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ON THE INVISCID ROLLED-UP STRUCTURE OF LIFT-GENERATED VORTICES

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by

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ABSTRACT

A simple form is presented of the relationships derived by Betz for the inviscid, fully developed structure of lift-generated vortices behind aircraft wings. Betz' method is then extended to arbitrary span-load distributions by inferring guidelines for the selection of rollup centers for the vortex sheet, along with rules for calculating the fully developed structure of the resulting multiple vortices. These techniques yield realistic estimates of the rolled-up structure of vortices produced by a wider variety of span-load distributions than possible with the original form of Betz' theory.

INTRODUCTION

Lift-generated vortices behind aircraft usually consist of two adjacent, well organized, oppositely rotating flow fields that persist longer than ordinary eddies. The amount of time that the region remains hazardous to other encountering aircraft is determined by how soon the vortices dissipate, move, or are blown out of the airspace to be used by a following aircraft. Since these factors are governed by the characteristics of the lift-generated vortices, it is highly desirable to predict, easily and accurately, the structure of the vortex pair for a wide variety of lift configurations. It will then be possible to assess better the potential hazard produced by the wake vortices of one aircraft on a second encountering aircraft and to explore ways to alleviate dangerous situations.

This paper presents several improvements made by the author on such a theory by A. Betz[1] for the vortex structure behind wings. The theory, as presented by Betz, uses the three conservation equations for vortex systems to relate the structure of the vortex sheet behind an isolated wingtip (isolated half span) to the structure of a single, fully developed vortex. Although this theory does not appear to have been used extensively in the past, it has recently been demonstrated by Donaldson[2] to be useful and often more accurate than more complex methods. The favorable publicity given to Betz' method by Donaldson led to an elaboration of the theory and more examples by Mason and Marchman[3] and to use by Brown[4] of the rollup theory to predict the axial flow velocity in the vortex. These papers used the rollup equations in about the same form presented by Betz. A new form of the rollup relationships is derived here that is simpler in form and easier to use. These equations are then applied to several conventional and several not-so-conventional span-load distributions to illustrate the variations in vortex structure that can be produced by various span-load distributions. The method is then generalized to include situations wherein the vortex sheet would roll up into several vortices on each side of the fuselage. Guidelines are then given for selecting likely vortex centers for the beginning of rollup along with a method for calculating the structure of the resulting fully developed vortices. A span-load distribution typical of current large aircraft is then analyzed by use of these techniques to illustrate how these extensions of Betz' theory provide a more reasonable estimate of the rolled-up structure of vortex sheets for a larger variety of span-load distributions.

DERIVATION OF SIMPLIFIED FORM OF ROLLUP EQUATIONS

The three-dimensional shape of the vortex sheet as it rolls up behind a lifting wing is often approximated by considering the sheet at its intersection of the so-called Trefftz plane, which is defined as a plane behind the wing that is perpendicular to the direction of, and that moves with, the freestream. See Fig. 1. This approximation makes it possible to treat the vortex sheet as a two-dimensional, time-dependent calculation without axial flow through the plane in which the motion is assumed to take place. For a modest expenditure of effort, this technique, although approximate, usually yields results that are in good agreement with experimental data.

The Betz method does not treat the transition or intermediate stages between the initial vortex sheet behind the wing and the final rolled-up vortex structure. It simply uses the three conservation relations for two-dimensional vortex distributions to relate the span-load distribution to the fully developed vortex structure. Such a calculation is well suited to analyses of vortex hazards because the vortex sheet is usually completely rolled up within several span lengths behind the aircraft, so that the region in which rollup occurs is not a part of the hazard volume. The inviscid relationships derived by Betz between the initial and final stages of the vortex sheet bypass the difficult calculation of the rollup process and provide the radial distribution of circulation in the trailing vortices. To derive the needed equations, it was necessary to assume not only that the flow was two-dimensional, inviscid, and incompressible, but also that one-half of the vortex sheet rolls up independently of the other half so that the final structure would be axially symmetric. This, in effect, assumes that the portion or segment of the vortex sheet that rolls into a given vortex does so as if it were isolated completely from the rest of the sheet and the other vortices.

To achieve a unique result, Betz also assumed that the rollup process is orderly so that the vorticity shed at the wingtip goes into the center of the vortex located at the spanwise centroid of vorticity. Each inboard portion of the sheet then forms a layer of vorticity around all of the previous wrappings until the entire sheet is rolled around the original center, as indicated in Fig. 1.

In summary, we assume that (1) the flow field is two-dimensional, inviscid, and incompressible and that there are no variations in streamwise velocity, (2) half of vortex sheet is isolated from other half so that developed vortex is axially symmetric, and (3) rollup of vorticity from sheet into vortex is orderly and sequential.

The spanwise variation of lift on the wing, $\ell(y)$, is taken to be represented by

$$\ell(y) = \rho U_\infty \Gamma_w(y) \tag{1}$$

where ρ is the air density, U_∞ the free-stream velocity, and $\Gamma_w(y)$ the spanwise variation of circulation or bound vorticity on the wing. The total lift is of course the integral of $\ell(y)$ over the wing span b. The vorticity shed by such a lift distribution into the wake is given by

$$\gamma_w(y) = -\frac{d\Gamma_w(y)}{dy} \tag{2}$$

and the total circulation on one side, or in one vortex, is given by

$$\Gamma_o = \int_o^{b/2} \gamma_w(y)dy = -\int_o^{b/2} \frac{d\Gamma_w(y)}{dy}\, dy = \Gamma_w(o) \tag{3}$$

since $\Gamma_w(b/2)$, the bound vorticity at the wingtip, is usually zero.

The three conservation laws that relate the circulation in the vortex sheet to that in the fully developed vortex state that:

I. The circulation is conserved,

$$\Gamma_o = \int_o^{b/2} \gamma_w(y)dy = 2\pi \int_o^{r_{max}} r\gamma_v(r)dr$$

or,

$$\Gamma_o = -\int_o^{b/2} \frac{d\Gamma_w(y)}{dy}\, dy = \int_o^{r_{max}} \frac{d\Gamma_v(r)}{dr}\, dr \tag{4}$$

where $\gamma_v(r)$ and $\Gamma_v(r)$ are, respectively, the vorticity and circulation distribution in the fully developed vortex.

II. The centroid of vorticity remains at a fixed spanwise location. That is, the first moment of vorticity is conserved so that the center of the final vortex, $r_1 = 0$, is located at $\bar{y}(0)$, where $\bar{y}(0)$ is the centroid of the portion of the vortex sheet that rolls up into the vortex and is given by

$$\bar{y}(0) = \frac{1}{\Gamma_o} \int_o^{b/2} y\gamma_w(y)dy = \frac{1}{\Gamma_o} \int_{b/2}^{o} y \frac{d\Gamma_w(y)}{dy} dy \tag{5}$$

III. The second moment of vorticity is conserved; $J_v = J_w = J$.

$$J = \int_o^{b/2} [\bar{y}(o) - y]^2 \gamma_w(y)dy = \int_o^{r_{max}} r^2 [2\pi r\gamma_v(r)]dr$$

or

$$J = \int_{b/2}^{o} [\bar{y}(o) - y]^2 \frac{d\Gamma_w(y)}{dy} dy = \int_o^{r_{max}} r^2 \frac{d\Gamma_v(r)}{dr} dr \tag{6}$$

where r_{max} is the radius within which all of the circulation is contained in the fully-developed vortex.

The three global relationships given by Eqs. (4), (5) and (6) are next assumed by Betz to apply piecewise, beginning at the wingtip, to successive portions of the sheet in toward the wing root. These segments of vortex sheet are assumed to be wrapped in the same sequence from the span loading onto the center of the vortex out to r_{max}. The equations that relate the vorticity in the sheet to that in the vortex may then be written as

$$\Gamma_w(y_1) = \int_{b/2}^{y_1} \frac{d\Gamma_w(y)}{dy}\, dy = \Gamma_v(r_1) = \int_0^{r_1} \frac{d\Gamma_v(r)}{dr}\, dr \tag{7}$$

$$\bar{y}(y_1) = \frac{1}{\Gamma_w(y_1)} \int_{b/2}^{y_1} y\, \frac{d\Gamma_w(y)}{dy}\, dy \tag{8}$$

$$J_w(y_1) = \int_{b/2}^{y_1} [\bar{y}(y_1) - y]^2\, \frac{d\Gamma_w(y)}{dy}\, dy$$

$$= J_v(r_1) = \int_0^{r_1} r^2\, \frac{d\Gamma_v(r)}{dr}\, dr \tag{9}$$

where the relationship between the two independent variables, r_1 and y_1, in the rollup process is yet to be determined. It should be noted that Eq. (9) becomes equivalent to Eq. (6) when $y_1 \to 0$. The second moment of each segment was referred to $\bar{y}(y_1)$ rather than $\bar{y}(0)$ so that the circulation shed at the wingtip would enter the vortex at $r_1 = 0$, and successive adjacent segments of the sheet would be wrapped in a continuous manner around previous layers of vorticity in the developed vortex. Since the rollup is assumed to occur continuously from the wingtip inboard to the wing root, Eqs. (7) and (9) can be applied piecewise so that

$$- \frac{d\Gamma_w(y_1)}{dy_1}\, dy_1 = \frac{d\Gamma_v(r_1)}{dr_1}\, dr_1$$

and

$$- [\bar{y}(y_1) - y_1]^2\, \frac{d\Gamma_w(y_1)}{dy_1}\, dy_1 = r_1^2\, \frac{d\Gamma_v(r_1)}{dr_1}\, dr_1$$

or, these two relationships can be combined to yield

$$r_1 = |\bar{y}(y_1) - y_1| \tag{10}$$

where $r_1 \geq 0$. Eq. (8) for $\bar{y}(y_1)$ can be simplified further by integrating by parts to yield

$$\bar{y}(y_1) = \frac{1}{\Gamma(y_1)} \left\{ y_1 \Gamma_w(y_1) - \frac{b}{2} \Gamma_w \left(\frac{b}{2}\right) - \int_{b/2}^{y_1} \Gamma_w(y) dy \right\}$$

or since $\Gamma_w\left(\frac{b}{2}\right) = 0$ in most cases,

$$\bar{y}(y_1) = y_1 - \frac{1}{\Gamma_w(y_1)} \int_{b/2}^{y_1} \Gamma_w(y) dy \tag{11}$$

The radius at which a given portion of the vortex sheet rolls up is then related to the spanwise coordinate by Eq. (10) as

$$r_1 = \left| \frac{1}{\Gamma_w(y_1)} \int_{b/2}^{y_1} \Gamma_w(y) dy \right| \tag{12}$$

This result expresses the same relationship originally put forth by Betz but it is much simpler in form and easier to use. A consequence of the postulated rollup that is immediately apparent from Eq. (10) is that $r_{max} = \bar{y}(0)$. Since the vortex is axially symmetric, the circumferential velocity is given by the definition of circulation as

$$v_\theta = \Gamma_v(r_1)/2\pi r_1 \tag{13}$$

The foregoing simplified form of the Betz rollup equations permits the evaluation in closed form of the vortex structures for a variety of span-load distributions. For example, the radial variation of circulation and velocity in the vortex for elliptic span loading are [with $\Gamma_v(r_1) = \Gamma_w(y_1)$ and $v_\theta = \Gamma_v(r_1)/2\pi r_1$]

$$\frac{\Gamma_w(y_1)}{\Gamma_o} = \left[1 - \left(\frac{2y_1}{b}\right)^2\right]^{1/2}$$

$$\frac{2r_1}{b} = \frac{\frac{\pi}{4} - \frac{1}{2}\sin^{-1}\frac{2y_1}{b}}{\left[1 - \left(\frac{2y_1}{b}\right)^2\right]^{1/2}} - \frac{y_1}{b}$$

and for parabolic span loading the results are

$$\frac{\Gamma_w(y_1)}{\Gamma_o} = 1 - \left(\frac{2y_1}{b}\right)^2$$

$$\frac{2r}{b} = \frac{2}{3}\,\frac{1 + \frac{2y_1}{b} + \left(\frac{2y_1}{b}\right)^2}{\left(1 + \frac{2y_1}{b}\right)} - \frac{2y_1}{b}$$

which agrees with the result presented by Brown.[4] The expressions for triangular span loading are

$$\frac{\Gamma_w(y_1)}{\Gamma_o} = 1 - \left|\frac{2y_1}{b}\right|$$

$$\frac{2r_1}{b} = \left(1 - \left|\frac{2y_1}{b}\right|\right)/2$$

so,

$$v_\theta = \Gamma_o/\pi$$

where, as mentioned previously, the radial velocity in the vortex is zero and the axial velocity is equal to the freestream value, U_∞.

To gain an understanding of how the structure of the rolled-up vortex changes with variations in span-load distribution, a series of cases were calculated using Eqs. (12) and (13). The function used for the span load is

$$\Gamma_w(y)/\Gamma_o = GAM(Y) = (1 - Y^N)^M \tag{14}$$

where $Y = 2y/b$, $V = bv_\theta/2\Gamma_o$ and $W(z = 0) = bw)_{z=0}/2\Gamma_o$ with $z = 0$ defined as the horizontal plane through the center of the developed vortex. The velocity components are related to the freestream velocity, U_∞, through the lift (or weight of the aircraft) by

$$L = \rho U_\infty \int_{-b/2}^{b/2} \Gamma_w(y)dy \, .$$

The curves in the various parts of Fig. 2 present the spanwise loading, the circulation, and the vertical velocity for the rolled-up vortex for several values of N and M, assuming that rollup begins at the wingtip as postulated in the original Betz theory. Instead of plotting $\Gamma_v(r)/\Gamma_o$ and v_θ as a function of radius $R = 2r/b$, the results are presented at their position on the $Y = 2y/b$ axis to better illustrate their spanwise location. The curves for velocity in these figures, and in the ones to follow, terminate at the edges of the region where vorticity is located and so do not present the velocity in the irrotational part of the flow field. Since Eq. (14) can be integrated in closed form for only a few values of N and M, the results shown in Fig. 2

were obtained by integrating Eq. (12) numerically. The double-valued portions (solid curve) of the results in Fig. 2e are a consequence of choosing the incorrect rollup center in the original vortex sheet. To eliminate these unrealistic results, guidelines are inferred in the next section of this paper for the rollup centers in order to yield a better estimate of the structure of fully developed vortices.

EXTENSION TO ARBITRARY SPAN LOADINGS

The double-valued character of the circulation and of the velocity in the developed vortex shown in Fig. 2e suggests that something is incorrect in Betz' theory, and a broader more general set of rules is required if realistic results are to be achieved consistently. Experience with, and numerical calculation of, the rollup of vortex sheets has shown that the beginning point for rollup in the sheet is not always at a given end (say the wingtip) but rather that it can occur at either end of the vortex sheet or somewhere in the middle. In this section, several approximate criteria are presented for finding suitable spanwise locations for these beginning points of rollup. Rules are then given for using the equations in the previous section to calculate the structure of the one or more vortices that result from these postulated rollup starting points.

An exact evaluation of the starting points for rollup would require a calculation of the complete time history of the change of the originally flat vortex sheet into the final individual vortices. This is probably unnecessary for most cases because time-dependent calculations of the rollup indicate that the centers of the vortices appear to form closely behind the wing and not to be altered greatly, if at all, as rollup proceeds to completion. Hence, an estimate based on the initial downwash velocity at the wing trailing edge would probably be a sufficient indicator for most purposes. It is, therefore, necessary to derive an expression that relates the downwash velocity $(-w(y))$ at the wing to the span-load distribution, and then to infer guidelines for choosing locations for the beginning of rollup that will conform with other ways for identifying these centers of rollup.

Since the vortex sheet is taken as flat when it leaves the wing trailing edge, the vertical velocity of the sheet is given by

$$w(y) = + \frac{1}{2\pi} \int_{-b/2}^{+b/2} \frac{\gamma_w(\eta)d\eta}{y - \eta} \tag{15}$$

Since the integral cannot be evaluated in the closed form for the general case, an estimate of the relationship between $w(y)$ and $\gamma_w(y)\{ = -[d\Gamma_w(g)/dy]\}$ can be obtained by expanding Eq. (15) in the series

$$w(y) = -\frac{1}{2\pi}\left\{\gamma_w(y)\ \ln\left(\frac{\frac{b}{2} - y}{\frac{b}{2} + y}\right) + \sum_{n=1}^{\infty} \frac{\left(\frac{b}{2} - y\right)^n - \left(-\frac{b}{2} - y\right)^n}{n \cdot n!}\ \frac{d^n\gamma_w(y)}{dy^n}\right\} \tag{16}$$

The most common sites for the beginning of rollup are at places where the vertical velocity of the sheet $w(y)$ changes sign or changes abruptly. Eq. (16) indicates that these locations are tied to the strength of the vortex sheet $\gamma_w(y)$ and its derivatives. Hence, vortices originate at those places on the sheet where $d\gamma_w(y)/dy$ is at maximum or changes abruptly. For example, $d\gamma_w(y)/dy$ is infinite at the wingtip for elliptic loading

(γ_w is also infinite there). For the triangularly loaded case, the derivative $d\gamma_w(y)/dy$ is infinite at both wingtips and at midspan $(y = 0)$ because $\gamma_w(y)$ is discontinuous there. It is known from experiments and from time-dependent calculations that these are the sites for the beginning of the rollup process. In Figs. 2e and 2f, the strength of the inboard or wing root portion of the vortex sheet is at maximum and the vortex strength would, in fact, be discontinuous across the junction of the left and right wing, so that rollup should begin at center span rather than at the wingtip in both of those cases. The parabolic and contoured loading shown in Figs. 2b and 2d have their maximum values of γ_w, respectively, at the tip and about half-way out (i.e., at $y/_{b/2} = 0.4444$). As already mentioned for parabolic loading, the wingtip is chosen as the rollup site because $\gamma_w(y)$ is discontinuous there, being finite for $y \leq \frac{b}{2}$ and zero for $y > \frac{b}{2}$.

If two or more vortices are produced on each side of the wing, the vortex sheet is divided into rollup segments at those places where $\gamma_w(y)$ or its derivative is zero. In triangular loading, the sheet is divided at $y = b/4$, so that the two vortices are of equal strength. Such a division is exact if the other half of the wing does not influence the rollup.

These considerations indicate that only the two curves shown in Figs. 2a and 2b for elliptic and parabolic span loading are correct; the others are incorrect because, in those four cases, the wingtip is either the incorrect rollup starting point or more than one vortex per side occurs. Correct application of the rollup Eqs. (12) and (13) involves changing the integration in Eq. (12) to begin at the correct rollup site, $y = y_B$, on the sheet rather than at the wingtip, $y = b/2$. That is, Eq. (12), in the general case, should be written as,

$$r_1 = \left| \frac{1}{\Gamma_w(y_1) - \Gamma_w(y_B)} \int_{y_B}^{y_1} [\Gamma_w(y) - \Gamma_w(y_B)]dy \right| \tag{17}$$

If the rollup site occurs at midpoint in the sheet, Eq. (17) is applied to the two segments of the sheet separately, and the two resulting curves for $\Gamma_v(r_1)$ are added. That is, it is assumed that the two parts of sheet roll up within one another without interacting. The final variation of $\Gamma_v(r_1)[= \Gamma_{left}(r_1) + \Gamma_{right}(r_1)]$ is then used in Eq. (13) for the calculation of the velocity. The center of the vortex is, of course, located at the centroid for the entire portion of the sheet that is rolled into the vortex.

Fig. 3 shows the application of these techniques to the four cases that do not have the entire rollup beginning at the wingtip. These new rules produce results that are more realistic and that are in better agreement with other theoretical and experimental findings.

The cases presented in Figs. 2 and 3 may each be considered parts or segments of the vortex sheet that trails behind the more general span loadings that occur on present day aircraft. A calculation of the vortices to be expected far behind a current large aircraft with flaps deflected is shown in Fig. 4. It was made by first choosing rollup sites and then dividing the span-loading or vortex sheet into the segments that enter each vortex. The rules presented for Eq. (17) are then used according to whether the rollup site is at an end or at the center of the segment in order to calculate the radial variation of the circulation and velocity for each vortex.

Since these calculations are made as if each vortex were isolated from all the others, overlap of the various vorticity distributions occurs in both of the cases shown in Fig. 4. As before, the velocity is shown only for that region where vorticity is nonzero and as if the vortex were isolated from all the others. An approximate correction for overlap of the individual vortices is to superimpose all of the individual vortex fields so that non-axially symmetric streamlines are obtained. The streamlines in their new locations each possess the same vorticity as before superposition, so that the centroids are no longer necessarily located at the centers of the vortices. This refinement usually ignores this shift and also ignores any relative motion of the vortices that occurs as they orbit about one another during their development and after they are fully formed. The superimposed flow field would then be applicable at only one instant of time, if at all. For these reasons, it is felt that the effort required to superimpose the vortex flow fields and to calculate new streamline paths is not warranted.

CONCLUDING REMARKS

The simplified form of the Betz rollup equations derived and the extensions suggested in this paper make it possible to estimate easily the structure of vortices that trail behind wings with arbitrary span-load distributions. The rules inferred for subdividing the vortex sheet into separate segments and for identifying the beginning points of rollup for each segment can be summarized as follows:

1. Vortex rollup sites are located at maxima of sheet strength and at abrupt changes in sheet strength.

2. The edges of the segment of vortex sheet that rolls into a vortex occur where the sheet strength vanishes or changes sign, or where the sheet strength is at minimum.

These improvements in the estimate for the rollup structure of vortex sheets are still approximate in that the interaction of the vortices with one another is ignored, along with viscosity and variations in the axial velocity. Also, it is assumed that vorticity from the sheet enters the vortex in sequence from its position relative to its neighbors in the sheet, so that orbiting or interchanging of positions of elements of the sheet is ruled out.

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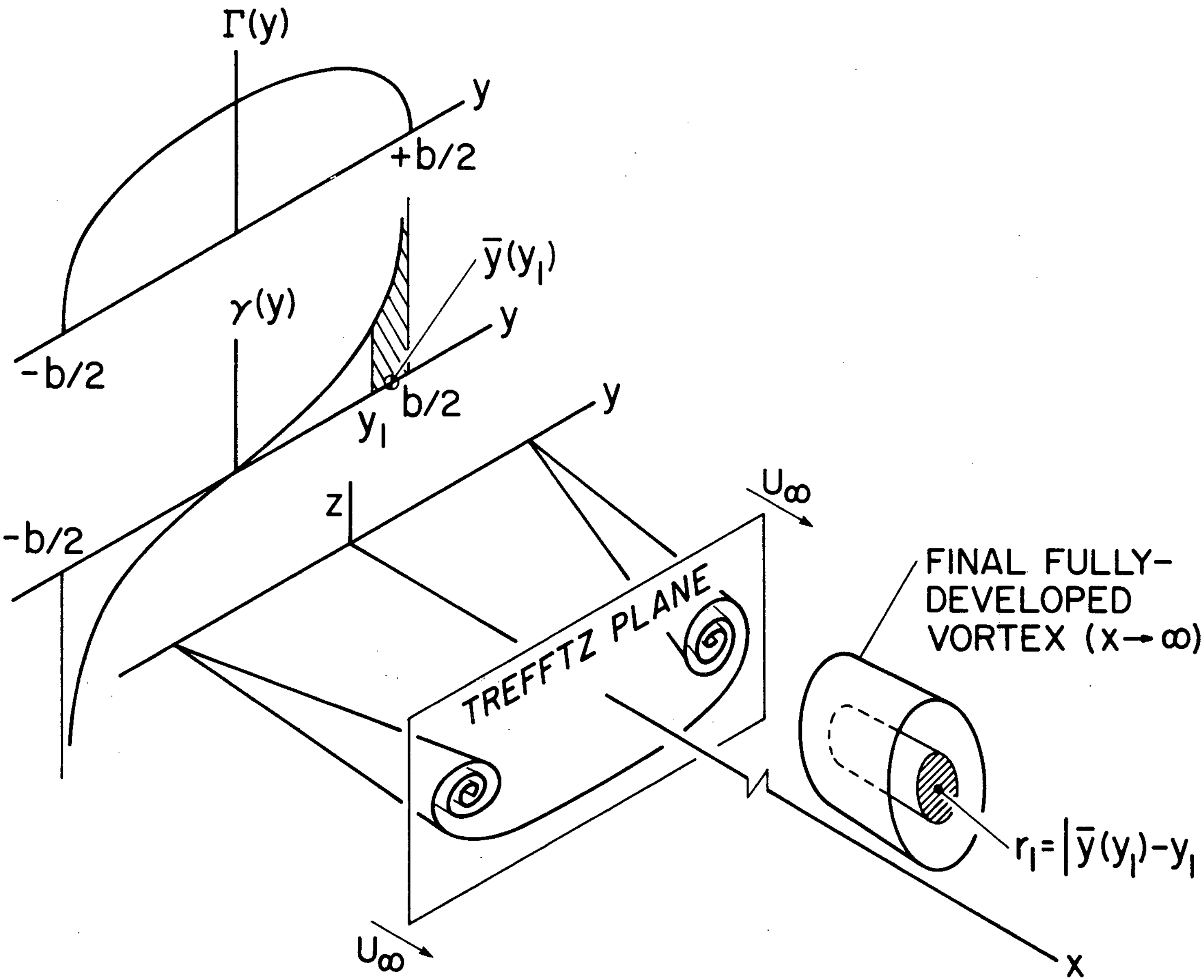


Figure 1.—Schematic diagram of relationship between span loading, $\Gamma_w(y)$, vortex sheet, $\gamma_w(y)$, Trefftz plane, and final rolled-up vortex for one side.

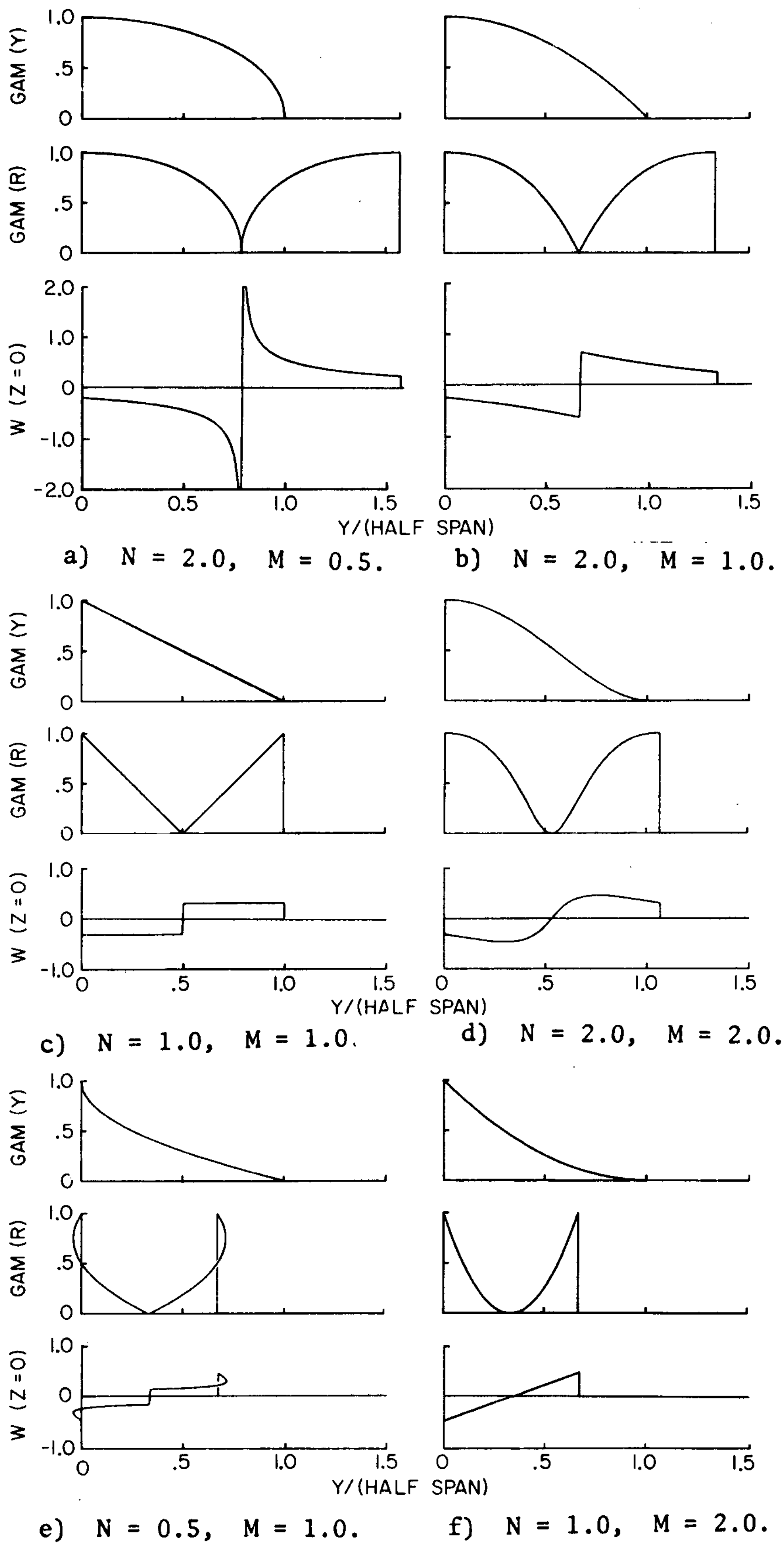


Figure 2.—Structure of fully developed vortex assuming that rollup begins at the wingtip end of the vortex sheet (Eq. (12)) for various span loadings represented by $\Gamma_w(y)/\Gamma_o = \left[1 - \left(\frac{2y}{b}\right)^N\right]^M$.

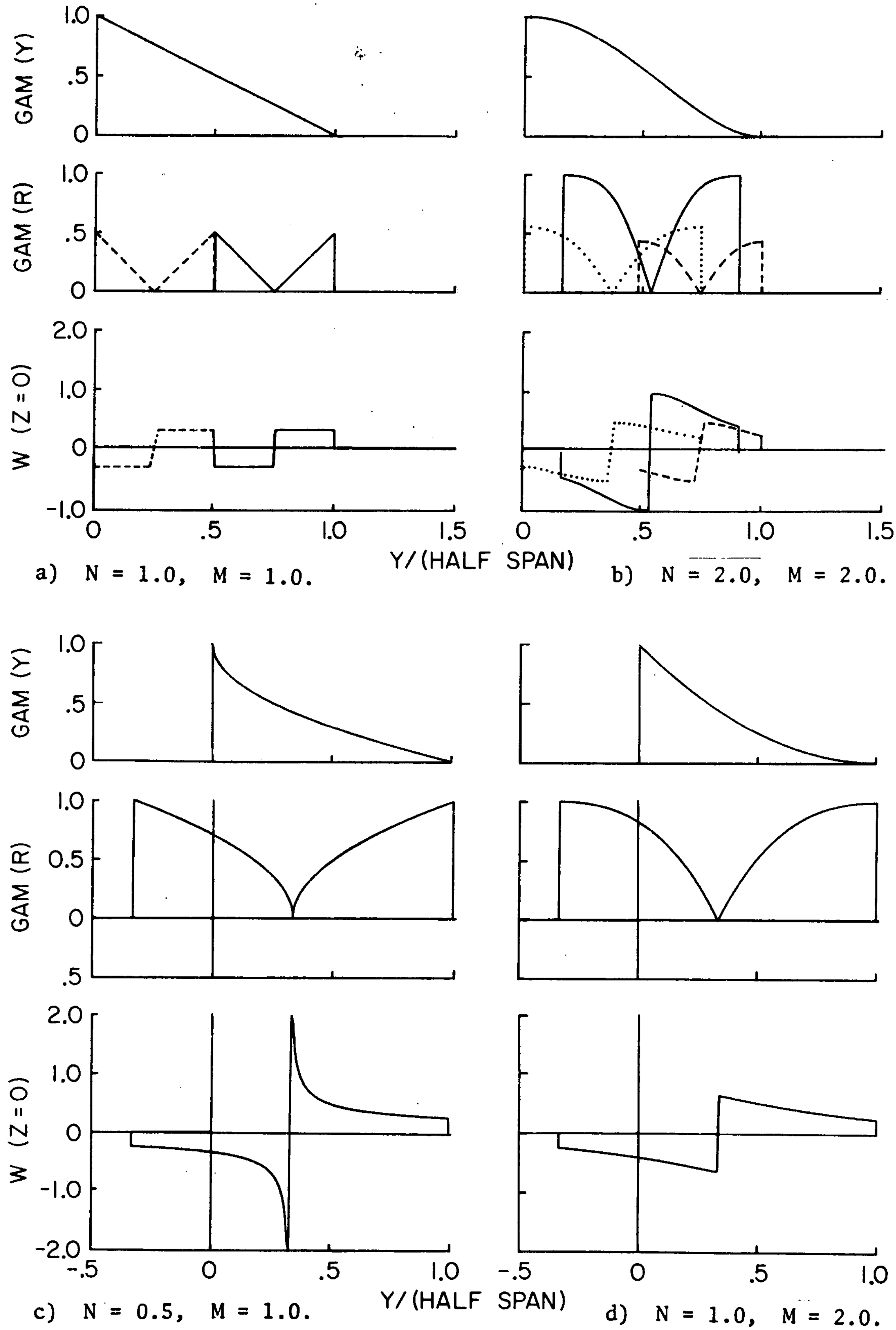
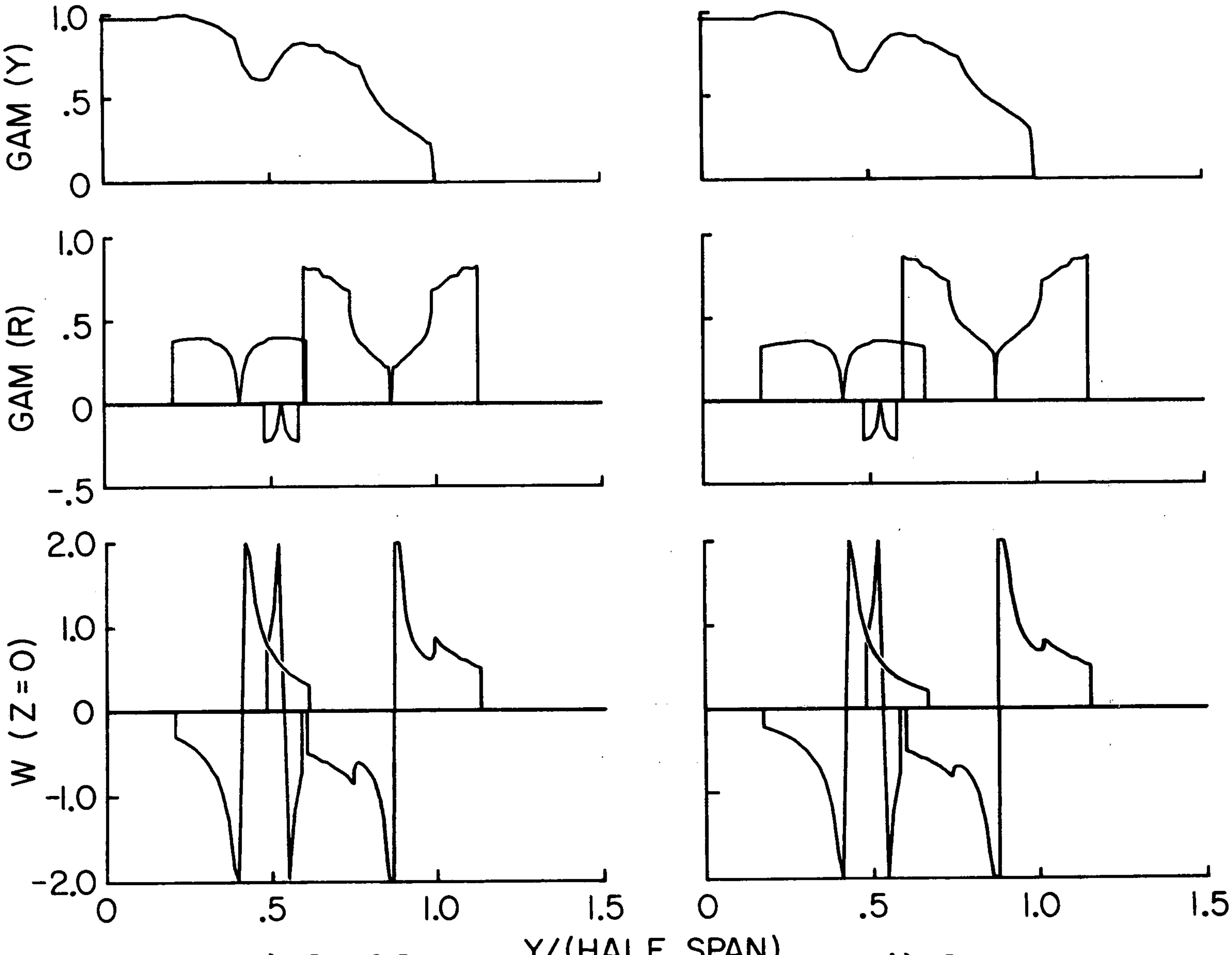


Figure 3.—Structure of fully developed vortices for the same span loadings presented in Figs. 2c—2f but with rollup beginning points determined by rules inferred in this paper and then using Eq. (17).



a) C_L = 1.5. b) C_L = 2.0.

Figure 4.—Structure of multiple vortices that are estimated to form behind span loadings typical of current large aircraft in landing configuration.