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ON THE JOINT DISTRIBUTION OF THE SUPREMUM, INFIMUM, AND THE VALUE OF A SEMICONTINUOUS PROCESS WITH INDEPENDENT INCREMENTS

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T. V. KADANKOVA

Abstract. The joint distribution of the supremum, infimum, and the value of a homogeneous lower semicontinuous process with independent increments is found in this paper.

The weak convergence of the boundary distribution to the corresponding distribution of the Wiener process is proved in the case of $\mathsf{E}\xi(1) = 0$ and $\mathsf{E}\xi^2(1) < \infty$. Exact and asymptotic relations are obtained for this distribution.

Let $\xi(t) \in \mathbf{R}$, $t \geq 0$, be a homogeneous lower semicontinuous process with independent increments [1] and let $k(p)$ be its cumulant:

$$\xi(0) = 0, \qquad \mathsf{E}\left[e^{-p\xi(t)}\right] = e^{tk(p)}, \qquad \operatorname{Re} p = 0.$$

The aim of this paper is to determine the joint distribution

$$Q^t(-y, \alpha, \beta, x) = \mathsf{P}\left[-y \leq \inf_{u \leq t} \xi(u),\, \xi(t) \in (\alpha, \beta),\, \sup_{u \leq t} \xi(u) \leq x\right] \tag{1}$$

where

$$x, y > 0, \qquad -y \leq \alpha < \beta \leq x.$$

This problem is solved in [2] for homogeneous processes with independent increments. The problem for semicontinuous processes with independent increments can be solved in the closed form in terms of the resolvent

$$R^s(x) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{px} \frac{1}{k(p) - s}\, dp, \qquad \gamma > c(s) \tag{2}$$

(see [2]–[6]) where $c(s) > 0$ for $s > 0$ is a unique in the half-plane $\operatorname{Re} p > 0$ positive root of the equation

$$k(p) - s = 0$$

(see [2]).

Now we state the main results of the paper.

Theorem 1. *Let $\xi(t)$, $t \geq 0$, be a homogeneous lower semicontinuous process, ν_s an exponential random variable with parameter $s > 0$, and let*

$$\tilde{Q}^s(-y, \alpha, \beta, x) = \int_0^\infty e^{-st} \mathsf{P}\left[-y \leq \inf_{u \leq t} \xi(u),\, \xi(t) \in (\alpha, \beta),\, \sup_{u \leq t} \xi(u) \leq x\right] dt,$$

$$Q_p^s(-y, x) = \int_{-y}^{x} e^{-pu} \mathsf{P}\left[-y \leq \inf_{u \leq \nu_s} \xi(u),\, \xi(\nu_s) \in du,\, \sup_{u \leq \nu_s} \xi(u) \leq x\right] du$$

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be the integral transforms of the joint distribution (1).

Then

$$Q_p^s(-y,x) = U_p^s(x) - e^{py}\frac{R^s(x)}{R^s(B)}U_p^s(B), \qquad B = x+y, \tag{3}$$

$$\tilde{Q}^s(-y,\alpha,\beta,x) = \frac{R^s(x)}{R^s(B)}\int_\alpha^\beta R^s(y+u)\,du - \int_{\max\{0,\alpha\}}^{\max\{0,\beta\}} R^s(u)\,du \tag{4}$$

where

$$U_p^s(x) = \mathsf{E}\left[e^{-p\xi(\nu_s)}; \xi^+(\nu_s) \le x\right] = \frac{c(s)}{c(s)-p}\,\mathsf{E}\left[e^{-p\xi^+(\nu_s)}; \xi^+(\nu_s) \le x\right], \tag{5}$$

$$\xi^+(t) = \sup_{u\le t}\xi(u), \qquad \xi^-(t) = \inf_{u\le t}\xi(u).$$

Corollary 1. *Let $w(t)$, $t \ge 0$, be the Wiener process with cumulant $k(p) = \frac{1}{2}\sigma^2p^2$ and let*

$$\chi = \inf\{t > 0\colon w(t) \notin (-y,x)\}$$

be the first exit time of the process $w(t)$, $t \ge 0$, from the interval $(-y,x)$.

Then

1) the following equalities hold:

$$\begin{aligned}\mathsf{P}\left[-y \le \inf_{u\le t} w(u),\ w(t)\in(\alpha,\beta),\ \sup_{u\le t} w(u) \le x\right] &\stackrel{\text{def}}{=} \bar{Q}^t(-y,\alpha,\beta,x)\\ &= \frac{4}{\pi}\sum_{\nu=1}^\infty \frac{1}{\nu}e^{-t(\pi\nu\sigma/B)^2/2}\sin\left(\frac{x}{B}\pi\nu\right)\\ &\qquad\times\sin\left(\frac{2x-\alpha-\beta}{2B}\pi\nu\right)\sin\left(\frac{\beta-\alpha}{2B}\pi\nu\right),\end{aligned} \tag{6}$$

$$\mathsf{P}[\chi > t] = \frac{4}{\pi}\sum_{\nu=0}^\infty \frac{1}{2\nu+1}e^{-t(\pi(2\nu+1)\sigma/B)^2/2}\sin\left(\frac{x}{B}(2\nu+1)\pi\right); \tag{7}$$

2) the first two moments of the random variable χ are given by

$$\mathsf{E}\,\chi = \frac{1}{\sigma^2}xy, \qquad \mathsf{E}\,\chi^2 = \frac{1}{3\sigma^4}xy\left(x^2+3xy+y^2\right),$$
$$\operatorname{Var}\chi = \frac{1}{3\sigma^4}xy\left(x^2+y^2\right).$$

Moreover, if $x = y$, then

$$\mathsf{E}\,\chi^n = \frac{1}{(2n-1)!!}\left(\frac{x}{\sigma}\right)^{2n}E_n, \qquad n > 0,$$

where E_n, $n>0$, are the Euler numbers;

3) the probability $\bar{Q}^t(-y,\alpha,\beta,x)$ is such that

$$\begin{aligned}&\bar{Q}^t(-y,\alpha,\beta,x)\\ &\quad= \frac{1}{\sigma\sqrt{2\pi t}}\int_\alpha^\beta\left(\sum_{k=-\infty}^\infty e^{-(2Bk+u)^2/2\sigma^2t} - \sum_{k=-\infty}^\infty e^{-(2Bk+2x-u)^2/2\sigma^2t}\right)du\end{aligned} \tag{8}$$

(see [2]*).*

Theorem 2. *Let $\mathsf{E}\,\xi(1) = 0$, $\mathsf{E}\,\xi^2(1) = \sigma^2 < \infty$, and*

$$x, y > 0, \qquad x+y = 1, \qquad -y \le \alpha < \beta \le x.$$

Then the joint distribution

$$\mathsf{P}\left[-yB \le \xi^-\left(tB^2\right), \xi\left(tB^2\right) \in (\alpha B, \beta B), \xi^+\left(tB^2\right) \le xB\right] \stackrel{\text{def}}{=} Q^t(-y,\alpha,\beta,x,B) \tag{9}$$

weakly converges as $B \to \infty$ to the joint distribution

$$\mathsf{P}\left[-y \le \inf_{u\le t} w(u),\ w(t) \in (\alpha,\beta), \sup_{u \le t} w(u) \le x\right] \tag{10}$$

of the supremum, infimum, and the value of the symmetric Wiener process $w(t)$, $t \ge 0$, with the cumulant

$$k(p) = \frac{1}{2}\sigma^2 p^2.$$

Moreover,

$$\begin{aligned}&\lim_{B\to\infty} Q^t(-y,\alpha,\beta,x,B)\\ &\quad= \frac{4}{\pi}\sum_{\nu=1}^{\infty}\frac{1}{\nu}e^{-t(\pi\nu\sigma)^2/2}\sin(x\pi\nu)\sin\left(\frac{2x-\alpha-\beta}{2}\pi\nu\right)\sin\left(\frac{\beta-\alpha}{2}\pi\nu\right).\end{aligned} \tag{11}$$

The limit distribution (11) *is such that*

$$\begin{aligned}&\lim_{B\to\infty} Q^t(-y,\alpha,\beta,x,B)\\ &\quad= \frac{1}{\sigma\sqrt{2\pi t}}\int_\alpha^\beta\left(\sum_{k=-\infty}^{\infty} e^{-(2k+u)^2/2\sigma^2 t} - \sum_{k=-\infty}^{\infty} e^{-(2k+2x-u)^2/2\sigma^2 t}\right)du.\end{aligned}$$

Corollary 2. *Let $\mathsf{E}\,\xi(1) = 0$, $\mathsf{E}\,\xi^2(1) = \sigma^2 < \infty$, $x, y > 0$, $x + y = 1$, and let*

$$\chi(B) = \inf\{t > 0 \colon \xi(t) \notin (-yB, xB)\}$$

be the first exit time of the process $\xi(t)$, $t \ge 0$, from the interval $(-yB, xB)$.

Then the random variable $\frac{1}{B^2}\chi(B)$ weakly converges as $B \to \infty$ to the first exit time

$$\chi = \inf\{t > 0 \colon w(t) \notin (-y, x)\}$$

of the Wiener process from the interval $(-y, x)$. Moreover,

$$\begin{aligned}\lim_{B\to\infty}\mathsf{P}\left[\frac{\chi(B)}{B^2} > t\right] &= \frac{4}{\pi}\sum_{\nu=0}^{\infty}\frac{1}{2\nu+1}e^{-t(\pi(2\nu+1)\sigma)^2/2}\sin(x(2\nu+1)\pi)\\ &= \mathsf{P}[\chi > t].\end{aligned} \tag{12}$$

Remark. The right-hand sides of equalities (6), (7), (11), and (12) can be used to determine the asymptotic expansions for the probabilities on the corresponding left-hand sides. For example, considering only the first terms in expansions (6) and (7) we get

$$\begin{aligned}&\mathsf{P}\left[-y \le \inf_{u\le t} w(u),\ w(t) \in (0x), \sup_{u\le t} w(u) \le x\right]\\ &\qquad= \frac{4}{\pi}e^{-t(\pi\sigma/B)^2/2}\sin\left(\frac{x}{B}\pi\right)\sin^2\left(\frac{x}{2B}\pi\right) + o\left(e^{-t(\pi\sigma/B)^2/2}\right),\\ &\mathsf{P}[\chi > t] = \frac{4}{\pi}e^{-t(\pi\sigma/B)^2/2}\sin\left(\frac{x}{B}\pi\right) + o\left(e^{-t(\pi\sigma/B)^2/2}\right)\end{aligned}$$

as $t \to \infty$.

Proof of Theorem 1. Let

$$\chi = \inf\{t > 0 \colon \xi(t) \notin (-y, x)\}$$

be the first exit time of the process $\xi(t)$, $t \geq 0$, from the interval $(-y, x)$. According to the total probability formula we have

$$
\begin{aligned}
&\mathsf{E}\left[e^{-p\xi(\nu_s)}; \xi^+(\nu_s) \leq x\right] \\
&\quad = \mathsf{E}\left[e^{-p\xi(\nu_s)}; -y \leq \xi^-(\nu_s),\, \xi^+(\nu_s) \leq x\right] \\
&\quad + \mathsf{E}\left[e^{-s\chi} e^{-p\xi(\chi)}; \xi(\chi) = -y\right] \mathsf{E}\left[e^{-p\xi(\nu_s)}; \xi^+(\nu_s) \leq B\right],
\end{aligned}
\tag{13}
$$

since the event that the process $\xi(t)$ does not exceed the upper level x on the interval $[0, \nu_s]$ (the left-hand side of the equality) occurs if either $\xi(t)$ does not cross the lower level $-y$ (the first term on the right-hand side of (13)) or it crosses the lower level $-y$ but then its increments do not exceed the upper level $x + y = B$ (the second term on the right-hand side of (13)). According to results of the paper [7],

$$
\mathsf{E}\left[e^{-s\chi} e^{-p\xi(\chi)}; \xi(\chi) = -y\right] = e^{py} \frac{R^s(x)}{R^s(B)},
$$

whence

$$
\mathsf{E}\left[e^{-p\xi(\nu_s)}; -y \leq \xi^-(\nu_s),\, \xi^+(\nu_s) \leq x\right] = Q_p^s(-y, x).
$$

Thus (13) implies (3). Now we prove (5).

Let $\xi(t)$, $t \geq 0$, be a homogeneous process with independent increments such that $\xi(0) = 0$. For $x > 0$ put

$$
\tau_x = \inf\{t > 0 \colon \xi(t) > x\}.
$$

Applying the total probability formula we obtain

$$
\mathsf{E}\, e^{-p\xi(\nu_s)} = \mathsf{E}\left[e^{-p\xi(\nu_s)}; \xi^+(\nu_s) \leq x\right] + \mathsf{E}\left[e^{-s\tau_x} e^{-p\xi(\tau_x)}\right] \mathsf{E}\, e^{-p\xi(\nu_s)},
\tag{14}
$$

since the behavior of the process $\xi(t)$, $t \geq 0$, on the interval $[0, \nu_s]$ (the left-hand side of the equality) is such that either ξ does not exceed the upper level x (the first term on the right-hand side of (14)) or it exceeds the level x but then it varies in an exponentially distributed time $[0, \nu_s]$ (the second term on the right-hand side of (14)).

Using the Spitzer–Rogozin equality

$$
\mathsf{E}\, e^{-p\xi(\nu_s)} = \mathsf{E}\left[e^{-p\xi^+(\nu_s)}\right] \mathsf{E}\left[e^{-p\xi^-(\nu_s)}\right], \qquad \operatorname{Re} p = 0,
$$

we rewrite (14) as follows:

$$
\begin{gathered}
\left(\mathsf{E}\, e^{-p\xi^-(\nu_s)}\right)^{-1} \mathsf{E}\left[e^{-p(\xi(\nu_s) - x)}; \xi^+(\nu_s) \leq x\right] - \mathsf{E}\left[e^{-p(\xi^+(\nu_s) - x)}; \xi^+(\nu_s) \leq x\right] \\
= \mathsf{E}\left[e^{-p(\xi^+(\nu_s) - x)}; \xi^+(\nu_s) > x\right] - \mathsf{E}\, e^{-p\xi^+(\nu_s)}\, \mathsf{E}\left[e^{-s\tau_x} e^{-p(\xi(\tau_x) - x)}\right], \\
\operatorname{Re} p = 0.
\end{gathered}
$$

Following the standard reasoning based on the factorization (see [8]) we obtain from the latter equality that

$$
\begin{gathered}
\mathsf{E}\left[e^{-s\tau_x} e^{-p\xi(\tau_x)}\right] = \left(\mathsf{E}\, e^{-p\xi^+(\nu_s)}\right)^{-1} \mathsf{E}\left[e^{-p\xi^+(\nu_s)}; \xi^+(\nu_s) > x\right], \\
\operatorname{Re} p \geq 0;
\end{gathered}
$$

$$
\begin{gathered}
\mathsf{E}\left[e^{-p\xi(\nu_s)}; \xi^+(\nu_s) \leq x\right] = \mathsf{E}\, e^{-p\xi^-(\nu_s)}\, \mathsf{E}\left[e^{-p\xi^+(\nu_s)}; \xi^+(\nu_s) \leq x\right], \\
\operatorname{Re} p \leq 0.
\end{gathered}
\tag{15}
$$

Note that the above approach is only one of several methods that can be used to get the distributions of $(\xi(\cdot), \xi^+(\cdot))$ and $(\tau_x, \xi(\tau_x))$. The joint distribution of a homogeneous process with independent increments and its maximum is studied in [9].

Since

$$\mathsf{E}\, e^{-p\xi^-(\nu_s)} = \frac{c(s)}{c(s)-p}, \quad \operatorname{Re} p \le 0,$$
$$\mathsf{E}\, e^{-p\xi^+(\nu_s)} = \frac{s}{c(s)}\frac{p-c(s)}{k(p)-s}, \quad \operatorname{Re} p \ge 0,$$

for lower semicontinuous processes, relation (15) implies (5).

Now we find the resolvent representation of the right-hand side of (15). Multiplying both sides of this equality by $e^{-\lambda x}$ and integrating with respect to $x > 0$, we get

$$\int_0^\infty e^{-\lambda x}\,\mathsf{E}\left[e^{-p\xi(\nu_s)}; \xi^+(\nu_s) \le x\right] dx = \frac{1}{\lambda}\,\mathsf{E}\left[e^{-(p+\lambda)\xi^+(\nu_s)}\right]\mathsf{E}\left[e^{-p\xi^-(\nu_s)}\right]$$
$$= \frac{s}{c(s)-p}\frac{1}{k(p+\lambda)-s} - \frac{s}{\lambda}\frac{1}{k(p+\lambda)-s},$$
$$\operatorname{Re}\lambda, p \ge 0.$$

Using definition (2) of the resolvent, we obtain from the latter equality that

$$\mathsf{E}\left[e^{-p\xi(\nu_s)}; \zeta^+(\nu_s) \le x\right] = \frac{s}{c(s)-p} e^{-px} R^s(x) - s\int_0^x e^{-pu} R^s(u)\, du. \tag{16}$$

Now we find the resolvent representation for the function $Q_p^s(-y,x)$.

Substituting (16) into (3) we get

$$Q_p^s(-y,x) = s\frac{R^s(x)}{R^s(B)} e^{py}\int_0^B e^{-pu} R^s(u)\, du - s\int_0^x e^{-pu}R^s(u)\, du$$

or

$$\int_{-y}^x e^{-pu}\mathsf{P}\left[-y \le \xi^-(\nu_s),\, \xi(\nu_s)\in du,\, \xi^+(\nu_s) \le x\right]$$
$$= \int_{-y}^x e^{-pu}\left(s\frac{R^s(x)}{R^s(B)} R^s(y+u) - sR^s(u)\right) du,$$
$$R^s(u) \overset{\text{def}}{=} 0, \qquad u \le 0,$$

whence

$$\mathsf{P}\left[-y \le \xi^-(\nu_s),\, \xi(\nu_s)\in du,\, \xi^+(\nu_s) \le x\right] = s\left\{\frac{R^s(x)}{R^s(B)} R^s(y+u) - sR^s(u)\right\}, \tag{17}$$
$$u \in [-y, x].$$

Integrating the latter equality in the interval (α,β) and taking into account that

$$R^s(u) = 0$$

for $u \le 0$ we prove that

$$s\tilde{Q}^s(-y,\alpha,\beta,x) = s\frac{R^s(x)}{R^s(B)}\int_\alpha^\beta R^s(y+u)\, du - s\int_{\max\{0,\alpha\}}^{\max\{0,\beta\}} R^s(u)\, du.$$

This result coincides with (4). Thus Theorem 1 is proved. Another method to obtain equality (17) for a Poisson process with two-sided reflection is described in [10]. □

Proof of Corollary 1. If $w(t)$, $t \ge 0$, is a Wiener process with cumulant $\frac{1}{2}p^2\sigma^2$, then

$$R^s(x) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{px}\frac{dp}{\frac{1}{2}p^2\sigma^2 - s} = \frac{2}{\sigma\sqrt{2s}}\operatorname{sh}\left(\frac{x}{\sigma}\sqrt{2s}\right), \qquad \gamma > \frac{1}{\sigma}\sqrt{2s}, \tag{18}$$

where

$$\operatorname{sh} u = \frac{1}{2}(e^u - e^{-u}).$$

Then relation (4) yields

$$
\begin{aligned}
\tilde{Q}^s(-y,\alpha,\beta,x) = \frac{1}{s}\frac{\operatorname{sh}\frac{x}{\sigma}\sqrt{2s}}{\operatorname{sh}\frac{B}{\sigma}\sqrt{2s}}\left[\operatorname{ch}\left(\frac{y+\beta}{\sigma}\sqrt{2s}\right) - \operatorname{ch}\left(\frac{y+\alpha}{\sigma}\sqrt{2s}\right)\right] \\
- \frac{1}{s}\left[\operatorname{ch}\left(\frac{\beta^+}{\sigma}\sqrt{2s}\right) - \operatorname{ch}\left(\frac{\alpha^+}{\sigma}\sqrt{2s}\right)\right]
\end{aligned}
\tag{19}
$$

where

$$
\alpha^+ = \max\{0,\alpha\}, \qquad \beta^+ = \max\{0,\beta\},
$$

$$
\operatorname{ch} u = \frac{1}{2}\left(e^u + e^{-u}\right).
$$

Considering the cases $0 \le \alpha < \beta$, $\alpha < 0 < \beta$, and $\alpha < \beta \le 0$ separately we obtain from (19) that

$$
\tilde{Q}^s(-y,\alpha,\beta,x) = \frac{2}{s}\frac{\operatorname{sh}\frac{y}{\sigma}\sqrt{2s}}{\operatorname{sh}\frac{B}{\sigma}\sqrt{2s}}\operatorname{sh}\left(\frac{2x-\alpha-\beta}{2\sigma}\sqrt{2s}\right)\operatorname{sh}\left(\frac{\beta-\alpha}{2\sigma}\sqrt{2s}\right), \quad 0 \le \alpha < \beta,
\tag{20}
$$

$$
\begin{aligned}
\tilde{Q}^s(-y,\alpha,\beta,x) = \frac{2}{s}\frac{\operatorname{sh}\frac{y}{\sigma}\sqrt{2s}}{\operatorname{sh}\frac{B}{\sigma}\sqrt{2s}}\operatorname{sh}\left(\frac{2x-\beta}{2\sigma}\sqrt{2s}\right)\operatorname{sh}\left(\frac{\beta}{2\sigma}\sqrt{2s}\right) \\
+ \frac{2}{s}\frac{\operatorname{sh}\frac{x}{\sigma}\sqrt{2s}}{\operatorname{sh}\frac{B}{\sigma}\sqrt{2s}}\operatorname{sh}\left(\frac{2y+\alpha}{2\sigma}\sqrt{2s}\right)\operatorname{sh}\left(\frac{-\alpha}{2\sigma}\sqrt{2s}\right), \\
\alpha < 0 < \beta,
\end{aligned}
\tag{20*}
$$

$$
\tilde{Q}^s(-y,\alpha,\beta,x) = \frac{2}{s}\frac{\operatorname{sh}\frac{x}{\sigma}\sqrt{2s}}{\operatorname{sh}\frac{B}{\sigma}\sqrt{2s}}\operatorname{sh}\left(\frac{2y+\alpha+\beta}{2\sigma}\sqrt{2s}\right)\operatorname{sh}\left(\frac{\beta-\alpha}{2\sigma}\sqrt{2s}\right), \quad \alpha < \beta \le 0.
\tag{20**}
$$

Now we turn to equality (20). According to the inversion formula

$$
Q^t(-y,\alpha,\beta,x) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}\tilde{Q}^s(-y,\alpha,\beta,x)\,ds, \qquad \gamma > 0
$$

(see [3]). The integrand is an analytic function everywhere in the plane except for the points

$$
s_\nu = -\frac{1}{2}\left(\frac{\pi\nu\sigma}{B}\right)^2, \qquad \nu \in \mathbf{N}^+ = \{1,2,\dots\}
$$

where it has simple poles. Considering appropriate contours for integration (see [3]) and performing necessary transforms for the probability $Q^t(-y,\alpha,\beta,x)$ we obtain

$$
\begin{aligned}
Q^t(-y,\alpha,\beta,x) &= \sum_{\nu=1}^{\infty} \operatorname{Re} s_{s=s_\nu}\left(e^{st}\tilde{Q}^s(-y,\alpha,\beta,x)\right) \\
&= \frac{4}{\pi}\sum_{\nu=1}^{\infty}\frac{1}{\nu}e^{-t(\pi\nu\sigma/B)^2/2}\sin\left(\frac{x}{B}\nu\pi\right)\sin\left(\frac{2x-\alpha-\beta}{2B}\nu\pi\right)\sin\left(\frac{\beta-\alpha}{2B}\nu\pi\right).
\end{aligned}
\tag{21}
$$

Applying the same method to equalities (20*) and (20**) we prove that the joint distribution $Q^t(-y,\alpha,\beta,x)$ is given by (21) for all cases under consideration. Thus equality (6) is proved.

Since

$$
\mathsf{P}[\chi > t] = \mathsf{P}\left[-y \le \inf_{u\le t} w(u),\ \sup_{u\le t} w(u) \le x\right],
$$

we put $\alpha = -y$ and $\beta = x$ in (21) and obtain

$$\mathsf{P}[\chi > t] = \frac{4}{\pi} \sum_{\nu=0}^{\infty} \frac{1}{2\nu+1} e^{-t(\pi(2\nu+1)\sigma/B)^2/2} \sin\left(\frac{x}{B}(2\nu+1)\pi\right) \tag{22}$$

which in fact coincides with equality (7). Putting $t = 0$ in (22) we evaluate the series

$$\sum_{\nu=0}^{\infty} \frac{1}{2\nu+1} \sin\left(\frac{x}{B}(2\nu+1)\pi\right) = \frac{\pi}{4}, \qquad x, y > 0, \quad B = x + y. \tag{23}$$

Now we evaluate the first two moments of the random variable χ.

It follows from (20^*) for $\beta = x$ and $\alpha = -y$ that

$$\mathsf{E}\, e^{-s\chi} = \operatorname{ch}\left(\frac{x-y}{2\sigma}\sqrt{2s}\right) \Big/ \operatorname{ch}\left(\frac{x+y}{2\sigma}\sqrt{2s}\right). \tag{24}$$

Expanding the right-hand side of the latter equality into a series in powers of s, we find

$$\mathsf{E}\,\chi = \frac{1}{\sigma^2}xy, \qquad \mathsf{E}\,\chi^2 = \frac{1}{3\sigma^4}xy\left(x^2+3xy+y^2\right), \qquad \operatorname{Var}\chi = \frac{1}{3\sigma^4}xy\left(x^2+y^2\right),$$
$$x, y > 0.$$

On the other hand, integrating equality (22) we obtain

$$\mathsf{E}\,\chi = \frac{8B^2}{\pi^3\sigma^2} \sum_{\nu=0}^{\infty} \frac{1}{(2\nu+1)^3} \sin\left(\frac{x}{B}(2\nu+1)\pi\right),$$
$$\mathsf{E}\,\chi^2 = \frac{32B^4}{\pi^5\sigma^4} \sum_{\nu=0}^{\infty} \frac{1}{(2\nu+1)^5} \sin\left(\frac{x}{B}(2\nu+1)\pi\right).$$

The term-by-term integration is justified by the uniform convergence of series (22) for $t \geq 0$, which in turn follows from (23). Comparing the latter equalities with the preceding ones we get for $x, y > 0$ and $B = x + y$ that

$$\begin{aligned} \sum_{\nu=0}^{\infty} \frac{1}{(2\nu+1)^3} \sin\left(\frac{x}{B}(2\nu+1)\pi\right) &= \frac{\pi^3}{8B^2}xy, \\ \sum_{\nu=0}^{\infty} \frac{1}{(2\nu+1)^5} \sin\left(\frac{x}{B}(2\nu+1)\pi\right) &= \frac{\pi^5}{96B^4}xy\left(x^2+3xy+y^2\right). \end{aligned} \tag{25}$$

In particular, we put $x = y > 0$ in (23) and (25) and obtain

$$\sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{2\nu+1} = \frac{\pi}{4}, \qquad \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{(2\nu+1)^3} = \frac{\pi^3}{32}, \qquad \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{(2\nu+1)^5} = \frac{5\pi^5}{1536}.$$

Using (22) and (24) one can evaluate higher moments of the random variable χ.

Further, relation (24) for $x = y$ implies that

$$\mathsf{E}\, e^{-s\chi} = \frac{1}{\operatorname{ch}\left(\frac{x}{\sigma}\sqrt{2s}\right)} = \operatorname{sech}\left(\frac{x}{\sigma}\sqrt{2s}\right).$$

Using the Taylor expansion of the function $\operatorname{sech}(\cdot)$ we get

$$\sum_{n=0}^{\infty} (-1)^n \frac{s^n}{n!}\,\mathsf{E}\,\chi^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(\frac{x}{\sigma}\sqrt{2s}\right) E_n$$

where $E_0 = 1$ and E_n, $n > 0$, are Euler numbers.

Comparing the coefficients of the powers of s we conclude that

$$\mathsf{E}\,\chi^n = \frac{1}{(2n-1)!!}\left(\frac{x}{\sigma}\right)^{2n} E_n, \qquad n > 0.$$

Now we derive representation (8) for the probability $\bar{Q}^t(-y,\alpha,\beta,x)$. Equalities (17) and (18) imply

$$\begin{aligned}(26)\qquad &\int_0^\infty e^{-st}\mathsf{P}\left[-y\le\inf_{u\le t}w(u),\, w(u)\in du,\,\sup_{u\le t}w(u)\le x\right]\stackrel{\text{def}}{=}\tilde{q}^s(-y,u,x)\,du\\ &=\frac{2}{\sigma\sqrt{2s}}\left[\frac{\operatorname{sh}\frac{x}{\sigma}\sqrt{2s}}{\operatorname{sh}\frac{B}{\sigma}\sqrt{2s}}\operatorname{sh}\left(\frac{y+u}{\sigma}\sqrt{2s}\right)-\operatorname{sh}\left(\frac{|u|+u}{2\sigma}\sqrt{2s}\right)\right]du,\\ &\qquad u\in[-y,x].\end{aligned}$$

It follows from (26) that

$$(27)\qquad \tilde{q}^s(-y,u,x)=\frac{1}{\sigma\sqrt{2s}}\frac{1}{\operatorname{sh}(\frac{B}{\sigma}\sqrt{2s})}\left[\operatorname{ch}\left(\frac{B-|u|}{\sigma}\sqrt{2s}\right)-\operatorname{ch}\left(\frac{x-y-u}{\sigma}\sqrt{2s}\right)\right],\quad u\in[-y,x].$$

To invert the Laplace transform on the right-hand side of (27) we use the following expansion:

$$\frac{1}{2\operatorname{sh}(B\sqrt{2s}/\sigma)}=\sum_{k=0}^\infty e^{-B(2k+1)\sqrt{2s}/\sigma}$$

(see [2]) and the equality

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}e^{st}\frac{1}{\sqrt{s}}e^{-a\sqrt{s}}\,ds=\frac{1}{\sqrt{\pi t}}e^{-a^2/4t},\qquad \gamma>0,\ a>0.$$

Then

$$\begin{aligned}&\mathsf{P}\left[-y\le\inf_{u\le t}w(u),\, w(t)\in du,\,\sup_{u\le t}w(u)\le x\right]=\left(\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}e^{st}\tilde{q}^s(-y,u,x)\,ds\right)du\\ &=\frac{1}{\sigma\sqrt{2\pi t}}\left\{\sum_{k=-\infty}^\infty e^{-(2Bk+|u|)^2/2\sigma^2t}-\sum_{k=-\infty}^\infty e^{-(2Bk+2x-u)^2/2\sigma^2t}\right\}du,\\ &\qquad u\in[-y,x].\end{aligned}$$

Integrating this equality in the interval (α,β) we prove equality (8). This completes the proof of Corollary 1. □

Proof of Theorem 2. Suppose the assumptions of Theorem 2 hold. Since

$$\begin{aligned}&\frac{s}{B^2}\tilde{Q}^{s/B^2}(-yB,\alpha B,\beta B,xB)\\ &=\frac{s}{B^2}\int_0^\infty e^{-us/B^2}\mathsf{P}[-yB\le\xi^-(u),\,\xi(u)\in(\alpha B,\beta B),\,\xi^+(u)\le xB]\,du\\ &=s\int_0^\infty e^{-st}\mathsf{P}\left[-yB\le\xi^-(tB^2),\,\xi(tB^2)\in(\alpha B,\beta B),\,\xi^+(tB^2)\le xB\right]dt\\ &=s\int_0^\infty e^{-st}Q^t(-y,\alpha,\beta,x,B)\,dt,\end{aligned}$$

we get

$$\begin{aligned}(28)\qquad &\lim_{B\to\infty}s\int_0^\infty e^{-st}Q^t(-y,\alpha,\beta,x,B)\,dt=\lim_{B\to\infty}\frac{s}{B^2}\tilde{Q}^{s/B^2}(-yB,\alpha B,\beta B,xB,B)\\ &=\lim_{B\to\infty}\frac{s}{B^2}\left(\frac{R^{s/B^2}(xB)}{R^{s/B^2}(B)}\int_{B(y+\alpha)}^{B(y+\beta)}R^{s/B^2}(u)\,du-\int_{B\max\{0,\alpha\}}^{B\max\{0,\beta\}}R^{s/B^2}(u)\,du\right)\end{aligned}$$

by relation (4) and the preceding chain of equalities.

Asymptotic properties of the resolvent and potential of a semicontinuous process with independent increments are studied in [5, 6]. In particular, it is proved in [5, 6] that

$$\lim_{B\to\infty} \frac{1}{B} R^{s/B^2}(xB) = \frac{1}{\sigma}\sqrt{\frac{2}{s}}\,\mathrm{sh}\left(\frac{x}{\sigma}\sqrt{2s}\right),$$
$$\lim_{B\to\infty} \frac{s}{B^2}\int_0^{xB} R^{s/B^2}(u)\,du = \mathrm{ch}\left(\frac{x}{\sigma}\sqrt{2s}\right) - 1$$

under the assumptions of the theorem. Using the latter equalities and evaluating the limits on the right-hand side of (28) we deduce

$$\begin{aligned}\lim_{B\to\infty}\int_0^\infty e^{-st}Q^t(-y,\alpha,\beta,x,B)\,dt &= \frac{1}{s}\frac{\mathrm{sh}\,\frac{x}{\sigma}\sqrt{2s}}{\mathrm{sh}\,\frac{1}{\sigma}\sqrt{2s}}\left[\mathrm{ch}\left(\frac{y+\beta}{\sigma}\sqrt{2s}\right) - \mathrm{ch}\left(\frac{y+\alpha}{\sigma}\sqrt{2s}\right)\right]\\ &\quad - \frac{1}{s}\left[\mathrm{ch}\left(\frac{\beta^+}{\sigma}\sqrt{2s}\right) - \mathrm{ch}\left(\frac{\alpha^+}{\sigma}\sqrt{2s}\right)\right],\\ \alpha^+ = \max\{0,\alpha\}, &\qquad \beta^+ = \max\{0,\beta\}.\end{aligned}$$

The right-hand side of this equality coincides with the right-hand side of equality (19) for $B = 1$. Thus the left-hand sides of these equalities coincide as well. Therefore

$$\begin{aligned}&\lim_{B\to\infty}\int_0^\infty e^{-st}\mathsf{P}\left[-yB \le \xi^-(tB^2),\, \xi(tB^2)\in(\alpha B,\beta B),\, \xi^+(tB^2)\le xB\right]\,dt\\ &\quad = \int_0^\infty e^{-st}\mathsf{P}\left[-y \le \inf_{u\le t} w(u),\, w(u)\in du,\, \sup_{u\le t} w(u) \le x\right]dt,\\ &\qquad\qquad x,y>0, \qquad x+y=1,\end{aligned}$$

and the weak convergence as $B\to\infty$ of the joint distribution $Q^t(-y,\alpha,\beta,x,B)$ to the corresponding joint distribution of the Wiener process is proved. Equality (11) follows from (6) for $B = 1$.

To prove Corollary 2 we note that

$$\mathsf{P}\left[\frac{1}{B^2}\chi(B) > t\right] = \mathsf{P}\left[\chi(B) > tB^2\right] = Q^t(-y,-y,x,x,B),$$

whence

$$\begin{aligned}\lim_{B\to\infty}\mathsf{P}\left[\frac{1}{B^2}\chi(B) > t\right] &= \lim_{B\to\infty} Q^t(-y,-y,x,x,B)\\ &= \mathsf{P}\left[-y \le \inf_{u\le t} w(u), \sup_{u\le t} w(u) \le x\right]dt = \mathsf{P}[\chi > t]\\ &= \frac{4}{\pi}\sum_{\nu=0}^\infty \frac{1}{2\nu+1} e^{-t(\pi(2\nu+1)\sigma/B)^2/2}\sin(x(2\nu+1)\pi).\end{aligned}$$

Thus Theorem 2 and Corollary 2 are proved. □

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Mechanics and Mathematics Faculty, National Taras Shevchenko University, Academician Glushkov Avenue 6, Kyiv 03127, Ukraine

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