choice of ϵ. The usual procedure is to divide the range of integration into two parts, integrate outwards for a solution satisfying one boundary condition, integrate inwards for a solution satisfying the other boundary condition, match the solutions at an intermediate point and adjust ϵ so that the derivatives also agree [1], [2]. The inward integration may be avoided with the procedure described earlier. A convenient way of dividing the range is according to the sign of $f(r)$. For some r, $f(r) < 0$ so that condition (ii) is not satisfied: the procedure described here is not always numerically stable when $f(r) < 0$ [3]; in fact, for some values of i, $| d_i | < 1$. Of a series of standard methods, the Numerov method,

$$y_{n+1} = \left(\left(2 + \frac{10}{12}\ h^2 f_n\right) y_n - \left(1 - \frac{h^2}{12} f_{n-1}\right) y_{n-1} + \frac{h^2}{12}\left(g_{n+1} + 10 g_n + g_{n-1}\right)\right) \Big/ \left(1 - \frac{h^2}{12} f_{n+1}\right), \tag{5}$$

was found to be most accurate in this case, for a given number of evaluations of f for the outward integration. The procedure used successfully was to integrate outwards according to (5) until $f(r) > 0$, then, with the last value computed as a boundary condition, to solve for the "tail" of the wave function by the method described here. The energy adjustment will be the same as before.

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On the Inversion of Sparse Matrices

By A. L. Dulmage and N. S. Mendelsohn

1. Introduction. There are a number of problems in applied mathematics involving many equations in many unknowns, but for which each equation involves only a small fraction of the unknowns. If such problems are linear or are approximated by linearization, one is involved with a matrix, a large proportion of whose entries are zero. To invert such a matrix A it is sometimes advantageous to permute the rows and columns of A yielding PAQ where P and Q are permutation matrices. If

$$PAQ = \begin{vmatrix} A_1 & & & 0 \\ & A_2 & & \\ & & \ddots & \\ * & & & A_r \end{vmatrix}$$

where $A_1, A_2, \cdots A_r$ are square matrices, the problem of inverting PAQ is reduced

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to inverting the blocks A_r followed by matrix multiplication and addition. On permuting the rows and columns of $(PAQ)^{-1}$ one obtains the matrix A^{-1}.

F. Harary [4] gave a method based on the connectivity theory of directed graphs. The blocks A_i turn out to be the matrix representatives of the strong components of an associated directed graph. Harary's method requires that $Q = P^{-1}$, a restriction which is quite unnecessary for matrix inversion. As a result, many matrices which reduce under independent permutations of rows and columns will not reduce if one insists that $Q = P^{-1}$. To remove this latter restriction, the authors have replaced a directed graph by a bipartite graph. The strong components of a directed graph become the irreducible components of a bipartite graph (see [1] for the definition of an irreducible component of a bipartite graph, [2] for the connection between strong components of directed graphs and irreducible components of bipartite graphs).

2. The Method of Reduction. Let A be a square matrix of order n with entries a_{ij}. Associated with A is a bipartite graph K_A with two sets of vertices $S = s_1, s_2, \cdots, s_n$ and $T = t_1, t_2, \cdots, t_n$. A pair (s_i, t_j) is an edge of K_A if and only if $a_{ij} \neq 0$ (one obtains Harary's directed graph if one identifies the vertices s_i and t_i).

Suppose we can find matrices P and Q such that

$$PAQ = \begin{vmatrix} A_1 & & & 0 \\ & A_2 & & \\ & & \ddots & \\ * & & & A_r \end{vmatrix}$$

and such that $A_1, A_2, \cdots, A_r$ cannot be further reduced by permutations of their rows and columns. Then the graphs corresponding to $A_1, A_2, \cdots, A_r$ are the irreducible components of K_A.

In [3], the authors give an algorithm for obtaining the irreducible components of a bipartite graph. This algorithm is easily programmable for machine computation and yields the permutations P and Q. In the case where A is a non-singular matrix the graph K_A has no tails (see [3] for definition of a tail) and the algorithm described in [3] can be considerably simplified, since the steps needed to isolate and identify the tails can be omitted. An alternative procedure is the following. First, locate a nonzero term in the expansion of the determinant of A. This can readily be done using the algorithm of Marshall Hall [5] or Fulkerson and Ford [6]. Next, permute the rows of A until the entries of this term occupy the main diagonal. Call this new matrix A^*. Finally, apply the method of F. Harary given in [4] to A^*.

We append an example:

$$\text{Let } A = \begin{vmatrix} 0 & 0 & 5 & 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 4 \\ 1 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 4 & 0 & 3 & 0 & 0 & 0 \\ 3 & 0 & 1 & 0 & 3 & 2 & 0 \\ 3 & 2 & 0 & 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 4 & 0 & 0 & 0 \end{vmatrix}$$

Note that the corresponding directed graph has the Hamiltonian circuit $1 \to 6 \to 5 \to 3 \to 4 \to 2 \to 7 \to 1$ and so is strongly connected. Hence, for no permutation P does PAP^{-1} reduce. Using the algorithm described in [3] one obtains the permutations $P = (1\ 7\ 2\ 5\ 6\ 4\ 3)$ and $Q = (1)(2\ 3\ 6\ 7\ 5\ 4)$. Applying P and Q to the rows and columns of A, one obtains:

$$PAQ = \left|\begin{array}{cc:ccc:cc} 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 4 & 0 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 3 & 4 & 0 & 0 & 0 & 0 \\ 3 & 0 & 2 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 4 & 0 & 0 \\ \hdashline 3 & 0 & 0 & 3 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 5 & 1 \end{array}\right|$$

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Missing Data Correlation Computations

By R. I. Jennrich

In correlation analysis or in any multivariate analysis based on the computation of a correlation or covariance matrix, the applied statistician often runs into the problem of missing data. To avoid complication in computing the correlation matrix, a complete observation vector is often discarded when only one or more of its components are missing. If a correlation matrix is computed by means of a standard electronic computer program, this procedure is often necessary. A large percentage of data may be thrown away when only a small percentage is missing. This note describes a modification in the standard computing scheme which eliminates this waste of data.

Let $x_{n1}, x_{n2}, \cdots, x_{np}$ denote the p components of the nth observation vector, $n = 1, 2, \cdots, N$. It is customary to add an $n + 1$st component to this vector which is identically equal to one. That is $x_{n,p+1} = 1$. The cross product matrix

$$a_{ij} = \sum_{n=1}^{N} x_{ni}x_{nj} \qquad i = 1, \cdots, p + 1; \quad j = 1, \cdots, p + 1$$

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