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Research Article

On the Jacobson Radical of an (m, n)-Semiring

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The notion of n-ary semimodules is introduced so that the Jacobson radical of an (m, n)-semiring is studied and some well-known results concerning the Jacobson radical of a ring (a semiring or a ternary semiring) are generalized to an (m, n)-semiring.

1. Introduction

The concept of semigroups [1] was generalized to that of ternary semigroups [2], that of n-ary semigroups [3–6], and even to that of (n, m)-semigroups [7]. Similarly, it was natural to generalize the notion of rings to that of ternary semirings, that of n-ary semirings, and even that of (m, n)-semirings.

Indeed, there were some research articles on semirings, (see, for example, [8–14]), specially on the radical of a semiring; see [15–18]. Semigroups over semirings were studied in [19] and semimodules over semirings were studied in [14]. The notion of semirings can be generalized to ternary semirings [20] and Γ-semirings [21], even to (m, n)-semirings [22–24]. The radicals of ternary semirings and of Γ-semirings were studied in [20, 21], respectively. The concept of (m, n)-semirings was introduced and accordingly some simple properties were discussed in [22–24], where the concept of radicals was not mentioned.

The notion of the Jacobson radicals was first introduced by Jacobson in the ring theory in 1945. Jacobson [25] defined the radical of R, which we call the Jacobson radical, to be the join of all quasi-regular right ideals and verified that the radical is a two-sided ideal and can also be defined to be the join of the left quasi-regular ideals.

The concept of the Jacobson radical of a semiring has been introduced internally by Bourne [15], where it was proved that the right and left Jacobson radicals coincide; thus one could say the Jacobson radical briefly. These and some other results were generalizations of well-known results of Jacobson [25].

In 1958, by associating a suitable ring with the semiring, Bourne and Zassenhaus defined the semiradical of the semiring [16]. In [18] it was proved that the concepts of the Jacobson radical and the semiradical coincide.

Iizuka [17] considered the Jacobson radical of a semiring from the point of view of the representation theory [15] without reducing it to the ring theory. The external notion of the radical was proved to be related to internal one; at the same time, it was shown that the radical defined in [17] coincides with the Jacobson radical and with the semiradical of the semiring.

In the present paper, we investigate (n, m)-semirings by means of n-ary semimodules so that we can define externally the Jacobson radical of an (n, m)-semiring, and then we establish the radical properties of the Jacobson radical of an (n, m)-semiring. Some necessary notions such as irreducible n-ary semimodules over an (n, m)-semiring are adequately defined. All results in this paper generalize the corresponding ones concerning the radical of a ring [25], of a semiring [15–18], or of a ternary semiring [20].

2. Preliminaries

We used following convention as followed by [4]: The sequence $x_i, x_{i+1}, \ldots, x_m$ is denoted by x_i^m. Thus the following expression

$$f\left(x_1, \ldots, x_i, y_{i+1}, \ldots, y_j, z_{j+1}, \ldots, z_m\right) \tag{1}$$

is represented as

$$f\left(x_1^i, y_{i+1}^j, z_{j+1}^m\right). \tag{2}$$

In the case when $y_{i+1} = \cdots = y_j = y$, then (2) is expressed as

$$f\left(x_1^i, \overset{(j-i)}{y}, z_{j+1}^m\right). \tag{3}$$

If $x_1 = \cdots = x_i = y_{i+1} = \cdots = y_j = z_{j+1} = \cdots = z_m = f(a_1^m)$, then (2) can be written as $f(\overset{(m)}{f(a_1^m)})$.

Recall that an n-ary semigroup (S, f) is defined as a nonempty set S with an n-ary associative operation $f : S^n = \underbrace{S \times \cdots \times S}_{n} \to S$; that is,

$$f\left(x_1^{i-1}, f\left(x_i^{n+i-1}\right), x_{n+i}^{2n-1}\right) = f\left(x_1^{j-1}, f\left(x_j^{n+j-1}\right), x_{n+j}^{2n-1}\right) \tag{4}$$

for all $x_1^{2n-1} \in S$ and all $1 \le i < j \le n$. Whence we may denote $f(f(x_1^n), x_{n+1}^{2n-1})$ by $f(x_1^{2n-1})$ briefly. Generally, we have the notation $f(x_1^{t(m-1)+1})$ for each positive t and all $x_j \in S$. Thus for positive integer k, $f(x_1^k)$ is well defined if and only if $k \equiv 1 \mod (n-1)$; see [7, Lemma 1.1]. An n-ary semigroup (S, f) is called cancellative if

$$f\left(x_1^{i-1}, a, x_{i+1}^n\right) = f\left(x_1^{i-1}, b, x_{i+1}^n\right) \Longrightarrow a = b \tag{5}$$

for all $a, b, x_j \in S$.

The next definition is a generalization of the concept of ternary semirings in [20] and similar to the notion of the (m, n)-semirings in [24].

Definition 1. A nonempty set R together with an m-ary operation f, called *addition,* and an n-ary operation g, called *multiplication*, is said to be an (m, n)*-semiring* if the following conditions are satisfied.

(1) (R, f) is an m-ary semigroup and (R, g) is an n-ary semigroup.

(2) g is distributive with respect to operation f; that is, for every $a_1^{i-1}, b_1^m, a_{i+1}^n \in R$,

$$\begin{aligned} &g\left(a_1^{i-1}, f\left(b_1^m\right), a_{i+1}^n\right) \\ &\quad = f\left(g\left(a_1^{i-1}, b_1, a_{i+1}^n\right), \ldots, g\left(a_1^{i-1}, b_m, a_{i+1}^n\right)\right). \end{aligned} \tag{6}$$

(3) (R, f) is commutative; that is, for every permutation p of $\{1, 2, \ldots, m\}$ and all $x_1^m \in R$,

$$f\left(x_1, x_2, \ldots, x_m\right) = f\left(x_{p(1)}, x_{p(2)}, \ldots, x_{p(m)}\right). \tag{7}$$

(4) There is an element 0, called the *zero* of (R, f, g), satisfying the following two properties:

(4A) 0 is an f-identity; that is, for every $x \in R f(\overset{(m-1)}{0}, x) = x$;

(4B) 0 is a g-zero; that is, for all $x_1^m \in R$, $g(x_1^n) = 0$ whenever there exists i such that $x_i = 0$.

It is clear that the zero of an (m, n)-semiring R is necessarily unique.

Definition 2. An (m, n)-semiring (R, f, g) is called *additively cancellative* if the m-ary semigroup (R, f) is cancellative and *multiplicatively cancellative* if the n-ary semigroup (R, g) is cancellative.

Recall that for an n-ary semigroup (S, f), a nonempty subset T of S is called a subsemigroup of (S, f) if $f(a_1^n) \in T$ whenever all $a_1^n \in T$. For $i \le n$, we call T an i-ideal of S if $f(a_1^n) \in T$ whenever $a_i \in T$. T is called an ideal of S if and only if it is an i-ideal of S. See, for example, [7, Definition 1.6].

Definition 3. A nonempty subset T of an (m, n)-semiring (R, f, g) is called an n-ary *subsemiring* of (R, f, g) if T is a subsemigroup of (R, f) as well as a subsemigroup of (R, g) and an *(i-)ideal* of (R, f, g) if T is a subsemigroup of (R, f) as well as an *(i-)ideal* of (R, g) (where $i \le n$). An 1-ideal is also called a *right ideal* and an n-ideal is also called a *left ideal*. An ideal T of (R, f, g) is called a *k-ideal* if $f(x, y_2^m) \in T$; $x \in R$ and $y_2^m \in T$ imply that $x \in T$. An ideal T of (R, f, g) is called an h-ideal if $f(x, y_2^m, z_2^m) = f(y_1, z_2^m)$; $x, z_2^m \in R$ and $y_1^m \in T$ imply that $x \in T$.

Let A be an ideal of (R, f, g). Then the k-closure of A, denoted by $\overline{A}$, is defined by $\overline{A} = \{x \in R : f(x, a_2^m) = a_1 \text{ for some } a_1^m \in A\}$. Similarly, the h-closure of A, denoted by $\widehat{A}$, is defined by $\widehat{A} = \{x \in R : f(x, a_2^m, y_2^m) = f(a_1, y_2^m)$ for some $a_1^m \in A$ and some $y_2^m \in R\}$. One can show that $\overline{A}$ is a k-ideal and $\widehat{A}$ is an h-ideal. Furthermore, it is shown that an ideal A of (R, f, g) is a k-ideal if and only if $\overline{A} = A$ and that A is an h-ideal if and only if $\widehat{A} = A$.

Definition 4. An equivalence relation ρ on an (m, n)-semiring (R, f, g) is said to be a *congruence relation* or simply a *congruence* of (R, f, g) if the following conditions are satisfied:

(1) $a_1^m \rho b_1^m \Rightarrow f(a_1^m) \rho f(b_1^m)$ for all $a_1^m, b_1^m \in R$,

(2) $a_1^n \rho b_1^n \Rightarrow g(a_1^n) \rho g(b_1^n)$ for all $a_1^n, b_1^n \in R$.

Let I be a proper ideal of an (m, n)-semiring (R, f, g). Then the congruence on (R, f, g), denoted by k_I, and defined by setting $x k_I y$ if and only if $f(x, a_2^m) = f(y, b_2^m)$ for some $a_2^m, b_2^m \in I$, is called the *Bourne congruence* on (R, f, g) defined by the ideal I. We denote the Bourne congruence class of an element x by x/I and denote the set of all such congruence classes of (R, f, g) by R/I. If the Bourne congruence k_I is proper, that is, $0/I \neq R$, then we can define two operations, m-ary addition and n-ary multiplication on R/I by $\overline{f}(a_1^m/I) = f(a_1^m)/I$ and $\overline{g}(b_1^n/I) = g(b_1^n)/I$ for all $a_1^m, b_1^n \in R$. Then $(R/I, \overline{f}, \overline{g})$ is an (m, n)-semiring and is called the *Bourne factor* (m, n)*-semiring*.

Similarly, the congruence on (R, f, g), denoted by h_I, and defined by setting $x h_I y$ if and only if $f(x, a_2^m, y_2^m) = f(y, b_2^m, y_2^m)$ for some $a_2^m, b_2^m \in I$ and some $y_2^m \in R$, is called the *Iizuka congruence* on (R, f, g) defined by the ideal I. We denote the Iizuka congruence class of an element x by $x[/]I$ and denote the set of all such congruence classes of (R, f, g) by $R[/]I$. If the Iizuka congruence h_I is proper, that is, $0[/]I \neq R$, then we can define two operations, m-ary addition and n-ary multiplication on $R[/]I$ by $\hat{f}(a_1^m[/]I) = f(a_1^m)$ and $\hat{g}(b_1^n[/]I) = g(b_1^n)$ for all $a_1^m, b_1^n \in R$. Then $(R[/]I, \hat{f}, \hat{g})$ is an (m,n)-semiring and we call it the *Iizuka factor* (m,n)-*semiring*.

The next definition is a generalization of [20, Definition 2.13].

Definition 5. A commutative m-ary semigroup (M, f_0) with an identity 0_M (operation f_0 to be called addition) is called a *right n-ary semimodule* over an (m,n)-semiring (R, f, g) or simply an n-ary R-semimodule if there exists a mapping $M \times \underbrace{R \times \cdots \times R}_{n-1} \to M$ (images to be denoted by $ar_2 \cdots r_n$ or briefly by ar_2^n for all $a \in M$ and $r_2^n \in R$) satisfying the following conditions:

(1) $f_0(a_1^m)r_2 \cdots r_n = f_0(a_1^m r_2 \cdots r_n)$ for all $a_1^m \in M$ and all $r_2^n \in R$;

(2) $ar_2 \cdots (f(s_1^m)) \cdots r_n = f_0(ar_2 \cdots s_1^m \cdots r_n)$ for all $a \in M$ and all $r_2^n, s_1^m \in R$;

(3) $(ar_2 \cdots r_n)r_{n+1} \cdots r_{2n-1} = ar_2 \cdots g(r_{k+1}^{k+n}) \cdots r_{2n-1}$ for all $a \in M$, $k \in \{1, \dots, n-1\}$, and $r_{k+1}^{k+n} \in R$;

(4) $0_M r_2 \cdots r_n = 0_M$ for all $r_2^n \in R$;

(5) $ar_2 \cdots r_n = 0_M$ whenever $a \in M$, $r_2^n \in R$, and $r_i = 0$ for some i.

Definition 6. A nonempty subset N of a right n-ary semimodule (M, f_0) over an (m,n)-semiring (R, f, g) is called an n-ary *subsemimodule* of M if (i) $f_0(a_1^m) \in N$ and (ii) $a_1 x_2^n \in N$ for all $a_1^m \in N$ and $x_2^n \in R$.

An n-ary subsemimodule N of M is called an n-ary k-subsemimodule if $f_0(a, b_2^m) \in N$; $a \in M$ and $b_2^m \in N$ imply that $a \in N$. An n-ary subsemimodule N of M is called an n-ary h-subsemimodule if $f(a, b_2^m, c_2^m) = f(b_1, c_2^m)$; $a, c_2^m \in R$ and $b_1^m \in N$ imply that $a \in N$.

For example, an (m,n)-semiring (R, f, g) can be regarded as a right n-ary R-semimodule naturally. Then if I is a k-ideal (an h-ideal) of the (m,n)-semiring (R, f, g), then I is also an n-ary k-(h-)subsemimodule of this right n-ary R-semimodule R.

Definition 7. A right n-ary R-semimodule (M, f_0) is said to be *cancellative* if (M, f_0) is a cancellative m-ary semigroup.

Definition 8. An equivalence relation ρ on right n-ary R-semimodule (M, f_0) is said to be a *congruence relation* or simply a *congruence* of (M, f_0) if the following conditions are satisfied:

(1) $a_1^m \rho b_1^m \Rightarrow f_0(a_1^m) \rho f_0(b_1^m)$ for all $a_1^m, b_1^m \in M$,

(2) $a \rho b \Rightarrow ar_2^n \rho br_2^n$ for all $a, b \in M$ and all $r_2^n \in R$.

We say that a congruence ρ of (M, f_0) admits the cancellation law (of addition) if

(3) $f_0(x, a_2^m) \rho f_0(y, b_2^m)$ and $a_2^m \rho b_2^m$ imply $x \rho y$.

Let N be an n-ary subsemimodule of an n-ary right semimodule (M, f_0) over an (m,n)-semiring R. Then the congruence on (M, f_0), denoted by k_N, and defined by setting

$$x k_N y \text{ iff } f_0(x, a_2^m) = f_0(y, b_2^m) \quad \text{for some } a_2^m, b_2^m \in N, \tag{8}$$

is called the *Bourne congruence* on M defined by the n-ary subsemimodule N. We denote the Bourne congruence class of an element x by x/N and denote the set of all such congruence classes of M by M/N. Define two operations, m-ary addition and n-ary scalar multiplication on M/N, by $\overline{f_0}(a_1^m/N) = f_0(a_1^m)/N$ and $(a_1/N)r_2^n = (a_1 r_2^n)/N$ for all $a_1^m \in M$ and all $r_2^n \in R$. With these two operations, M/N is an n-ary right semimodule over R and we call it the *Bourne factor n-ary semimodule*.

Similarly, we can define the *Iizuka congruence* h_N and the *Iizuka factor n-ary semimodule* $M[/]N$. It is easy to show that $M[/]N$ is cancellative.

In what follows, we always assume that the n-ary right semimodule is cancellative.

3. Primitive (m,n)-Semirings

Definition 9. Let (R, f, g) be an (m,n)-semiring with zero 0. Then the *zeroid* of R, denoted by $Z(R)$, is defined as

$$Z(R) = \{x \in R : f(x, y_2^m) = f(0, y_2^m) \text{ for some } y_2^m \in R\}. \tag{9}$$

Clearly, the zero element 0 of R belongs to $Z(R)$. Furthermore, we have the following.

Lemma 10. *The zeroid $Z(R)$ of an (m,n)-semiring (R, f, g) is the smallest h-ideal of (R, f, g).*

Proof. It is easily verified that $Z(R)$ is an ideal of R. To show $Z(R)$ is an h-ideal of R, we suppose $f(x, y_2^m, z_2^m) = f(y_1, z_2^m)$, where $x, z_2^m \in R$ and $y_1^m \in Z(R)$. For each $i \in \{1, \dots, m\}$ there exist $u(i)_2^m \in R$ such that $f(y_i, u(i)_2^m) = f(0, u(i)_2^m)$, so we have

$$f(f(x, y_2^m, z_2^m), u(1)_2^m, \dots, u(m)_2^m) = f(f(y_1, z_2^m), u(1)_2^m, \dots, u(m)_2^m), \tag{10}$$

that is,

$$f(x, u(1)_2^m, f(y_2, u(2)_2^m), \dots, f(y_m, u(m)_2^m), z_2^m) = f(f(y_1, u(1)_2^m), u(2)_2^m, \dots, u(m)_2^m, z_2^m). \tag{11}$$

It follows that

$$f\left(x, u(1)_2^m, f\left(0, u(2)_2^m\right), \ldots, f\left(0, u(m)_2^m\right), z_2^m\right) = f\left(f\left(0, u(1)_2^m\right), u(2)_2^m, \ldots, u(m)_2^m, z_2^m\right). \tag{12}$$

Hence we obtain

$$f\left(x, u(1)_2^m, u(2)_2^m, \ldots, u(m)_2^m, z_2^m\right) = f\left(0, u(1)_2^m, u(2)_2^m, \ldots, u(m)_2^m, z_2^m\right), \tag{13}$$

which shows that $x \in Z(R)$, so that $Z(R)$ is an h-ideal of R.

At last, suppose that I is an arbitrary h-ideal of R. We aim to show $Z(R) \subseteq I$. For this, let $x \in Z(R)$. Then there exist $y_2^m \in R$ such that $f(x, y_2^m) = f(0, y_2^m)$, so $f(x, \overset{m-1}{0}, y_2^m) = f(0, y_2^m)$. It follows that $x \in I$ since I is an h-ideal and $0 \in I$. Thus $Z(R) \subseteq I$. □

Definition 11. Let M be a right n-ary R-semimodule. The *annihilator* of M in R, denoted by $(0 : M)$ or $A_R(M)$, is defined as the subset

$$\{r \in R : ar_2^m = 0_M \text{ whenever } a \in M, r_2^m \in R \text{ and } r_i = r \text{ for some } i\}. \tag{14}$$

Lemma 12. *$A_R(M)$ is an h-ideal of R.*

Proof. It is obvious that $A_R(M)$ is an ideal of R. To show that it is an h-ideal, suppose $f(x, y_2^m, z_2^m) = f(y_1, z_2^m)$, where $x, z_2^m \in R$ and $y_1^m \in A_R(M)$. Then for all $r_i \in R$,

$$ar_2 \cdots f\left(x, y_2^m, z_2^m\right) \cdots r_n = ar_2 \cdots f\left(y_1, z_2^m\right) \cdots r_n, \tag{15}$$

that is,

$$f_0\left(ar_2 \cdots x \cdots r_n, ar_2 \cdots y_2^m \cdots r_n, ar_2 \cdots z_2^m \cdots r_n\right) = f_0\left(ar_2 \cdots y_1 \cdots r_n, ar_2 \cdots z_2^m \cdots r_n\right), \tag{16}$$

which deduces that

$$f_0\left(ar_2 \cdots x \cdots r_n, \overset{m-1}{0}, ar_2 \cdots z_2^m \cdots r_n\right) = f_0\left(0, ar_2 \cdots z_2^m \cdots r_n\right) \tag{17}$$

since $ar_2 \cdots y_j \cdots r_n = 0$ for each $j \in \{1, \ldots, m\}$. Thus we have

$$f_0\left(ar_2 \cdots x \cdots r_n, ar_2 \cdots z_2^m \cdots r_n\right) = f_0\left(0, ar_2 \cdots z_2^m \cdots r_n\right). \tag{18}$$

By cancellation law of M, $ar_2 \cdots x \cdots r_n = 0$. Hence $x \in A_R(M)$, as required. □

Definition 13. A right n-ary R-semimodule M is said to be *faithful* if $Z(R) = A_R(M)$.

One of difficulties when studying the radical of an (m, n)-semiring R is how to give an appropriate definition of the irreducibility of a right n-ary R-semimodule. The next definition is a generalization of [20, Definition 3.9].

Definition 14. A right n-ary R-semimodule M is said to be *irreducible* if for every arbitrary fixed pair $u_2^m, v_2^m \in M$ with $u_j \neq v_j$ for some j and for any $x \in M$, there exist $a(i)_2^m, b(i)_2^m \in R$ with $i = 2, \ldots, m$ such that

$$f_0\left(x, u_2 a(2)_2^n, \ldots, u_m a(m)_2^n, v_2 b(2)_2^n, \ldots, v_m b(m)_2^n\right) = f_0\left(0_M, u_2 b(2)_2^n, \ldots, u_m b(m)_2^n, v_2 a(2)_2^n, \ldots, v_m a(m)_2^n\right). \tag{19}$$

Remark 15. Since M is cancellative, it is easily seen that a right n-ary R-semimodule M is irreducible if and only if for every arbitrary fixed pair $u_2^m, v_2^m \in M$ with $u_j \neq v_j$ for all j and for any $x \in M$, there exist $a(i)_2^m, b(i)_2^m \in R$ with $i = 2, \ldots, m$ such that equality (2) holds.

Lemma 16. *Let I be an h-ideal of an (m, n)-semiring R. If M is an irreducible right n-ary R/I-semimodule, then M is an irreducible right n-ary R-semimodule.*

Proof. Let M be an irreducible right n-ary R-semimodule. Then we can define an n-ary action on M by $ar_2 \cdots r_n = a(r_2/I) \cdots (r_n/I)$ for all $a \in M$ and for all $r_2^n \in R$, and this makes M into an irreducible right n-ary R-semimodule. □

The converse of Lemma 16 is not necessarily true. But in particular we have the following lemma.

Lemma 17. *If M is a right n-ary R-semimodule then M is a right n-ary $R/A_R(M)$-semimodule, where $R/A_R(M)$ is the Bourne factor semiring. Moreover, if M is an irreducible right n-ary R-semimodule, then M is also an irreducible right n-ary $R/A_R(M)$-semimodule.*

Proof. Suppose M is a right n-ary R-semimodule. We define an n-ary action on M as follows: $ar_2/I \cdots r_n/I = ar_2 \cdots r_n$ where $I = A_R(M)$, for all $a \in M$ and for all $r_2^n \in R$. We now show that this definition is well-defined. If for each $i = 2, \ldots, n$, $r_i/I = s_i/I$, then $r_i k_I s_i$, that is, there exist $x(i)_2^m, y(i)_2^m \in I$ such that $f(r_i, x(i)_2^m) = f(s_i, y(i)_2^m)$. It follows that $af(r_2, x(2)_2^m) \cdots f(r_n, x(n)_2^m) = af(s_2, y(2)_2^m) \cdots f(s_n, y(n)_2^m)$, which implies that $ar_2 \cdots r_n = as_2 \cdots s_n$ since $x(i)_2^m, y(i)_2^m \in A_R(M)$. Thus $ar_2/I \cdots r_n/I = as_2/I \cdots s_n/I$, as required. It is easy to see that the above definition makes M into a right n-ary R-semimodule.

Moreover, if M is an irreducible right n-ary R-semimodule then it is routine to verify that M is also an irreducible right n-ary $R/A_R(M)$-semimodule by (2). □

Lemma 18. *Let M be a right n-ary R-semimodule. Then $A_{R/A_R(M)}(M) = \{0/A_R(M)\}$.*

Proof. Let $x/I \in A_{R/I}(M)$, where $I = A_R(M)$. Then for any $a \in M$ $ar_2/I \cdots r_n/I = 0_M$ whenever $r_2^n \in R$ and $r_i = x$ for some $i \in \{2, \ldots, n\}$. It follows that $ar_2 \cdots r_n = 0_M$ where $r_i = x$

for some $i \in \{2, \ldots, n\}$. This shows that $x \in I$ and so that $x/I = 0/I$. Consequently, $A_{R/A_R(M)}(M) = \{0/A_R(M)\}$. □

Lemma 19. *Any right n-ary R-semimodule M is a faithful $R/A_R(M)$-semimodule.*

Proof. Let M be a right n-ary R-semimodule. Then in view of Lemma 17, M is an $R/A_R(M)$-semimodule. On the one hand, by Lemma 12, $A_{R/A_R(M)}(M)$ is an h-ideal of $R/A_R(M)$. On the other hand, by Lemma 10, $Z(R/A_R(M))$ is the smallest h-ideal of $R/A_R(M)$. Thus $Z(R/A_R(M)) \subseteq A_{R/A_R(M)}(M)$. According to Lemma 18, $A_{R/A_R(M)}(M) = \{0/A_R(M)\}$. So $Z(R/A_R(M)) = \{0/A_R(M)\} = A_{R/A_R(M)}(M)$, which means that M is a faithful $R/A_R(M)$-semimodule. □

Lemma 20. *If I is an h-ideal of an (m, n)-semiring R then $Z(R/I) = \{0/I\}$ where R/I is the Bourne factor semiring.*

Proof. Suppose $x/I \in Z(R/I)$. Then we have $f(x/I, y_2^m/I) = f(0/I, y_2^m/I)$ for some $y_2^m/I \in R/I$. Thus we have $f(x, y_2^m)/I = f(0, y_2^m)/I$ which implies that $f(x, y_2^m, a_2^m) = f(0, y_2^m, b_2^m)$ for some $a_2^m, b_2^m \in I$. Hence $f(x, a_2^m, y_2^m) = f(0, b_2^m, y_2^m)$. This shows that $x \in I$ since I is an h-ideal of R. Consequently, $x/I = 0/I$. Thus $Z(R/I) = \{0/I\}$. □

Definition 21. An (m, n)-semiring R is said to be *primitive* if it has a faithful irreducible cancellative n-ary R-semimodule. An ideal P is said to be *primitive* if the Bourne factor semiring R/P is primitive.

Evidently, an (m, n)-semiring R is primitive if and only if $\{0\}$ is a primitive ideal of R. The following theorem characterizes primitive ideals of an (m, n)-semiring.

Theorem 22. *An h-ideal P of (m, n)-semiring R is primitive if and only if $P = A_R(M)$ for some irreducible right n-ary R-semimodule M.*

Proof. Let P be an h-ideal of R such that $P = A_R(M)$ for some irreducible right n-ary R-semimodule M. Then by Lemmas 17 and 19 M is a faithful irreducible n-ary R/P-semimodule. This shows that R/P is primitive and hence P is a primitive h-ideal of R.

Conversely, let P be a primitive h-ideal of R. Then R/P is a primitive (m, n)-semiring. So there exists a faithful irreducible n-ary R/P-semimodule M. Now by Lemma 16 M is an irreducible n-ary R-semimodule. It remains to show that $P = A_R(M)$. Now $x \in A_R(M) \Leftrightarrow$ for all $a \in M$ and $r_2^n \in R$, $ar_2 \cdots r_n = 0_M$ whenever $x = r_i$ for some $i \in \{2, \ldots, n\} \Leftrightarrow ar_2/P \cdots r_n/P = 0_M/P$ whenever $x/P = r_i/P$ for some $i \in \{2, \ldots, n\} \Leftrightarrow x/P \in A_{R/P}(M) = Z(R/P)$ since M is a faithful n-ary R/P-semimodule $\Leftrightarrow x/P \in A_{R/P}(M) = \{0/P\}$, by Lemma 20 $\Leftrightarrow x/P = 0/P \Leftrightarrow x \in P$. Thus $P = A_R(M)$ as desired. □

4. Jacobson Radical of an (m, n)-Semiring

Let us begin this section by defining the semi-irreducibility of a right n-ary R-semimodule.

Definition 23. A right n-ary R-semimodule M is said to be *semi-irreducible* if $MR^{n-1} \neq \{0_M\}$; that is, $ar_2 \cdots r_n \neq 0_M$ for some $a \in M$ and some $r_2^n \in R$, and M does not contain any n-ary k-subsemimodule other than $\{0_M\}$ and M.

Lemma 24. *Let I be a subset of an (m, n)-semiring R and M a right n-ary R-semimodule with $MR^{i-2}IR^{n-i} \neq \{0_M\}$ for some $i \in \{2, \ldots, n\}$. In the case where $i = 2$, we assume further that I is a left ideal of R. Then the following statements are true:*

(1) *If M is semi-irreducible and $a \in M$, then $a = 0$ if and only if $aR^{i-2}IR^{n-i} = \{0_M\}$;*

(2) *If M is irreducible and $a, b \in M$, then $a = b$ if and only if $ar_2^{i-1}sr_{i+1}^n = br_2^{i-1}sr_{i+1}^n$ for all $r_j \in R$ and all $s \in I$.*

Proof. Suppose that (M, f_0) is a semi-irreducible right n-ary semimodule over an (m, n)-semiring (R, f, g), and that I is a subset of R such that $MR^{i-2}IR^{n-i} \neq \{0_M\}$ for some $i \in \{2, \ldots, n\}$. In the case where $i = 2$, we further assume that I is a left ideal of R.

(1) Assume that M is semi-irreducible. Let $a \in M$ be such that

$$aR^{i-2}IR^{n-i} = \{0_M\}. \tag{20}$$

Set

$$M_0 = \left\{x \in M : xR^{i-2}IR^{n-i} = \{0_M\}\right\}. \tag{21}$$

It is clear that $a \in M_0$, and it is easy to show that M_0 is a subsemimodule of M. Let $f_0(x, y_2^m) \in M_0$ and $y_2^m \in M_0$. Then $f_0(x, y_2^m)R^{i-2}IR^{n-i} = \{0_M\}$ and $y_2^m R^{i-2}IR^{n-i} = \{0_M\}$. Thus $xR^{i-2}IR^{n-i} = \{0_M\}$; that is, $x \in M_0$. This shows that M_0 is a k-subsemimodule of M. Since $MR^{i-2}IR^{n-i} \neq \{0_M\}$, $M_0 \neq M$. Since M is semi-irreducible, $M_0 = \{0_M\}$ and therefore $a = 0$.

The converse part is obvious.

(2) Assume that M is irreducible. Let $a, b \in M$ be such that $a \neq b$. Set $u_j = a$, $v_j = b$ for $j = 2, \ldots, m$. Since $MR^{i-2}IR^{n-i} \neq \{0_M\}$, we have $xr_2^{i-1}sr_{i+1}^{n-i} \neq 0_M$ for some $x \in M, s \in I$ and $r_j \in R$. Since M is irreducible, according to Definition 14, there exist $a(i)_2^m, b(i)_2^m \in R$ with $i = 2, \ldots, m$ such that

$$\begin{aligned} &f_0\left(x, u_2 a(2)_2^n, \ldots, u_m a(m)_2^n,\right.\\ &\quad\left. v_2 b(2)_2^n, \ldots, v_m b(m)_2^n\right)\\ &\quad = f_0\left(0_M, u_2 b(2)_2^n, \ldots, u_m b(m)_2^n,\right.\\ &\qquad\left. v_2 a(2)_2^n, \ldots, v_m a(m)_2^n\right). \end{aligned} \tag{22}$$

Hence

$$f_0\left(xr_2^{i-1}sr_{i+1}^{n-i}, u_2a(2)_2^n r_2^{i-1}sr_{i+1}^{n-i}, \dots, u_m a(m)_2^n r_2^{i-1}sr_{i+1}^{n-i},\right.$$
$$\left. v_2b(2)_2^n r_2^{i-1}sr_{i+1}^{n-i}, \dots, v_m b(m)_2^n r_2^{i-1}sr_{i+1}^{n-i}\right)$$
$$= f_0\left(0_M r_2^{i-1}sr_{i+1}^{n-i}, u_2b(2)_2^n r_2^{i-1}sr_{i+1}^{n-i}, \dots,\right.$$
$$u_m b(m)_2^n r_2^{i-1}sr_{i+1}^{n-i},$$
$$\left. v_2a(2)_2^n r_2^{i-1}sr_{i+1}^{n-i}, \dots, v_m a(m)_2^n r_2^{i-1}sr_{i+1}^{n-i}\right)$$
$$= f_0\left(0_M, u_2b(2)_2^n r_2^{i-1}sr_{i+1}^{n-i}, \dots, u_m b(m)_2^n r_2^{i-1}sr_{i+1}^{n-i},\right.$$
$$\left. v_2a(2)_2^n r_2^{i-1}sr_{i+1}^{n-i}, \dots, v_m a(m)_2^n r_2^{i-1}sr_{i+1}^{n-i}\right). \tag{23}$$

Since M is cancellative and $xr_2^{i-1}sr_{i+1}^{n-i} \neq 0_M$, at least one of the following $2(m-1)$ equalities does not hold:

$$u_j a(j)_2^n r_2^{i-1}sr_{i+1}^{n-i} = v_j a(2)_2^n r_2^{i-1}sr_{i+1}^{n-i}, \quad \text{where } j = 2, \dots, m;$$
$$u_j b(j)_2^n r_2^{i-1}sr_{i+1}^{n-i} = v_j b(2)_2^n r_2^{i-1}sr_{i+1}^{n-i}, \quad \text{where } j = 2, \dots, m. \tag{24}$$

So we conclude that if $ar_2^{i-1}sr_{i+1}^n = br_2^{i-1}sr_{i+1}^n$ for all $r_j \in R$ and all $s \in I$, then $a = b$.

The converse part follows easily. □

Lemma 25. *Let M be a right n-ary R-semimodule and $M \neq \{0_M\}$. Then M is semi-irreducible if and only if for every nonzero $a \in M$, $\overline{aR^{n-1}} = M$.*

Proof. Assume that M is a semi-irreducible right n-ary R-semimodule and $M \neq \{0_M\}$. Let $a \in M$ be such that $a \neq 0_M$. Then by Lemma 24 $aR^{n-1} \neq \{0_M\}$. Since $\overline{aR^{n-1}}$ is an n-ary k-subsemimodule of M, $\overline{aR^{n-1}} = M$.

Conversely, suppose that for any nonzero $a \in M$, $\overline{aR^{n-1}} = M$. Let $N \neq \{0_M\}$ be an n-ary k-subsemimodule of M. Then there exists $b \in N$ such that $b \neq 0_M$. So by hypothesis, $\overline{bR^{n-1}} = M$. Hence for any $x \in M$, there exist $a_1^m \in bR^{n-1}$ such that $f_0(x, a_2^m) = a_1$. Since $bR^{n-1} \subseteq N$, we have $a_1^m \in N$. Since N is an n-ary k-subsemimodule, $f_0(x, a_2^m) = a_1$ implies that $x \in N$. This shows that $N = M$. Now if $MR^{n-1} = \{0_M\}$ then $aR^{n-1} = \{0_M\}$ for all $a \in M$. Hence $\overline{aR^{n-1}} = \{0_M\}$. So we have $M = \{0_M\}$, a contradiction. Therefore, $MR^{n-1} \neq \{0_M\}$. Thus M is semi-irreducible. □

Corollary 26. *If a right n-ary R-semimodule M is irreducible, then it is semi-irreducible and $\overline{MR^{n-1}} = M$.*

Proof. Assume that M is an irreducible right n-ary R-semimodule. Then $M \neq \{0_M\}$ and, consequently, there exists a nonzero $y \in M$. In view of (2) with $u_j = y$ and $v_j = 0_M$ for $j = 2, \dots, m$, we obtain that for any $x \in M$ there exist $a(j)_2^n \in R$ with $j = 2, \dots, m$ such that

$$f_0\left(x, ya(2)_2^n, \dots, ya(m)_2^n, \overset{(m-1)}{0_M}\right) = f_0\left(0_M, yb(2)_2^n, \dots, yb(m)_2^n, \overset{(m-1)}{0_M}\right), \tag{25}$$

so that

$$f_0\left(x, ya(2)_2^n, \dots, ya(m)_2^n\right) = f_0\left(0_M, yb(2)_2^n, \dots, yb(m)_2^n\right) \in yR^{n-1}. \tag{26}$$

It follows that $x \in \overline{yR^{n-1}}$. Thus $\overline{yR^{n-1}} = M$. By Lemma 25, M is semi-irreducible.

Furthermore, $MR^{n-1} \neq \{0_M\}$, which implies that $\overline{MR^{n-1}} \neq \{0_M\}$. Since $\overline{MR^{n-1}}$ is an n-ary k-subsemimodule of M, $\overline{MR^{n-1}} = M$ as required. □

Now we can define the Jacobson radical of an (m, n)-semiring in an external way.

Definition 27. Let R be an (m, n)-semiring and Δ be the set of all irreducible right n-ary R-semimodules. Then $J(R) = \bigcap_{M \in \Delta} A_R(M)$ is called the *Jacobson radical* of R. If Δ is empty then R itself is considered as $J(R)$; that is, $J(R) = R$, and in this case, we say that R is a *radical (m, n)-semiring*. An (m, n)-semiring R is said to be *Jacobson semisimple* or *J-semisimple* if $J(R) = \{0\}$.

By Lemma 12, $A_R(M)$ is an h-ideal of R. Note that the intersection of any family of h-ideals is again an h-ideal. Consequently, we obtain the following.

Lemma 28. *$J(R)$ is an h-ideal of R.*

Lemma 29. *If M is a right n-ary R-semimodule then M is a right n-ary $R/J(R)$-semimodule, where $R/J(R)$ is the Bourne factor semiring. Moreover, if M is an irreducible right n-ary R-semimodule, then M is also an irreducible right n-ary $R/J(R)$-semimodule.*

Proof. This lemma can be proved by the same method as in proving Lemma 17. □

Theorem 30. *If R is an (m, n)-semiring, then the Bourne factor semiring $R/J(R)$ is Jacobson semisimple.*

Proof. By Δ and Λ, we denote the set of all irreducible right n-ary R-semimodules and the set of all irreducible right n-ary $R/J(R)$-semimodules, respectively. Then according to Lemmas 28, 16, and, 29, we obtain that $\Delta = \Lambda$. For any $x \in J(R/J(R))$ and any $M \in \Delta$, we have $x \in A_{R/J(R)}(M)$, which means that for any $a \in M$, $ar_2/J(R) \cdots r_n/J(R) = 0_M$ whenever $r_2^n \in R$ and $r_i = x$ for some $i \in \{2, \dots, n\}$. Thus $ar_2 \cdots r_n = 0_M$ whenever $r_2^n \in R$ and $r_i = x$ for some $i \in \{2, \dots, n\}$, so $x \in A_R(M)$ for all $M \in \Lambda$. That is, $x \in J(R)$. Hence $x/J(R) = 0/J(R)$. We have shown that

$J(R/J(R)) = \{0/J(R)\}$. By Definition 27, $R/J(R)$ is Jacobson semisimple. □

The next theorem is a direct corollary of Theorem 22, giving an internal characterization of the Jacobson radical of an (m,n)-semiring.

Theorem 31. *$J(R)$ is the intersection of all primitive h-ideals of R.*

Definition 32. Let P be an i-ideal of an (m,n)-semiring R for some $i \in \{1,\dots,n\}$. Then P is said to be *strongly seminilpotent* if there exists a positive integer t such that $(PR^{n-2})^{t-1}P \subseteq Z(R)$, where $R^{n-2} = \overbrace{R\cdots R}^{n-2}$, $(PR^{n-2})^{t-1} = (PR^{n-2})(PR^{n-2})\cdots t-1$ times, $(PR^{n-1})^0P = P$. P is said to be *strongly nilpotent* if there exists a positive integer t such that $(PR^{n-2})^{t-1}P = \{0\}$.

Theorem 33. *If P is a strongly semi-nilpotent left ideal of R, then $P \subseteq J(R)$.*

Proof. Suppose on the contrary that

$$P \nsubseteq J(R) = \bigcap_{M\in\Delta} A_R(M), \tag{27}$$

where R is an (m,n)-semiring and Δ is the set of all irreducible right n-ary R-semimodules. Then there exists an $M \in \Delta$ such that $P \nsubseteq A_R(M)$. Thus there exists $i \in \{2,\dots,n\}$ such that

$$MR^{i-2}PR^{n-i} \neq \{0_M\}. \tag{28}$$

Since P is strongly semi-nilpotent, there exists a positive integer t such that $(PR^{n-2})^{t-1}P \subseteq Z(R)$. By Lemmas 10 and 12, $Z(R) \subseteq A_R(M)$. It follows that $(PR^{n-2})^{t-1}P \subseteq A_R(M)$, which implies that

$$MR^{i-2}\left(PR^{n-2}\right)^{t-1}PR^{n-i} = \{0_M\}. \tag{29}$$

If (29) holds for all positive integers t's, then in particular it is true for $t = 1$ and in this case we have $MR^{i-2}PR^{n-i} = \{0_M\}$, a contradiction to (28). If (29) does not hold for all t, then there exist $x \in M$ and positive s such that

$$\begin{gathered} xR^{i-2}\left(PR^{n-2}\right)^{s-1}PR^{n-1} \neq \{0_M\}, \\ xR^{i-2}\left(PR^{n-2}\right)^{s}PR^{n-i} = \{0_M\}. \end{gathered} \tag{30}$$

Thus $x \neq 0_M$ and there exists $a \in xR^{i-2}(PR^{n-2})^{s-1}PR^{n-i}$ such that $a \neq 0_M$. It follows that

$$\begin{aligned} aR^{i-2}PR^{n-i} &\subseteq xR^{i-2}\left(PR^{n-2}\right)^{s-1}PR^{n-i}R^{i-2}PR^{n-i} \\ &= xR^{i-2}\left(PR^{n-2}\right)^{s}PR^{n-i} = \{0_M\}, \end{aligned} \tag{31}$$

so we have

$$aR^{i-2}PR^{n-i} = \{0_M\}. \tag{32}$$

By Lemma 24, we obtain $a = 0_M$, again a contradiction. This completes the proof. □

The next result is a direct corollary of Theorem 33.

Corollary 34. *If an (m,n)-semiring R is Jacobson semisimple then R does not contain any non-zero strongly semi-nilpotent left ideal and hence R does not contain any nontrivial strongly nilpotent left ideal.*

Lemma 35. *If M is a (semi-)irreducible right n-ary R-semimodule and $N \neq \{0_M\}$ is an arbitrary R-subsemimodule (and $NR^{n-1} \neq \{0_M\}$), then N is (semi-)irreducible, and for any $x_2^n, y_2^n \in R$ the following statement is true: the equality $ux_2^n = uy_2^n$ holds for all $u \in M$ if and only if the same equality holds for all $u \in N$. Furthermore, $A_R(M) = A_R(N)$.*

Proof. Assume M is an irreducible right n-ary R-semimodule. Then from (2), it follows that N is irreducible. If M is a semi-irreducible and $NR^{n-1} \neq \{0_M\}$, then N is semi-irreducible by Definition 23 since any subsemimodule of N is clearly a subsemimodule of M.

Let $x_2^n, y_2^n \in R$ be such that the equality $ux_2^n = uy_2^n$ holds for all $u \in M$. Since M is semi-irreducible, for any $a(\neq 0_M) \in M$ and any $b(\neq 0_M) \in N$, there exist positive p, q and $x(i)_2^p, y(j)_2^q \in R$ such that $p \equiv q \equiv 0 \mod (m-1)$ and

$$f_0\left(a, bx(2)_2^n, \dots, bx(p)_2^n\right) = f_0\left(0_M, by(2)_2^n, \dots, by(q)_2^n\right). \tag{33}$$

Thus we have the following two equalities:

$$\begin{aligned} & f_0\left(ax_2^n, bx(2)_2^n x_2^n, \dots, bx(p)_2^n x_2^n\right) \\ &\quad = f_0\left(0_M, by(2)_2^n x_2^n, \dots, by(q)_2^n x_2^n\right), \\ & f_0\left(ay_2^n, bx(2)_2^n y_2^n, \dots, bx(p)_2^n y_2^n\right) \\ &\quad = f_0\left(0_M, by(2)_2^n y_2^n, \dots, by(q)_2^n y_2^n\right). \end{aligned} \tag{34}$$

It follows that

$$\begin{aligned} & f_0\left(0_M, ax_2^n, bx(2)_2^n x_2^n, \dots, bx(p)_2^n x_2^n,\right. \\ &\quad \left. by(2)_2^n y_2^n, \dots, by(q)_2^n y_2^n\right) \\ &\quad = f_0\left(0_M, ay_2^n, bx(2)_2^n y_2^n, \dots, bx(p)_2^n y_2^n,\right. \\ &\quad \left. by(2)_2^n x_2^n, \dots, by(q)_2^n x_2^n\right). \end{aligned} \tag{35}$$

Observing that $bx(i)_2^n, by(j)_2^n \in N$, since N is a submodule, we have $bx(i)_2^n x_2^n = bx(i)_2^n y_2^n$ and $by(2)_2^n x_2^n = by(2)_2^n y_2^n$ for all i, j by the assumption. Hence by cancellation law, (35) deduces that $ax_2^n = ay_2^n$. The converse implication is clear.

Furthermore, letting $y_i = 0$ for some i, we get that the equality $ux_2^n = 0_M$ holds for all $u \in M$ if and only if the same equality holds for all $u \in N$. Thus $A_R(M) = A_R(N)$. □

Lemma 36. *Let I be an ideal of an (m,n)-semiring R.*

(1) *If M is an (semi-)irreducible right n-ary R-semimodule (and $MR^{n-1} \neq \{0_M\}$), then M is an (semi-)irreducible right n-ary I-semimodule.*

(2) *If M is an irreducible right n-ary I-semimodule, then there exists an irreducible right n-ary R-semimodule M^*, which can be regarded as an I-subsemimodule N of M.*

Proof. (1) Let M be an irreducible R-semimodule and $u_2^m, v_2^m \in M$ be such that $u_j \neq v_j$ for some j. Without loss of generality, we suppose that $M \neq \{0_M\}$. From (2) we deduce that $MR^{n-1} \neq \{0_M\}$. By Lemma 24, $u_j c_2^n \neq v_j c_2^n$ for some $c_2^n \in I$. Since M is irreducible, by (19) there exist $a(i)_2^m, b(i)_2^m \in R$ with $i = 2, \ldots, m$ such that

$$\begin{aligned} f_0 & (x, (u_2 c_2^n) a(2)_2^n, \ldots, (u_m c_2^n) a(m)_2^n, \\ & (v_2 c_2^n) b(2)_2^n, \ldots, (v_m c_2^n) b(m)_2^n) \\ & = f_0 (0_M, (u_2 c_2^n) b(2)_2^n, \ldots, (u_m c_2^n) b(m)_2^n, \\ & \quad (v_2 c_2^n) a(2)_2^n, \ldots, (v_m c_2^n) a(m)_2^n), \end{aligned} \tag{36}$$

that is,

$$\begin{aligned} f_0 & (x, u_2 (c_2^n a(2)_2^n), \ldots, u_m (c_2^n a(m)_2^n), \\ & v_2 (c_2^n b(2)_2^n), \ldots, v_m (c_2^n b(m)_2^n)) \\ & = f_0 (0_M, u_2 (c_2^n b(2)_2^n), \ldots, u_m (c_2^n b(m)_2^n), \\ & \quad v_2 (c_2^n a(2)_2^n), \ldots, v_m (c_2^n a(m)_2^n)), \end{aligned} \tag{37}$$

which means that M is an irreducible I-semimodule by (19) again since for all $j \in \{2, \ldots, n\}$ $c_2^n a(j)_2^n, c_2^n b(j)_2^n \in I^{n-1}$.

Assume that M is a semi-irreducible R-semimodule and $MI^{n-1} \neq \{0_M\}$. According to Lemma 24, for any $u(\neq 0_M) \in M$ there exist $b_2^n \in R$ such that $ub_2^n \neq 0_M$. By Lemma 25, $\overline{(ub_2^n)R^{n-1}} = M$, so for any $x \in M$ there exist positive integers p, q and $x(i)_2^n, y(j)_2^n \in R$ such that $p \equiv q \equiv 0 \mod (m-1)$ and

$$\begin{aligned} f_0 & \left(x, (ub_2^n) x(2)_2^n, \ldots, (ub_2^n) x(p)_2^n\right) \\ & = f_0 \left(0_M, (ub_2^n) y(2)_2^n, \ldots, (ub_2^n) y(q)_2^n\right), \end{aligned} \tag{38}$$

which shows that

$$\begin{aligned} f_0 & \left(x, (ub_2^n x(2)_2^n), \ldots, u\left(b_2^n x(p)_2^n\right)\right) \\ & = f_0 \left(0_M, u\left(b_2^n x(2)_2^n\right), \ldots, u\left(b_2^n x(q)_2^n\right)\right). \end{aligned} \tag{39}$$

Note that for all i, j, $(b_2^n x(i)_2^n), (b_2^n x(j)_2^n) \in I^{n-1}$. Thus we obtain $\overline{uI^{n-1}} = M$. By Lemma 25 again, M is a semi-irreducible right n-ary I-semimodule.

(2) Let M be an irreducible right n-ary I-semimodule, and let $N = MI^{n-1}$. Then $N \neq \{0_M\}$ and N is an I-subsemimodule of M. Thus by Lemma 35, N is irreducible and for any $x_2^n, y_2^n \in R$ the following conclusion is true: the equality $ux_2^n = uy_2^n$ holds for all $u \in M$ if and only if the same equality holds for all $u \in N$. If $f_0(a(1)x(1)_2^n, \ldots, a(p)x(p)_2^n) = f_0(b(1)y(1)_2^n, \ldots, b(q)y(q)_2^n)$ for some $x(i)_2^n, y(j)_2^n \in I$ and $a(i), b(j) \in N$, then for any $r_2^n \in R$ and any $z_2^n \in I$,

$$\begin{aligned} f_0 & \left(a(1)\left(x(1)_2^n r_2^n\right), \ldots, a(p)\left(x(p)_2^n r_2^n\right)\right) z_2^n \\ & = f_0 \left(a(1)\left(x(1)_2^n r_2^n z_2^n\right), \ldots, a(p)\left(x(p)_2^n r_2^n z_2^n\right)\right) \\ & = f_0 \left(a(1) x(1)_2^n, \ldots, a(p) x(p)_2^n\right)\left(r_2^n z_2^n\right) \\ & = f_0 \left(b(1) y(1)_2^n, \ldots, b(q) y(q)_2^n\right)\left(r_2^n z_2^n\right) \\ & = f_0 \left(b(1)\left(y(1)_2^n r_2^n z_2^n\right), \ldots, b(q)\left(y(q)_2^n r_2^n z_2^n\right)\right) \\ & = f_0 \left(b(1)\left(y(1)_2^n r_2^n\right), \ldots, b(q)\left(y(q)_2^n r_2^n\right)\right) z_2^n, \end{aligned} \tag{40}$$

which implies that

$$\begin{aligned} f_0 & \left(a(1)\left(x(1)_2^n r_2^n\right), \ldots, a(p)\left(x(p)_2^n r_2^n\right)\right) \\ & = f_0 \left(b(1)\left(y(1)_2^n r_2^n\right), \ldots, b(q)\left(y(q)_2^n r_2^n\right)\right) \end{aligned} \tag{41}$$

by Lemma 24 since M is an irreducible right n-ary I-semimodule. Thus we can define an operation on NR^{n-1} into N by setting

$$\begin{aligned} f_0 & \left(a(1) x(1)_2^n, \ldots, a(p) x(p)_2^n\right) r_2^n \\ & = f_0 \left(a(1)\left(x(1)_2^n r_2^n\right), \ldots, a(p)\left(x(p)_2^n r_2^n\right)\right), \end{aligned} \tag{42}$$

where $x(i)_2^n, y(j)_2^n \in I$ and $a(i), b(j) \in N$. Thus N with the addition and the above operation becomes a right n-ary R-semimodule M^* which, as a right n-ary I-semimodule, is isomorphic to the right n-ary I-semimodule N. It is clear that M^* is an irreducible right n-ary R-semimodule. □

Now we are ready to generalize [17, Theorem 2].

Theorem 37. *If I is an ideal of an (m, n)-semiring R, then $J(I) = J(R) \cap I$.*

Proof. Let R be an (m, n)-semiring and let Δ be the set of all irreducible right n-ary R-semimodules. Then by Definition 27 $J(R) = \bigcap_{M \in \Delta} A_R(M)$. If I is an ideal of an (m, n)-semiring R, then $J(I) = \bigcap_{M \in \Lambda} A_I(M)$, where Λ is the set of all irreducible right n-ary I-semimodules.

For any $M \in \Delta$, according to Lemma 36, we have $M \in \Lambda$. It is evident that $A_I(M) = A_R(M) \cap I$. This shows that $J(I) \subseteq J(R) \cap I$.

For any $M \in \Lambda$, according to Lemma 36, we have that $M^* \in \Delta$ and that M^* can be regarded as an I-subsemimodule N of M. By Lemma 35, we have that $A_I(M) = A_I(N) = A_I(M^*) = A_R(M^*) \cap I$. This shows that $J(I) \supseteq J(R) \cap I$.

Summarizing the above, we obtain that $J(I) = J(R) \cap I$. □

Consequently, we have.

Theorem 38. *For an (m, n)-semiring R, $J(R)$ is a radical (m, n)-semiring; that is, $J(J(R)) = J(R)$.*

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