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ON THE IRRATIONALITY OF CERTAIN SERIES

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A criterion is established for the rationality of series of the form $\sum b_n/(a_1, \cdots, a_n)$ where a_n, b_n are integers, $a_n \geqq 2$ and $\lim b_n/(a_{n-1}a_n) = 0$. This criterion is applied to prove irrationality and rational independence of certain special series of the above type.

1. Introduction.

In an earlier paper [2] we proved the following result:

THEOREM 1.1. *If $\{a_n\}$ is a monotonic sequence of positive integers with $a_n \geqq n^{11/12}$ for all large n, then the series*

$$(1.2) \qquad \sum_{n=1}^{\infty} \frac{\varphi(n)}{a_1 a_2 \cdots a_n} \quad and \quad \sum_{k=1}^{\infty} \frac{\sigma(n)}{a_1 a_2 \cdots a_n}$$

are irrational.

We conjectured that the series (1.2) are irrational under the single assumption that $\{a_n\}$ is monotonic and we observed that some such condition is needed in view of the possible choices $a_n = \varphi(n) + 1$ or $a_n = \sigma(n) + 1$. These particular choices do not satisfy the hypothesis $\liminf a_{n+1}/a_n > 0$ but we do not know whether that hypothesis which is weaker than that of the monotonicity of a_n would suffice.

In this note we obtain various improvements and generalizations of Theorem 1.1, in particular by relaxing the growth conditions on the a_n and using more precise results in the distribution of primes.

In § 2 we obtain some general conditions for the rationality of series of the form $\sum b_n/(a_1, \cdots, a_n)$ which are modifications of [2, Lemma 2.29]. In § 3 we use a result of A. Selberg [3] on the regularity of primes in intervals to obtain improvements and generalizations of Theorem 1.1.

2. Criteria for rationality.

THEOREM 2.1. *Let $\{b_n\}$ be a sequence of integers and $\{a_n\}$ a sequence of positive integers with $a_n > 1$ for all large n and*

$$(2.2) \qquad \lim_{n=1} \frac{|b_n|}{a_{n-1}a_n} = 0 .$$

Then the series

$$\sum_{n=1}^{\infty} \frac{b_n}{a_1 \cdots a_n} \tag{2.3}$$

is rational if and only if there exists a positive integer B and a sequence of integers $\{c_n\}$ so that for all large n we have

$$Bb_n = c_n a_n - c_{n+1}\,, \qquad |c_{n+1}| < a_n/2\,. \tag{2.4}$$

Proof. Assume that (2.4) holds beyond N. Then

$$\begin{aligned} Ba_1 \cdots a_{N-1} \sum_{n=1}^{\infty} \frac{b_n}{a_1 \cdots a_n} &= \text{integer} + \sum_{n=N}^{\infty} \frac{c_n a_n - c_{n+1}}{a_N \cdots a_n} \\ &= \text{integer} + c_N = \text{integer}\,. \end{aligned}$$

Thus condition (2.4) is sufficient for the rationality of the series (2.3).

To prove the necessity of (2.4) assume that the series (2.3) equals A/B and that N is so large that $a_n \geqq 2$ and $|b_n/(a_{n-1}a_n)| < 1/(4B)$ for all $n \geqq N$. Then

$$\begin{aligned} Aa_1 \cdots a_{N-1} &= Ba_1 \cdots a_{N-1} \sum_{n=1}^{\infty} \frac{b_n}{a_1 \cdots a_n} \\ &= \text{integer} + \frac{Bb_N}{a_N} + \sum_{n=N+1}^{\infty} \frac{Bb_n}{a_N \cdots a_n}\,. \end{aligned} \tag{2.5}$$

If we call the last sum R_N we get

$$\begin{aligned} |R_N| &\leqq \max_{n>N} \frac{|Bb_n|}{a_{n-1}a_n} \sum_{n=N+1}^{\infty} \frac{1}{a_N \cdots a_{n-2}} \\ &< \frac{1}{4} \sum_{k=0}^{\infty} \frac{1}{2^k} = \frac{1}{2}\,. \end{aligned} \tag{2.6}$$

Thus, if we choose c_N to be the integer nearest to Bb_N/a_N and write $Bb_N = c_N a_N - c_{N+1}$ then (2.5) yields that $-c_{N+1}/a_N + R_N$ is an integer of absolute value less than 1 and hence 0, so that

$$\frac{c_{N+1}}{a_N} = R_N = \frac{Bb_{N+1}}{a_N a_{N+1}} + \frac{1}{a_N} R_{N+1} \tag{2.7}$$

or

$$\frac{Bb_{N+1}}{a_{N+1}} = c_{N+1} - R_{N+1}\,. \tag{2.8}$$

From (2.8) it follows that c_{N+1} is the integer nearest to Bb_{N+1}/a_{N+1} and if we write $Bb_{N+1} = c_{N+1}a_{N+1} - c_{N+2}$ we get

$$\frac{Bb_{N+2}}{a_{N+2}} = c_{N+2} - R_{N+2}\,. \tag{2.9}$$

Proceeding in this manner we get the desired sequence $\{c_n\}$.

REMARK. Since (2.2) implies $R_n \to 0$ it follows that for rational values of the series (2.3) we get $c_{n+1}/a_n \to 0$. Thus either $a_n \to \infty$ or $c_n = 0$ and hence $b_n = 0$ for all large n.

COROLLARY 2.10. *Let* $\{a_n\}$, $\{b_n\}$ *satisfy the hypotheses of Theorem* 2.1 *and in addition the conditions that for all large* n *we have* $b_n > 0$, $a_{n+1} \geqq a_n$, $\lim (b_{n+1} - b_n)/a_n \leqq 0$ *and* $\liminf a_n/b_n = 0$. *Then the series* (2.3) *is irrational.*

Proof. According to Theorem 2.1 the rationality of (2.3) implies the existence of a positive integer B and a sequence of integers $\{c_n\}$ so that

$$Bb_n = c_n a_n - a_{n+1}$$

for all large n where $c_{n+1}/a_n \to 0$. Thus

$$\frac{b_{n+1}}{b_n} = \frac{c_{n+1}a_{n+1} - c_{n+2}}{c_n a_n - c_{n+1}} > \frac{(c_{n+1} - \varepsilon)}{c_n a_n} \geqq \frac{c_{n+1} - \varepsilon}{c_n}$$

for all $\varepsilon > 0$ and sufficiently large n. Thus $c_{n+1} > c_n$ would lead to

$$b_{n+1} > \left(1 + \frac{1-\varepsilon}{c_n}\right)b_n > b_n + (1-\varepsilon)\left(a_n - \frac{c_{n+1}}{c_n}\right)/B \tag{2.11}$$

$$> b_n + (1-\varepsilon)^2 a_n/B\,.$$

This contradicts our hypothesis for sufficiently large n. Thus we get $0 < c_{n+1} \leqq c_n$ for all large n and hence b_n/a_n is bounded contrary to the hypothesis that $\liminf a_n/b_n = 0$.

In fact, if we omit the hypothesis $\liminf a_n/b_n = 0$ then we get rational values for the series (2.3) only when $Bb_n = C(a_n - 1)$ with positive integers B, C for all large n.

3. Some special sequences.

THEOREM 3.1. *Let* p_n *be the* nth *prime and let* $\{a_n\}$ *be a monotonic sequence of positive integers satisfying* $\lim p_n/a_n^2 = 0$ *and* $\liminf a_n/p_n = 0$. *Then the series*

$$\sum_{n=1}^{\infty} \frac{p_n}{a_1 \cdots a_n} \tag{3.2}$$

is irrational.

Proof. Since the series (3.2) satisfies the hypotheses of Theorem

2.1 it follows that there is a sequence $\{c_n\}$ and an integers B so that for all large n we have

$$Bp_n = c_n a_n - c_{n+1} . \tag{3.3}$$

For large n an equality $c_n = c_{n+1}$ would imply $c_n \mid B$ and $a_n > p_n$. Since $\{c_n\}$ is unbounded there must exist an index $m \geqq n$ so that $c_m \leqq c_n < c_{m+1}$. But this implies by an argument analogous to (2.11) that

$$p_{m+1} > p_m + a_m/(2B) > \left(1 + \frac{1}{2B}\right)p_m \tag{3.4}$$

which is impossible for large m. Thus we may assume that $c_n \neq c_{n+1}$ for all large n. Now consider an interval $N \leqq n \leqq 2N$. If $c_{n+1} > c_n$ then as in (3.4) we get

$$p_{n+1} > p_n + a_n/(2B) > p_n + \sqrt{p_n}$$

which therefore happens for fewer than $(p_{2N} - p_N)/\sqrt{p_N} < N^{1/2+\varepsilon}$ values in the interval $(N, 2N)$. If $c_{n+1} < c_n$ then we get

$$1 > \frac{c_n a_n - c_{n+1}}{c_{n+1}a_{n+1} - c_{n+2}} > \frac{c_n(a_n - 1)}{c_{n+1}a_{n+1}} > \left(1 + \frac{1}{c_{n+1}}\right)\frac{a_n - 1}{a_{n+1}}$$

so that

$$a_{n+1} > a_n + \frac{a_n - 1}{c_{n+1}} > a_n + 1 . \tag{3.5}$$

Since case (3.5) holds for more than $N/2$ values of n in $(N, 2N)$ we get $a_{2N} > N/2$ and thus for all large n we have $a_n > n/4$, $c_n < p_n/a_n + 1 < \sqrt{n}/4$. Substituting these values in (3.5) we get

$$a_{n+1} > a_n + \sqrt{n} \quad \text{when} \quad c_{n+1} < c_n,\ n \quad \text{large} ; \tag{3.6}$$

so that $a_{2N} > N^{3/2}/2$, contradicting the hypothesis that $\liminf a_n/p_n = 0$.

THEOREM 3.7. *Let $\{a_n\}$ be a monotonic sequence of positive integers with $a_n > n^{1/2+\delta}$ for some positive $\delta > 0$ and all large n. Then the numbers 1, x, y, z are rationally independent. Here*

$$x = \sum_{n=1}^{\infty} \frac{\varphi(n)}{a_1 \cdots a_n} , \qquad y = \sum_{n=1}^{\infty} \frac{\sigma(n)}{a_1 \cdots a_n}$$

and

$$z = \sum_{n=1}^{\infty} \frac{d_n}{a_1 \cdots a_n}$$

where $\{d_n\}$ *is any sequence of integers satisfying* $|d_n| < n^{1/2-\delta}$ *for all large* n *and infinitely many* $d_n \neq 0$.

Proof. Assume that there exist integers A, B, C not all 0 so that setting $b_n = A\varphi(n) + B\sigma(n) + Cd_n$ we get that $S = \sum_{n=1}^{\infty} b_n/(a_1, \cdots, a_n)$ is an integer.

From Theorem 2.1 it follows directly that z is irrational and thus not both A and B can be zero. We consider first the case $A + B \neq 0$ so that without loss of generality we may assume $A + B = D > 0$. Since S satisfies the hypotheses of Theorem 2.1 there exist integers $\{c_n\}$ so that

$$b_n = c_n a_n - c_{n+1} \quad \text{for all large} \quad n .$$

Since $|b_n| < n^{1+\delta/2}$ for all large n we get

$$|c_n| < n^{(1-\delta)/2} \quad \text{for all large} \quad n .$$

Let p_n be the nth prime and set

$$a'_n = a_{p_n},\ b'_n = b_{p_n},\ c'_n = c_{p_n},\ c''_n = c_{p_n+1} ,$$

then

$$b'_n = A(p_n - 1) + B(p_n + 1) + Cd_{p_n} = D_{p_n} + d'_n$$

where

$$d'_n = Cd_{p_n} - A + B \quad \text{with} \quad |d'_n| < n^{(1-\delta)/2} \quad \text{for all large} \quad n .$$

Now

$$b'_n = c'_n a'_n - c''_n$$
$$b'_{n+1} = c'_{n+1} a'_{n+1} - c''_{n+1}$$

so that from

$$\frac{b'_{n+1}}{b'_n} = \frac{Dp_{n+1} + d'_{n+1}}{Dp_n + d'_n} = \frac{p_{n+1}}{p_n} \frac{1 + d'_{n+1}/(Dp_{n+1})}{1 + d'_n/(Dp_n)}$$
$$= \frac{p_{n+1}}{p_n}(1 + o(n^{-(1+\delta)/2}))$$

we get

$$\frac{p_{n+1}}{p_n} = \frac{c'_{n+1}a'_{n+1} - c''_{n+1}}{c'_n a'_n - c''_n}(1 + o(n^{-(1+\delta)/2}))$$
$$= \frac{c'_{n+1}}{c'_n} \frac{1 - c''_{n+1}/(a'_{n+1}c'_{n+1})}{1 - c''_n/(a'_n c'_n)}(1 + o(n^{-(1+\delta)/2})) \tag{3.8}$$
$$= \frac{c'_{n+1}}{c'_n}(1 + o(n^{-(1+\delta)/2})) .$$

Here the last inequality follows from the fact that

$$\left|\frac{c_{n+1}}{c_n}\right| = \left|\frac{(b_{n+1} + c_{n+2})/a_{n+1}}{(b_n + c_{n+1})/a_n}\right| = \frac{|A\varphi(n+1) + B\sigma(n+1)| + O(n^{(1-\delta)/2})}{|A\varphi(n) + B\sigma(n)| + O(n^{(1-\delta)/2})}$$
$$= o(n^{\delta/2}) .$$

From (3.8) we get that $c'_{n+1} > c'_n$ implies

$$p_{n+1} > p_n + \frac{p_n}{c'_n} - p_n^{1/2-\delta/4} > p_n + \frac{1}{2}p_n^{1/2+\delta} \tag{3.9}$$

for all large n.

We now use the following result of A. Selberg [3, Theorem 4].

THEOREM 3.10. *Let $\Phi(x)$ be positive and increasing and $\Phi(x)/x$ decreasing for $x > 0$, further suppose*

$$\Phi(x)/x \to 0 \quad \textit{and} \quad \liminf \log \Phi(x)/\log x > 19/77 \quad \textit{for} \quad x \to \infty .$$

Then for almost all $x > 0$,

$$\pi(x + \Phi(x)) - \pi(x) \sim \frac{\Phi(x)}{\log x} .$$

We now apply this theorem with the choice $\Phi(x) = x^{1/2+\delta}$ to inequality (3.9) and consider the primes $N \leqq p_m < p_{m+1} < \cdots < p_n < 2N$ in an interval $(N, 2N)$ with N large. According to Theorem 3.10 the union of the set of intervals (p_i, p_{i+1}) where p_i, p_{i+1} satisfy (3.9) and $m \leqq i < n$, form a set of total length $< \varepsilon N$ where $\varepsilon > 0$ is arbitrarily small. Also the number of indices i for which (3.9) holds is $o(\sqrt{N})$. Thus by (3.8) and (3.9) we have

$$\frac{c'_n}{c'_m} = \prod_{i=m}^{n-1} \frac{c'_{i+1}}{c'_i} = \prod_{\substack{i=m \\ c'_{i+1} c'_i}}^{n-1} \frac{c'_{i+1}}{c'_i} < \frac{N + \varepsilon N}{N}(1 + o(N^{-(+\delta)/2}))^{\sqrt{N}}$$
$$< 1 + 2\varepsilon < 2^{2\varepsilon} .$$

From the monotonicity of a_n it now follows that for any $\varepsilon > 0$ we have

$$|c_n| < n^{\varepsilon} \quad \text{for all large} \quad n. \tag{3.11}$$

Substituting this inequality in (3.9) we get that $c'_{n+1} > c'_n$ would imply

$$p_{n+1} > p_n + \frac{p_n}{c'_n} - p^{1/2+\delta/4} > p_n + \frac{1}{2}p_n^{1-\varepsilon} \tag{3.12}$$

which is impossible for large n when $\varepsilon < 5/12$. Thus $\{c'_n\}$ becomes nonincreasing for large n and hence constant, $c'_n = c$, for large n.

This implies $a_p > p/(c+1)$ for large primes p and by the monotonicity of a_n we get

$$\frac{a_n}{n} > \frac{a_p}{2p} > \frac{1}{4c}$$

where p is the largest prime $\leqq n$.

Now consider the successive equations

$$b_p = ca_p - c_{p+1}$$
$$b_{p+1} = c_{p+1}a_{p+1} - c_{p+2} .$$

Thus

$$A\varphi(p+1) + B\sigma(p+1) + O(p^{1/2-\delta}) = c_{p+1}a_{p+1}$$
$$Dp + O(p^{1/2-\delta}) = ca_p$$

for all large primes p. This leads to

$$\left|\frac{A}{D}\frac{\varphi(p+1)}{p+1} + \frac{B}{D}\frac{\sigma(p+1)}{p+1} - \frac{c_{p+1}}{c}\right| < p^{-1/2} , \tag{3.13}$$

and hence to the conclusion that the only limit points of the sequence

$$\left\{\frac{A}{D}\frac{\varphi(p+1)}{p+1} + \frac{B}{D}\frac{\sigma(p+1)}{p+1}\,\middle|\, p = \text{prime}\right\}$$

are rational numbers with denominator c. To see that this is not the case, consider first the case $B \neq 0$. Then by Dirichlet's theorem about primes in arithmetic progressions we see that $\sigma(p+1)/(p+1)$ is everywhere dense in $(1, \infty)$. Thus we can choose p so that the distance of $B\sigma(p+1)/D(p+1)$ to the nearest fraction with denominator c is greater that $1/(3c)$ while at the same time $\sigma(p+1)/(p+1)$ is so large that $|A\varphi(p+1)/D(p+1)| < 1/(3c)$, contradicting (3.13). If $B = 0$ we use the fact that $\varphi(p+1)/(p+1)$ is dense in $(0, 1)$ to get the same contradiction.

Finally we must consider the case $A + B = 0$. Here we can go through the same argument as before except that we consider the subsequence $b_{2p} = A\varphi(2p) + B\sigma(2p) + Cd_{2p} = 2Bp + (3B + Cd_{2p}) = 2Bp + O(p^{1/2-\delta})$. As before we get

$$b_{2p} = ca_{2p} - c_{2p+1} \quad \text{for all large primes} \quad p$$

which leads to the wrong conclusion that

$$\left\{\frac{\sigma(2p+1)}{2p+1} - \frac{\varphi(2p+1)}{2p+1}\,\middle|\, p = \text{prime}\right\}$$

has rational numbers with denominator c as its only limit points.

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