)^j f_{\lambda\mu}^j \\ x_\lambda^\alpha - e_{\lambda\mu}^{-1} x_\mu^\alpha = (y_\lambda)^\alpha g_{\lambda\mu}^\alpha \end{cases} \quad \text{in } V'_\lambda \cap V'_\mu.$$

It may be that we have to replace V'_λ by a smaller open set, in order to secure that $(z_\lambda^j, x_\lambda^\alpha, y_\lambda)$ form a local coordinate system. But any way $V' = \bigcup_\lambda V'_\lambda$ is an open neighbourhood of L_a in \tilde{X} , and it is enough for our purpose.

2.3. Since $\{f_{\lambda\mu}^j\}$, $\{g_{\lambda\mu}^\alpha\}$ are 1-cocycles in $Z^1(\mathfrak{B}, \mathcal{O}([S]^{-1}))$, $Z^1(\mathfrak{B}, \mathcal{O}([S]^{-l-1}))$ respectively, it is clear that we can choose C^∞ functions $f_\lambda^j, g_\lambda^\alpha$ in V'_λ which satisfy

$$\begin{aligned} f_\lambda^j - e_{\lambda\mu}^{-l} f_\mu^j &= f_{\lambda\mu}^j, \\ g_\lambda^\alpha - e_{\lambda\mu}^{-l-1} g_\mu^\alpha &= g_{\lambda\mu}^\alpha. \end{aligned}$$

We set

$$(2.12) \quad \begin{cases} Z^j = z_\lambda^j - (y_\lambda)^j f_\lambda^j = z_\mu^j - (y_\mu)^j f_\mu^j, \\ X_\lambda^\alpha = x_\lambda^\alpha - (y_\lambda)^\alpha g_\lambda^\alpha, \end{cases}$$

then we have

$$(2.13) \quad X_\lambda^\alpha = e_{\lambda\mu}^{-1} X_\mu^\alpha \quad \text{in } V'_\lambda \cap V'_\mu.$$

Define functions A_λ, F and ψ by

$$(2.14) \quad \begin{cases} A_\lambda = e_j^{2|z^j|^2} \left(\sum_{\alpha=1}^r |X_\lambda^\alpha|^2 \right) & \text{in } V'_\lambda, \\ F = A_\lambda \cdot |y_\lambda|^2 = A_\mu \cdot |y_\mu|^2, \end{cases}$$

$$(2.15) \quad \psi = \sum_j |Z^j|^2 + F.$$

Since $A_\lambda = |e_{\lambda\mu}|^{-2} A_\mu$ by (2.13), F is a well-defined function on V' . It is clear that $\sum |Z^j|^2$ and A_λ reduce, on S , to $\phi = \sum |c^j|^2$ and a_λ respectively.

Proposition 7. *Take $l \geq 3$, then ψ is plurisubharmonic on the*

part of V' where F is small enough. More specifically, the Levi form of ψ is not less than a positive definite Hermitian form in $(dz^1 \cdots dz^m, dy)$ on this part.

Proof. We consider the coordinate neighbourhood V'_λ and verify the assertion in this domain. So we shall omit the suffix λ for a moment. For definiteness, let us assume $\tau(\lambda) = r$, hence $(z^1, \dots, z^m, x^1, \dots, x^{r-1}, y)$ is the coordinate system, while in the expression $\sum |X^\alpha|^2$, the summation extends for $\alpha = 1, 2, \dots, r$.

Direct computation shows that

$$\begin{aligned} \frac{\partial^2 \psi}{\partial z^j \partial \bar{z}^k} &= \delta_{jk} + O(|y|), \\ \frac{\partial^2 \psi}{\partial y \partial \bar{y}} &= e^{\sum |z^j|^2} (1 + \sum_{\alpha=1}^{r-1} |x^\alpha|^2) + O(|y|), \\ \frac{\partial^2 \psi}{\partial x^\alpha \partial \bar{x}^\beta} &= e^{\sum |z^j|^2} |y|^2 (\delta_{\alpha\beta} + O(|y|)), \\ \frac{\partial^2 \psi}{\partial y \partial \bar{x}^\beta} &= e^{\sum |z^j|^2} \bar{y} (x^\beta + O(|y|)), \\ \frac{\partial^2 \psi}{\partial y \partial \bar{z}^k} &= e^{\sum |z^j|^2} (1 + \sum_{\alpha=1}^{r-1} |x^\alpha|^2) \{z^k \bar{y} + O(|y|^2)\}, \\ \frac{\partial^2 \psi}{\partial z^j \partial \bar{x}^\beta} &= O(|y|^2), \quad \text{and their conjugates.} \end{aligned}$$

By taking V' smaller if necessary, we may assume that $\sum_{\alpha=1}^{r-1} |x_\alpha|^2$ is bounded in V' , say $\sum |x_\alpha|^2 \leq G$. Choose $\eta > 0$ such that $\eta G < \frac{1}{3}$, then we have

$$\begin{aligned} &\frac{\partial^2 \psi}{\partial y \partial \bar{y}} (dy, d\bar{y}) + \sum_\beta \frac{\partial^2 \psi}{\partial y \partial \bar{x}^\beta} (dy, d\bar{x}^\beta) + \sum_\alpha \frac{\partial^2 \psi}{\partial x^\alpha \partial \bar{y}} (dx^\alpha, d\bar{y}) \\ &\quad + \sum_{\alpha, \beta} \frac{\partial^2 \psi}{\partial x^\alpha \partial \bar{x}^\beta} (dx^\alpha, d\bar{x}^\beta) \\ &= e^{\sum |z^j|^2} \left\{ \sum_{\alpha=1}^{r-1} (1 + \eta)^{-1/2} y dx^\alpha + (1 + \eta)^{1/2} x^\alpha dy \right\}^2 \\ &\quad + (1 - \eta \sum |x_\alpha|^2 + O(|y|)) (dy, d\bar{y}) + \sum \bar{y} a_\beta (dy, d\bar{x}^\beta) \\ &\quad + \sum y \bar{a}_\alpha (dx^\alpha, d\bar{y}) + |y|^2 \sum \frac{\eta}{1 + \eta} (\delta_{\alpha\beta} + b_{\alpha\beta}) (dx^\alpha, d\bar{x}^\beta) \end{aligned}$$

$$\begin{aligned} &\geq e^{2|z|^2} \left\{ \frac{2}{3} (dy, d\bar{y}) + \sum \bar{y} a_{\alpha\beta} (dy, d\bar{x}^\beta) + \sum y \bar{a}_{\alpha\beta} (dx^\alpha, d\bar{y}) \right. \\ &\quad \left. + |y|^2 \sum \left(\frac{\eta}{2} \delta_{\alpha\beta} + b_{\alpha\beta} \right) (dx^\alpha d\bar{x}^\beta) \right\}, \end{aligned}$$

where a_α and $b_{\alpha\beta}$ are quantities of $O(|y|)$. The last expression is ≥ 0 provided $|y|$ is small enough. We can deal with terms containing $\frac{\partial^2 \psi}{\partial y \partial \bar{z}^k}$ etc. similarly, and we see that the Levi form of ψ is equal to the sum of, say $\frac{1}{2} (\sum_j |dz^j|^2 + |dy|^2)$ and a non-negative Hermitian form, if $|y|$ is small enough.

Proposition 8. $\left(-\frac{\partial^2 \log A_\lambda}{\partial x^\alpha \partial \bar{x}^\beta} \right)$ is positive definite for $|y_\lambda|$ small enough.

Proof. This is clear since A_λ reduces to a_λ for $y_\lambda=0$.

Theorem 2. If \tilde{X}, S, M satisfy condition (α) of the Main Theorem, then, for any $a \in M$, we can find a neighbourhood V of L_a in \tilde{X} , such that the conditions $(a), (b), (c)$ in Theorem 1 hold for V and $\mathcal{B} = [S] \big|_V$. If we have condition (β) in addition, then condition (d) holds too.

Proof. We use the notations in the above. We choose a small positive number δ such that Propositions 7 and 8 hold for $V = \{x \in \tilde{X} \mid \psi(x) < \delta\}$ and such that V is relatively compact in V' . If we set $\Psi = (1 - \psi/\delta)^{-1}$, then conditions (a) and (b) are satisfied. If we note $\{e^{-m\Psi} A^{-1}\}$ gives a Hermitian metric on the fibres of $[S]$ for any constant m , and if we take m big enough, we see that condition (c) is satisfied in view of Propositions 7 and 8. The last assertion of the theorem is trivial.

Remark. It is important to note that each fibre of S is either contained in V or does not meet V .

§3. The Proof of the Main Theorem

3.1. First we prove the

Proposition 9. *Let V be an n -dimensional complex analytic manifold and S a submanifold of V of codimension 1. Suppose that S is analytically homeomorphic to $D \times \mathbb{P}^{r-1}$, where D is a domain in \mathbb{C}^m and $m+r=n$, and that $[S]_s = [e]^{-1}$, $[e]$ being the complex line bundle on S defined by $D \times (\text{hyperplane})$. If we have $H^1(V, \mathcal{O}([S]^{-\epsilon})) = 0$ for $\epsilon=1, 2$, then for any point $a \in D$, there exist a neighbourhood W of $L_a (= \text{the submanifold of } S \text{ corresponding to } a \times \mathbb{P})$ in V , and a holomorphic map π from W to $\Delta = \{(z, y) \in \mathbb{C}^m \times \mathbb{C}^r \mid |z^i| < \epsilon, |y^\alpha| < \epsilon\}$ such that (W, π) is the monoidal transform of Δ with centre $\Gamma = \text{the linear variety defined by } y^1 = \dots = y^r = 0$. We can identify Γ with a neighbourhood of a in D , and the restriction of π to S corresponds to the canonical projection $D \times \mathbb{P} \rightarrow D$ by these identifications.*

Proof. Let $(\zeta^1, \dots, \zeta^m)$ be coordinates \mathbb{C}^m which contains D , and let (η^1, \dots, η^r) be a set of homogeneous coordinates on \mathbb{P} . $H^1(V, \mathcal{O}([S]^{-1})) = 0$ implies that the restriction $\Gamma(V, \mathcal{O}) \rightarrow \Gamma(S, \mathcal{O}_S)$ is surjective. Hence there exist holomorphic functions z^1, \dots, z^m on V , whose restrictions on S are ζ^1, \dots, ζ^m . $H^1(V, \mathcal{O}([S]^{-2})) = 0$ implies that $\Gamma(V, \mathcal{O}([S]^{-1})) \xrightarrow{\rho} \Gamma(S, \mathcal{O}([e]))$ is surjective. The latter contains cross sections corresponding to η^1, \dots, η^r . (If we define $[e]$ by transition functions $\varepsilon_{\alpha\beta} = \eta^\beta / \eta^\alpha$ as in §2, the cross section corresponding to η^γ is represented by a system of holomorphic functions $\{\eta^\gamma / \eta^\alpha\}_{\alpha=1, \dots, r}$ satisfying $\eta^\gamma / \eta^\alpha = \varepsilon_{\alpha\beta} (\eta^\gamma / \eta^\beta)$.) Hence we can find cross sections f^1, \dots, f^r which are mapped to η^1, \dots, η^r by ρ .

We may assume we have coordinate neighbourhoods V'_λ in V , such that $\bigcup_\lambda V'_\lambda \supset L_a$, $V'_\lambda \cap S$ has the form $D_1 \times U'_\lambda$, and S is defined in V'_λ by a local equation $y_\lambda = 0$, where y_λ is a member of a local coordinate system in V'_λ . We also assume that (2.5) and (2.8) hold for these data. The cross section f^α can be expressed as a system $\{f^\alpha_\lambda\}$, where f^α_λ is holomorphic in V'_λ and

$$f_\lambda^\alpha = (y_\lambda/y_\mu)^{-1} f_\mu^\alpha \quad \text{in } V'_\lambda \wedge V'_\mu.$$

Hence we can associate the holomorphic function on $V' = \cup V'_\lambda$ which is given by $y_\lambda f_\lambda^\alpha$ in V'_λ . We denote this function by f^α . f^α is in the defining ideal for $S \cap V'$. The fact that the map ρ associates f^α to η^α is now expressed by

$$(3.1) \quad (f^\alpha/y^\lambda)|_S = \eta^\alpha/\eta^{\tau(\lambda)}.$$

Now take \mathbb{C}^n with linear coordinates $(Z^1, \dots, Z^m, Y^1, \dots, Y^r)$ and blow it up with centre \mathbb{C}^m defined by $Y^1 = \dots = Y^r = 0$. We denote the monoidal transform by $\tilde{\mathcal{C}}$, and define an analytic map θ from V' into $\tilde{\mathcal{C}}$ by

$$\theta(\mathbf{x}) = (z^1(\mathbf{x}), \dots, z^m(\mathbf{x}), f^1(\mathbf{x}), \dots, f^r(\mathbf{x})) \times (f^1(\mathbf{x}) : \dots : f^r(\mathbf{x})).$$

In view of (3.1), it is seen that θ maps L_a onto $(0) \times \mathbb{P}^{r-1}$ biholomorphically and the Jacobian of θ does not vanish at every point of L_a . Hence θ maps a neighbourhood W of L_a onto an open set of $\tilde{\mathcal{C}}$ isomorphically. We can assume that this open set is the transform of a domain Δ in \mathbb{C}^n containing the origin, with centre $\Gamma = \Delta \cap \mathbb{C}^m$. The rest of the Proposition is easy to see.

3.2. The proof of the main theorem is now easy. If we have \tilde{X}, S and M satisfying conditions (α) and (β) , then by Theorem 2, there exists a neighbourhood V of L_a for each $a \in M$, satisfying the conditions of Theorem 1 for $\mathcal{B} = [S]_V$. Hence we have the conditions of Proposition 9, and therefore, there exist, for each a , a neighbourhood W_a of L_a in \tilde{X} , a holomorphic map π_a from W_a onto a domain Δ_a in \mathbb{C}^n and a linear variety Γ_a of dimension m in Δ_a , such that $\pi_a: W_a \rightarrow \Delta_a$ is the monoidal transform Δ_a with centre Γ_a .

If $W_a \cap W_b \neq \emptyset$, we consider the diagram

$$\begin{array}{ccc} W_a \supset W_a \cap W_b & \xrightarrow{id} & W_a \cap W_b \subset W_b \\ \pi_a \downarrow & & \downarrow \pi_b \quad \downarrow \pi_b \\ \Delta_a \supset \pi_a(W_a \cap W_b) & \xrightarrow{\phi_{ab}} & \pi_b(W_a \cap W_b) \subset \Delta_b. \end{array}$$

Since π_a and π_b are analytic homeomorphism from $W_a - S$ to $\Delta_a - \Gamma_a$ and

$W_b - S$ to $\Delta_b - \Gamma_b$ respectively, $\phi_{ab} = \pi_b \circ id \circ \pi_a^{-1}$ is defined on $\pi_c(W_a \cap W_b) - \Gamma_a$ and maps it isomorphically onto $\pi_b(W_a \cap W_b) - \Gamma_b$. Because Γ_a is of codimension $r \geq 2$ in Δ_a , ϕ_{ab} can be extended to the whole $\pi_c(W_a \cap W_b)$ as a holomorphic mapping. Since we can interchange a and b , ϕ_{ab} is a biholomorphic homeomorphism from $\pi_a(W_a \cap W_b)$ onto $\pi_b(W_a \cap W_b)$.

Thus we can patch together Δ'_a s and obtain a possibly non-connected analytic manifold X_1 . In this procedure Γ'_a s are patched together and ϕ_{ab} is nothing but the coordinate transformation on M , when we identify Γ_a and Γ_b with coordinate neighbourhoods around a and b respectively. By a standard connectedness argument, we see that X_1 is connected and Γ'_a s form a submanifold biholomorphically homeomorphic to M . We identify this submanifold with M .

The open subsets $\bigcup_{a \in M} W_a - S$ of \tilde{X} and $X_1 - M$ of X_1 are biholomorphically homeomorphic with each other, the mapping being given by π_a on $W_a - S$. Hence we can patch together $\tilde{X} - S$ and X_1 on these parts, and obtain a manifold X . The map π from \tilde{X} to X , defined by $\pi = \pi_a$ on W_a and by $\pi = id.$ on $\tilde{X} - S$ makes X the monoidal transform of \tilde{X} with centre M , and we have $S = \pi^{-1}(M)$. This completes the proof of the sufficiency part of the Main Theorem.

3.3. The necessity part is trivial. As for condition (β) , we remark that, if (\tilde{X}, π) is the monoidal transform of X with centre M of codimension r , then we have $\mathcal{K}_{\tilde{X}} = \pi^* \mathcal{K}_X \otimes [S]^{r-1}$, where $\mathcal{K}_X, \mathcal{K}_{\tilde{X}}$ denote the canonical bundles of X and \tilde{X} respectively.

3.4. In our Theorem, conditions (α) and (β) are certainly necessary and sufficient, but, at the moment, the author doesn't know if these are independent. In fact in the works quoted in the introduction, condition (β) is not explicitly mentioned, and (β) was used in a technical way in our proof. Hence it may be conjectured that the condition (β) will follow from (α) (and with the value $k = r - 1$).

Note Added in Proof (March 15, 1971): After this paper was written, Mr. A. Fujiki and the author noticed that condition (a) alone is sufficient to derive the Main Theorem. This supplement will appear in the coming issue of these *Publications*.

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