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# On the k-Fibonacci words

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**Abstract.** In this paper we define the k-Fibonacci words in analogy with the definition of the k-Fibonacci numbers. We study their properties and we associate to this family of words a family of curves with interesting patterns.

## 1 Introduction

Fibonacci numbers and their generalizations have many interesting properties and applications to almost every fields of science and arts (e.g. see [13]). The Fibonacci numbers $F_n$ are the terms of the sequence $0, 1, 1, 2, 3, 5, \ldots$ wherein each term is the sum of the two previous terms, beginning with the values $F_0 = 0$, and $F_1 = 1$.

Besides the usual Fibonacci numbers many kinds of generalizations of these numbers have been presented in the literature. In particular, a generalization is the k-Fibonacci numbers [11].

For any positive real number $k$, the k-Fibonacci sequence, say $\{F_{k,n}\}_{n\in\mathbb{N}}$ is defined recurrently by

$$F_{k,0} = 0, \ F_{k,1} = 1 \ \text{ and } \ F_{k,n+1} = kF_{k,n} + F_{k,n-1}, \ n \geqslant 1. \quad (1)$$

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In [11], k-Fibonacci numbers were found by studying the recursive application of two geometrical transformations used in the four-triangle longest-edge (4TLE) partition. These numbers have been studied in several papers, see [5, 10, 11, 12, 18, 19, 23].

The characteristic equation associated to the recurrence relation (1) is $x^2 = kx + 1$. The roots of this equation are

$$r_{k,1} = \frac{k + \sqrt{k^2+4}}{2}, \quad \text{and} \quad r_{k,2} = \frac{k - \sqrt{k^2+4}}{2}.$$

Some of the properties that the k-Fibonacci numbers verify are (see [11, 12] for the proofs).

- Binet Formula: $F_{k,n} = \frac{r_{k,1}^n - r_{k,2}^n}{r_{k,1} - r_{k,2}}$.
- Combinatorial Formula: $F_{k,n} = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-i}{i} k^{n-1-2i}$.
- $\lim_{n\to\infty} \frac{F_{k,n}}{F_{k,n-1}} = r_{k,1}$.

On the other hand, there is a word-combinatorial interpretation of the Fibonacci sequence. Fibonacci words are words over $\{0,1\}$ defined recursively as follows:

$$f_0 = 1, \qquad f_1 = 0, \qquad f_n = f_{n-1} f_{n-2}, \quad n \geqslant 2.$$

The words $f_n$ are referred to as the finite Fibonacci words and it is clear that $|f_n| = F_{n+1}$. The limit

$$\mathbf{f} = \lim_{n\to\infty} f_n = 0100101001001010010100100101\cdots$$

is called the Fibonacci word. This word is certainly one of the most studied words in the combinatorics on words, (see, e.g., [2, 6, 7, 9, 15, 22]). It is the archetype of a Sturmian word [14]. This word can be associated with a curve, which has fractal properties obtained from combinatorial properties of $\mathbf{f}$ [3, 16, 21].

In this paper we introduce a family of words $\mathbf{f}_k$ that generalize the Fibonacci word. Specifically, the k-Fibonacci words are words over $\{0,1\}$ defined inductively as follows

$$f_{k,0} = 0, \qquad f_{k,1} = 0^{k-1}1, \qquad f_{k,n} = f_{k,n-1}^k f_{k,n-2},$$

for all $n \geq 2$ and $k \geq 1$. Then it is clear that $|f_{k,n}| = F_{k,n+1}$. The infinite word

$$\mathbf{f}_k := \lim_{n\to\infty} f_{k,n}$$

is called the k-Fibonacci word. In connection with this definition, we investigate some new combinatorial properties and we associate a family of curves with interesting patterns.

## 2 Definitions and notation

The terminology and notations are mainly those of Lothaire [14] and Allouche and Shallit [1].

Let $\Sigma$ be a finite alphabet, whose elements are called symbols. A word over $\Sigma$ is a finite sequence of symbols from $\Sigma$. The set of all words over $\Sigma$, i.e., the free monoid generated by $\Sigma$, is denoted by $\Sigma^*$. The identity element $\epsilon$ of $\Sigma^*$ is called the empty word and $\Sigma^+ = \Sigma^* \setminus \{\epsilon\}$. For any word $w \in \Sigma^*$, $|w|$ denotes its length, i.e., the number of symbols occurring in $w$. The length of $\epsilon$ is taken to be equal to 0. If $a \in \Sigma$ and $w \in \Sigma^*$, then $|w|_a$ denotes the number of occurrences of $a$ in $w$.

For two words $u = a_1a_2\cdots a_k$ and $v = b_1b_2\cdots b_s$ in $\Sigma^*$ we denote by $uv$ the concatenation of the two words, that is, $uv = a_1a_2\cdots a_kb_1b_2\cdots b_s$. If $v = \epsilon$ then $u\epsilon = \epsilon u = u$, moreover, by $u^n$ we denote the word $uu\cdots u$ ($n$ times). A word $v$ is a subword (or factor) of $u$ if there exist $x, y \in \Sigma^*$ such that $u = xvy$. If $x = \epsilon$ ($y = \epsilon$), then $v$ is called prefix (suffix) of $u$.

The reversal of a word $u = a_1a_2\cdots a_n$ is the word $u^R = a_n\cdots a_2a_1$ and $\epsilon^R = \epsilon$. A word $u$ is a palindrome if $u^R = u$.

An infinite word over $\Sigma$ is a map $\mathbf{u} : \mathbb{N} \to \Sigma$. It is written $\mathbf{u} = a_1a_2a_3\cdots$. The set of all infinite words over $\Sigma$ is denoted by $\Sigma^\omega$.

**Example 1** *The word* $\mathbf{p} = (p_n)_{n\geq 1} = 0110101000101\cdots$, *where* $p_n = 1$ *if* $n$ *is a prime number and* $p_n = 0$ *otherwise, is an example of an infinite word.* $\mathbf{p}$ *is called the characteristic sequence of the prime numbers.*

**Definition 2** *Let* $\Sigma$ *and* $\Delta$ *be alphabets. A morphism is a map* $h : \Sigma^* \to \Delta^*$ *such that* $h(xy) = h(x)h(y)$ *for all* $x, y \in \Sigma^*$. *It is clear that* $h(\epsilon) = \epsilon$.

There is a special class of words, with many remarkable properties, the so-called Sturmian words. These words admit several equivalent definitions (see, e.g. [1] or [14]).

**Definition 3** *Let* $\mathbf{w} \in \Sigma^\omega$. *We define* $P(\mathbf{w}, n)$, *the complexity function of* $\mathbf{w}$, *to be the map that counts, for all integer* $n \geq 0$, *the number of subwords of length* $n$ *in* $\mathbf{w}$. *An infinite word* $\mathbf{w}$ *is a Sturmian word if* $P(\mathbf{w}, n) = n + 1$ *for all integer* $n \geq 0$.

Since for any Sturmian word $P(\mathbf{w}, 1) = 2$, then Sturmian words are over two symbols. The word $\mathbf{p}$, in Example 1, is not a Sturmian word because $P(\mathbf{p}, 2) = 4$.

Given two real numbers $\alpha, \beta \in \mathbb{R}$ with $\alpha$ irrational and $0 < \alpha < 1$, $0 \leq \beta < 1$, we define the infinite word $\mathbf{w} = w_1 w_2 w_3 \cdots$ as

$$w_n = \lfloor (n+1)\alpha + \beta \rfloor - \lfloor n\alpha + \beta \rfloor.$$

The numbers $\alpha$ and $\beta$ are the slope and the intercept, respectively. This word is called mechanical. The mechanical words are equivalent to Sturmian words [14]. As special case, when $\beta = 0$, we obtain the characteristic words.

**Definition 4** *Let* $\alpha$ *be an irrational number with* $0 < \alpha < 1$. *For* $n \geq 1$, *define*

$$w_\alpha(n) := \lfloor (n+1)\alpha \rfloor - \lfloor n\alpha \rfloor,$$

*and*

$$\mathbf{w}(\alpha) := w_\alpha(1) w_\alpha(2) w_\alpha(3) \cdots,$$

*then* $\mathbf{w}(\alpha)$ *is called a characteristic word with slope* $\alpha$.

On the other hand, note that every irrational $\alpha \in (0, 1)$ has a unique continued fraction expansion

$$\alpha = [0, a_1, a_2, a_3, \ldots] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}},$$

where each $a_i$ is a positive integer. Let $\alpha = [0, 1 + d_1, d_2, \ldots]$ be an irrational number with $d_1 \geq 0$ and $d_n > 0$ for $n > 1$. To the directive sequence $(d_1, d_2, \ldots, d_n, \ldots)$, we associate a sequence $(s_n)_{n \geq -1}$ of words defined by

$$s_{-1} = 1, \quad s_0 = 0, \quad s_n = s_{n-1}^{d_n} s_{n-2}, \quad (n \geq 1).$$

Such a sequence of words is called a standard sequence. This sequence is related to characteristic words in the following way. Observe that, for any $n \geq 0$, $s_n$ is a prefix of $s_{n+1}$, which gives meaning to $\lim_{n\to\infty} s_n$ as an infinite word. In fact, one can prove [14] that each $s_n$ is a prefix of $\mathbf{w}(\alpha)$ for all $n \geq 0$ and

$$\mathbf{w}(\alpha) = \lim_{n\to\infty} s_n. \tag{2}$$

**Example 5** *The infinite Fibonacci word* $\mathbf{f} = 0100101001001010\cdots$ *is a Sturmian word [14], exactly* $\mathbf{f} = \mathbf{w}\left(\frac{1}{\phi^2}\right)$ *where* $\phi = \frac{1+\sqrt{5}}{2}$ *is the golden ratio.*

**Definition 6** *The Fibonacci morphism* $\sigma : \{0,1\} \to \{0,1\}$ *is defined by* $\sigma(0) = 01$ *and* $\sigma(1) = 0$*.*

The Fibonacci word $\mathbf{f}$ satisfies that $\lim_{n\to\infty} \sigma^n(1) = \mathbf{f}$ [1].

# 3 The k-Fibonacci words

**Definition 7** *The* $n$*th* $k$*-Fibonacci words are words over* $\{0,1\}$ *defined inductively as follows*

$$f_{k,0} = 0, \qquad f_{k,1} = 0^{k-1}1, \qquad f_{k,n} = f_{k,n-1}^k f_{k,n-2},$$

*for all* $n \geq 2$ *and* $k \geq 1$*. The infinite word*

$$\mathbf{f}_k := \lim_{n\to\infty} f_{k,n}$$

*is called the* $k$*-Fibonacci word.*

It is clear that $|f_{k,n}| = F_{k,n+1}$. For $k = 1$ we have the word $\bar{\mathbf{f}} = 1011010110110\ldots$, where $\bar{a}$ is a morphism, with $a \in \{0,1\}$, defined by $\bar{0} = 1, \bar{1} = 0$.

**Example 8** *The first* $k$*-Fibonacci words are*

$$\mathbf{f}_1 = 1011010110110\cdots = \overline{\mathbf{f}}, \quad \mathbf{f}_2 = 0101001010010\cdots, \quad \mathbf{f}_3 = 0010010010001\cdots,$$
$$\mathbf{f}_4 = 0001000100010\cdots, \quad \mathbf{f}_5 = 0000100001000\cdots, \quad \mathbf{f}_6 = 0000010000010\cdots.$$

**Definition 9** *The k-Fibonacci morphism* $\sigma_k : \{0,1\} \to \{0,1\}$ *is defined by* $\sigma_k(0) = 0^{k-1}1$ *and* $\sigma_k(1) = 0^{k-1}10$*.*

**Theorem 10** *For all* $n \geq 0$*,* $\sigma_k^n(0) = f_{k,n}$ *and* $\sigma_k^{n+1}(1) = f_{k,n+1}f_{k,n}$*. Hence, the* $k$*-Fibonacci word* $\mathbf{f}_k$ *satisfies that* $\lim_{n\to\infty} \sigma^n(0) = \mathbf{f}_k$*.*

**Proof.** We prove the two assertions about $\sigma_k^n$ by induction on $n$. They are clearly true for $n = 0, 1$. Assume for all $j < n$; we prove them for $n$:

$$\sigma_k^{n+1}(0) = \sigma_k^n(0^{k-1}1) = (\sigma_k^n(0))^{k-1}\sigma_k^n(1) = f_{k,n}^{k-1} f_{k,n} f_{k,n-1} = f_{k,n}^k f_{k,n-1} = f_{k,n+1}.$$

$$\sigma_k^{n+2}(1) = \sigma_k^{n+1}(0^{k-1}10) = (\sigma_k^{n+1}(0))^{k-1}\sigma_k^{n+1}(1)\sigma_k^{n+1}(0) = f_{k,n+1}^{k-1} f_{k,n+1} f_{k,n} f_{k,n+1}$$

$$= f_{k,n+1}^k f_{k,n} f_{k,n+1} = f_{k,n+2} f_{k,n+1}. \qquad \square$$

**Proposition 11**

1. $|f_{k,n}|_1 = F_{k,n}$ *and* $|f_{k,n+1}|_0 = F_{k,n+1} + F_{k,n}$ *for all* $n \geqslant 0$.
2. $\lim_{n\to\infty} \frac{|f_{k,n}|}{|f_{k,n}|_0} = \frac{r_{k,1}^2}{1 + r_{k,1}}$.
3. $\lim_{n\to\infty} \frac{|f_{k,n}|}{|f_{k,n}|_1} = r_{k,1}$.
4. $\lim_{n\to\infty} \frac{|f_{k,n}|_0}{|f_{k,n}|_1} = 1 + \frac{1}{r_{k,1}}$.

**Proof.**

1. It is clear by induction on $n$.

2. $$\lim_{n\to\infty} \frac{|f_{k,n}|}{|f_{k,n}|_0} = \lim_{n\to\infty} \frac{F_{k,n+1}}{F_{k,n} + F_{k,n-1}} = \lim_{n\to\infty} \frac{\frac{F_{k,n+1}}{F_{k,n}}}{1 + \frac{F_{k,n-1}}{F_{k,n}}} = \frac{r_{k,1}^2}{1 + r_{k,1}}.$$

3. $$\lim_{n\to\infty} \frac{|f_{k,n}|}{|f_{k,n}|_1} = \lim_{n\to\infty} \frac{F_{k,n+1}}{F_{k,n}} = r_{k,1}.$$

4. $$\lim_{n\to\infty} \frac{|f_{k,n}|_0}{|f_{k,n}|_1} = \lim_{n\to\infty} \frac{F_{k,n} + F_{k,n-1}}{F_{k,n}} = 1 + \frac{1}{r_{k,1}}.$$

$\square$

**Proposition 12** *The* $k$*-Fibonacci word and the* $n$*th* $k$*-Fibonacci word satisfy that*

1. *The word* $11$ *is not a subword of the* $k$*-Fibonacci word,* $k \geq 2$.
2. *Let* $ab$ *be the last two symbols of* $f_{k,n}$. *For* $n \geq 1$, *we have* $ab = 10$ *if* $n$ *is even and* $ab = 01$ *if* $n$ *is odd,* $k \geq 2$.
3. *The concatenation of two successive* $k$*-Fibonacci words is "almost commutative", i.e.,* $f_{k,n-1}f_{k,n-2}$ *and* $f_{k,n-2}f_{k,n-1}$ *have a common prefix the length* $F_{k,n} + F_{k,n-1} - 2$ *for all* $n \geq 2$.

**Proof.**

1. It suffices to prove that $11$ is not a subword of $f_{k,n}$, for all $n \geq 0$. By induction on $n$. For $n = 0, 1$ it is clear. Assume for all $j < n$; we prove it for $n$. We know that $f_{k,n} = f_{k,n-1}^{k} f_{k,n-2}$ so by the induction hypothesis we have that $11$ is not a subword of $f_{k,n-1}$ and $f_{k,n-2}$. Therefore, the only possibility is that $1$ is a suffix and prefix of $f_{k,n-1}$ or $1$ is a suffix of $f_{k,n-1}$ and a prefix of $f_{k,n-2}$, both there are impossible.

2. By induction on $n$. For $n = 1, 2$ it is clear. Assume for all $j < n$; we prove it for $n$. We know that $f_{k,n+1} = f_{k,n}^{k} f_{k,n-1}$, if $n + 1$ is even then by the induction hypothesis the last two symbols of $f_{k,n-1}$ are $10$, therefore the last two symbols of $f_{k,n+1}$ are $10$. Analogously, if $n + 1$ is odd.

3. By induction on $n$. For $n = 1, 2$ it is clear. Assume for all $j < n$; we prove it for $n$. By definition of $f_{k,n}$, we have

$$\begin{aligned} f_{k,n-1} f_{k,n-2} &= f_{k,n-2}^{k} f_{k,n-3} \cdot f_{k,n-3}^{k} f_{k,n-4} \\ &= (f_{k,n-3}^{k} f_{k,n-4})^{k} \cdot f_{k,n-3}^{k} f_{k,n-3} f_{k,n-4}, \end{aligned}$$

and

$$\begin{aligned} f_{k,n-2} f_{k,n-1} &= f_{k,n-3}^{k} f_{k,n-4} \cdot f_{k,n-2}^{k} f_{k,n-3} \\ &= f_{k,n-3}^{k} f_{k,n-4} \cdot (f_{k,n-3}^{k} f_{k,n-4})^{k} \cdot f_{k,n-3} \\ &= (f_{k,n-3}^{k} f_{k,n-4})^{k} f_{k,n-3}^{k} f_{k,n-4} f_{k,n-3}. \end{aligned}$$

Hence the words have a common prefix of length $k(kF_{k,n-2} + F_{k,n-3}) + kF_{n-2} = k(F_{k,n-1} + F_{k,n-2})$. By the induction hypothesis $f_{k,n-3} f_{k,n-4}$ and $f_{k,n-4} f_{k,n-3}$ have a common prefix of length $F_{k,n-2} + F_{k,n-3} - 2$. Therefore the words have a common prefix of length

$$k(F_{k,n-1} + F_{k,n-2}) + F_{k,n-2} + F_{k,n-3} - 2 = F_{k,n} + F_{k,n-1} - 2. \qquad \square$$

**Definition 13** *Let* $\Phi : \{0, 1\}^{*} \to \{0, 1\}^{*}$ *be a map such that* $\Phi$ *deletes the last two symbols, i.e.,* $\Phi(a_1 a_2 \cdots a_n) = a_1 a_2 \cdots a_{n-2}$ $(n \geq 2)$.

**Corollary 14** *The* $n$*th* $k$*-Fibonacci word, satisfy for all* $n \geqslant 2$ *that*

1. $\Phi(f_{k,n-1} f_{k,n-2}) = \Phi(f_{k,n-2} f_{k,n-1})$.
2. $\Phi(f_{k,n-1} f_{k,n-2}) = f_{k,n-2} \Phi(f_{k,n-1}) = f_{k,n-1} \Phi(f_{k,n-2})$.
3. *If* $f_{k,n} = \Phi(f_{k,n}) ab$, *then* $\Phi(f_{k,n-2}) ab \Phi(f_{k,n-1}) = f_{k,n-1} \Phi(f_{k,n-2})$.

*4. If* $f_{k,n} = \Phi(f_{k,n})ab$*, then* $\Phi(f_{k,n-2})(ab\Phi(f_{k,n-1}))^k = \Phi(f_{k,n})$*.*

**Proof.** Parts (a) and (b) follow immediately from Proposition 12-(3) and because of $|f_{k,n}| \geqslant 2$ for all $n \geqslant 2$. (c) In fact, if $f_{k,n} = \Phi(f_{k,n})ab$ then from Proposition 12-(2) we have $f_{k,n-2} = \Phi(f_{k,n-2})ab$. Hence $\Phi(f_{k,n-2})ab\Phi(f_{k,n-1}) = f_{k,n-2}\Phi(f_{k,n-1}) = f_{k,n-1}\Phi(f_{k,n-2})$. (d) It is clear from (c) and definition of $f_{k,n}$. □

**Theorem 15** $\Phi(f_{k,n})$ *is a palindrome for all* $n \geq 1$ *and* $k \geq 1$*.*

**Proof.** By induction on $n$. If $n = 2$ then $\Phi(f_{k,2}) = (0^{k-1}1)^{k-1}0^{k-1}$ is a palindrome. Now suppose that the result is true for all $j < n$; we prove it for $n$.

$$\begin{aligned}(\Phi(f_{k,n}))^R &= (\Phi(f_{k,n-1}^k f_{k,n-2}))^R = (f_{k,n-1}^k \Phi(f_{k,n-2}))^R = \Phi(f_{k,n-2})^R (f_{k,n-1}^k)^R \\ &= \Phi(f_{k,n-2})(f_{k,n-1}^R)^k.\end{aligned}$$

If $n$ is even then $f_{k,n} = \Phi(f_{k,n})10$ and from Corollary 14-(4), we have that

$$\begin{aligned}(\Phi(f_{k,n}))^R &= \Phi(f_{k,n-2})((\Phi(f_{k,n-1})01)^R)^k = \Phi(f_{k,n-2})(10(\Phi(f_{k,n-1}))^R)^k \\ &= \Phi(f_{k,n-2})(10\Phi(f_{k,n-1}))^k = \Phi(f_{k,n}).\end{aligned}$$

If $n$ is odd, the proof is analogous. □

**Corollary 16** *1. If* $f_{k,n} = \Phi(f_{k,n})ab$ *then* $ba\Phi(f_{k,n})ab$ *is a palindrome.*
*2. If* $u$ *is a subword of the* k*-Fibonacci word, then so is its reversal,* $u^R$ *.*

**Theorem 17** *Let* $\alpha = [0, \overline{k}]$ *be an irrational number, with* $k$ *a positive integer, then*

$$\mathbf{w}(\alpha) = \mathbf{f}_k.$$

**Proof.** Let $\alpha = [0, \overline{k}]$ an irrational number, then its associated standard sequence is

$$s_{-1} = 1, \quad s_0 = 0, \quad s_1 = s_0^{k-1}s_{-1} = 0^{k-1}1 \text{ and } s_n = s_{n-1}^k s_{n-2}, \; n \geq 2.$$

Hence $\{s_n\}_{n\geq 0} = \{f_{k,n}\}_{n\geq 0}$ and from equation (2), we have

$$\mathbf{w}(\alpha) = \lim_{n\to\infty} s_n = \mathbf{f}_k.$$

□

**Remark.** Note that

$$[0,\overline{k}] = \cfrac{1}{k+\cfrac{1}{k+\cfrac{1}{k+\cfrac{1}{\ddots}}}} = \frac{-k+\sqrt{k^2+4}}{2} = -r_{k,2}$$

From the above theorem, we conclude that k-Fibonacci words are Sturmian words.

A fractional power is a word of the form $z = x^n y$, where $n \in \mathbb{Z}^+$ and $x \in \Sigma^+$, and $y$ is power power prefix of $x$. If $|z| = p$ and $|x| = q$, we say that $z$ is a $p/q$-power, or $z = x^{p/q}$. In the expression $x^{p/q}$, the number $p/q$ is the power's exponent. For example, 01201201 is a $8/3$-power, $01201201 = (012)^{8/3}$. The index of an infinite word $\mathbf{w} \in \Sigma^\omega$ is defined by

$$\mathrm{Ind}(w) := \sup\{r \in \mathbb{Q}_{\geqslant 1} : w \text{ contains an } r\text{-power}\}$$

For example Ind($\mathbf{f}$)> 3 because the cube $(010)^3$ occurs in $\mathbf{f}$ at position 6. In [15] the authors proof that Ind($\mathbf{f}$)=$2+\phi \approx 3.618$. A general formula for the index of a Sturmian word was given in [8].

**Theorem 18** *If* $\mathbf{u}$ *is a Sturmian word of slope* $\alpha = [0, a_1, a_2, a_3, \ldots]$*, then*

$$\mathrm{Ind}(w) = \sup_{n \geqslant 0}\left\{2 + a_{n+1} + \frac{q_{n-1} - 2}{q_n}\right\},$$

*where* $q_n$ *is the denominator of* $\alpha = [0, a_1, a_2, a_3, \ldots, a_n]$ *and satisfies* $q_{-1} = 0, q_0 = 1, q_{n+1} = a_{n+1} q_n + q_{n-1}$.

**Corollary 19** *The index of* k*-Fibonacci words is* $Ind(\mathbf{f}_k) = 2 + k + \frac{1}{r_{k,1}}$.

**Proof.** $\mathbf{f}_k$ is a Sturmian word of slope $\alpha = [0, \overline{k}]$, then it is clear that $q_n = F_{k,n+1}$, and from above theorem

$$\mathrm{Ind}(\mathbf{f}_k) = \sup_{n \geqslant 0}\left\{2 + k + \frac{F_{k,n} - 2}{F_{k,n+1}}\right\} = 2 + k + \frac{1}{r_{k,1}}. \qquad \square$$

# 4 The k-Fibonacci Word Curve

The Fibonacci word can be associated to a curve from a drawing rule. We must travel the word in a particular way, depending on the symbol read a particular action is produced, this idea is the same as that used in the L-Systems [17]. In this case, the drawing rule is called "odd-even drawing rule" [16], this is defined as shown in the following table.

| Symbol | Action |
|---|---|
| 1 | Draw a line forward. |
| 0 | Draw a line forward and if the symbol 0 is in a position even then turn $\theta$ degree and if 0 is in a position odd then turn $-\theta$ degrees. |

**Definition 20** *The* $n$*th-curve of Fibonacci, denoted by* $\mathcal{F}_n$*, is obtained by applying the odd-even drawing rule to the word* $f_n$*. The Fibonacci word fractal* $\mathcal{F}$*, is defined as*

$$\mathcal{F} := \lim_{n\to\infty} \mathcal{F}_n.$$

**Example 21** *In Figure 1 we show the curve* $\mathcal{F}_{10}$ *and* $\mathcal{F}_{17}$*. The graphics in this paper were generated using the software* Mathematica 9.0*,* [20].

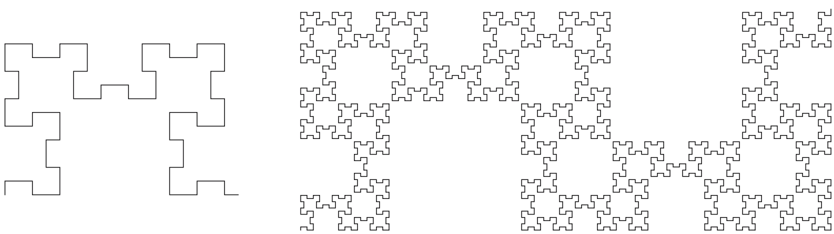


Figure 1: Fibonacci curves $\mathcal{F}_{10}$ and $\mathcal{F}_{17}$ corresponding to the words $f_{10}$ and $f_{17}$

Properties of Fibonacci Word Fractal can be found in [3, 4, 16].

**Definition 22** *The* $n$*th* $k$*-curve of Fibonacci, denoted by* $\mathcal{F}_{k,n}$*, is obtained by applying the odd-even drawing rule to the word* $f_{k,n}$*. The* $k$*-Fibonacci word curve* $\mathcal{F}_k$ *is defined as*

$$\mathcal{F}_k := \lim_{n\to\infty} \mathcal{F}_{k,n}.$$

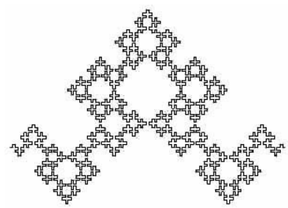
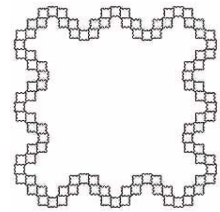
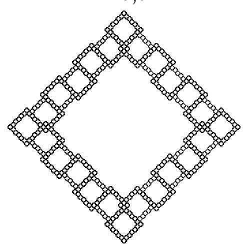
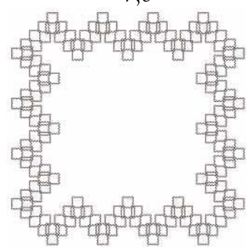
| $\mathcal{F}_{1,18}$ | $\mathcal{F}_{5,6}$ |
|---|---|
|  |  |
| $\mathcal{F}_{6,6}$ | $\mathcal{F}_{7,6}$ |
|  |  |

Table 1: Some curves $\mathcal{F}_{k,n}$ with $\theta = 90°$

In Table 1, we show some curves $\mathcal{F}_{k,n}$ with an angle $\theta = 90°$.

In Table 2, we show some curves $\mathcal{F}_{k,n}$ with an angle $\theta = 60°$. In general these curves have a lot of patterns because the index is large, see Corollary 19.

**Proposition 23** *The* $k$*-Fibonacci word curve and the curve* $\mathcal{F}_{k,n}$ *have the following properties:*

1. *The* $k$*-Fibonacci curve* $\mathcal{F}_k$ *is composed only of segments of length 1 or 2.*
2. *The* $\mathcal{F}_{k,n}$ *is symmetric.*
3. *The number of turns in the curve* $\mathcal{F}_{k,n}$ *is* $F_{k,n} + F_{k,n-1}$.
4. *If* $n$ *is even then the* $\mathcal{F}_{k,n}$ *curve is similar to the curve* $\mathcal{F}_{k,n-2}$ *and if* $n$ *is odd then the* $\mathcal{F}_{k,n}$ *curve is similar to the curve* $\mathcal{F}_{k,n-3}$.

**Proof.**

1. It is clear from Proposition 12-1, because 110 and 111 are not subwords of $\mathbf{f}_k$.
2. It is clear from Theorem 15, because $f_{k,n} = \Phi(f_{k,n})ab$, where $\Phi(f_{k,n})$ is a palindrome.
3. It is clear from definition of odd-even drawn rule and because $|f_{k,n+1}|_0 = F_{k,n+1} + F_{k,n}$.

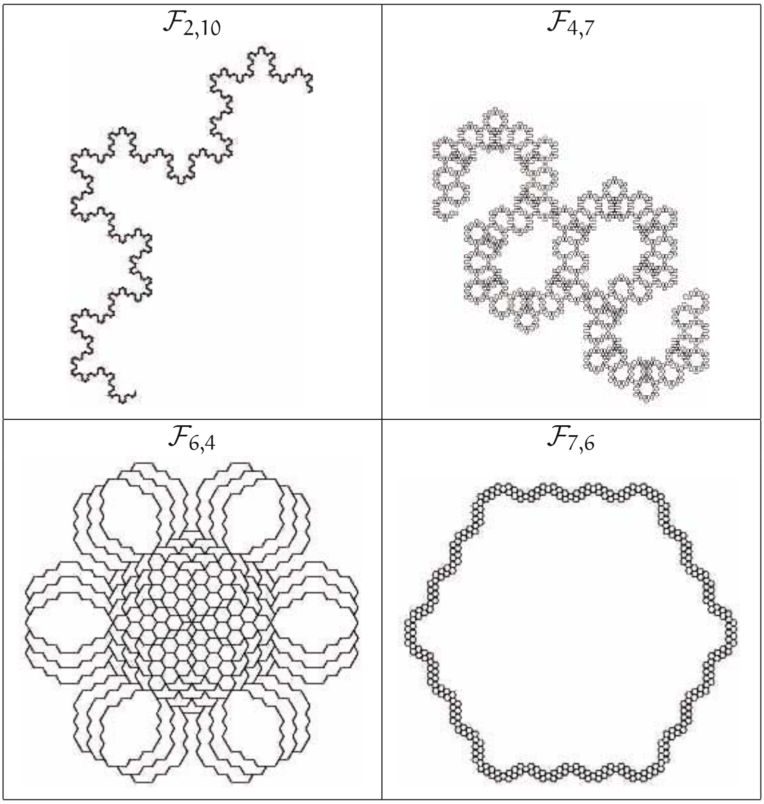



Table 2: Some curves $\mathcal{F}_{k,n}$ with $\theta = 60°$

4. If $n$ is even. It is clear that $\sigma_k^2(f_{k,n-2}) = f_{k,n}$. We are going to proof that $\sigma_k^2$ guaranties the odd-even alternation required by the odd-even drawing rule. In fact, $\sigma_k^2(0) = \sigma_k(0^{k-1}1) = (0^{k-1}1)^k 0$ and $\sigma_k^2(1) = (0^{k-1}1)^k 0^k 1$. As $k$ is even, then $|\sigma_k^2(0)|$ and $|\sigma_k^2(1)|$ are odd. Hence if $|w|$ is even (odd) then $|\sigma_k^2(w)|$ is even (odd). Since $\sigma_k^2$ preserves the parity of length then any subword in the k-Fibonacci word preserves the parity of position.

Finally, we have to proof that the resulting angle of a pattern must be preserved or inverted by $\sigma_k^2$. Let $\alpha(w)$ be the function that gives the resulting angle of a word $w$ through the odd-even drawing rule of angle

$\theta$. Note that $a(00) = 0^\circ$, $a(01) = -\theta^\circ$ and $a(10) = \theta^\circ$. Therefore

$$\begin{aligned}
a(\sigma_k^2(00)) &= a((0^{k-1}1)^k 0 (0^{k-1}1)^k 0) \\
&= a((0^{k-1}1)^k) a(00^{k-1}) a(1(0^{k-1}1)^{k-1}0) \\
&= -k\theta^\circ + 0^\circ + k\theta^\circ = 0^\circ \\
a(\sigma_k^2(01)) &= a((0^{k-1}1)^k 0 (0^{k-1}1)^k 0^k 1) \\
&= a((0^{k-1}1)^k) a(00^{k-1}) a(1(0^{k-1}1)^{k-1}0) a(0^{k-1}1) \\
&= -k\theta^\circ + 0^\circ + k\theta^\circ - \theta^\circ = -\theta^\circ \\
a(\sigma_k^2(10)) &= a((0^{k-1}1)^k 0^k 1 (0^{k-1}1)^k 0) \\
&= a((0^{k-1}1)^k) a(0^k) a(1(0^{k-1}1)^k 0) \\
&= -k\theta^\circ + 0^\circ + (k+1)\theta^\circ = \theta^\circ
\end{aligned}$$

Then $\sigma_k^2$ inverts the resulting angle, i.e., $a(w) = -a(\sigma_k^2(w))$ for any word $w$. Therefore the image of a pattern by $\sigma_k^2$ is the rotation of this pattern by a rotation of $-\theta^\circ$. Since $\sigma_k^2(f_{k,n-2}) = f_{k,n}$, then the curve $\mathcal{F}_{k,n}$ is similar to the curve $\mathcal{F}_{k,n-2}$.

If $n$ is odd the proof is similar, but using $\sigma_k^3$. □

**Example 24** *In Figure 2 $\mathcal{F}_{4,4}$ looks similar to $\mathcal{F}_{4,6}, \mathcal{F}_{4,8}$ and so on.*

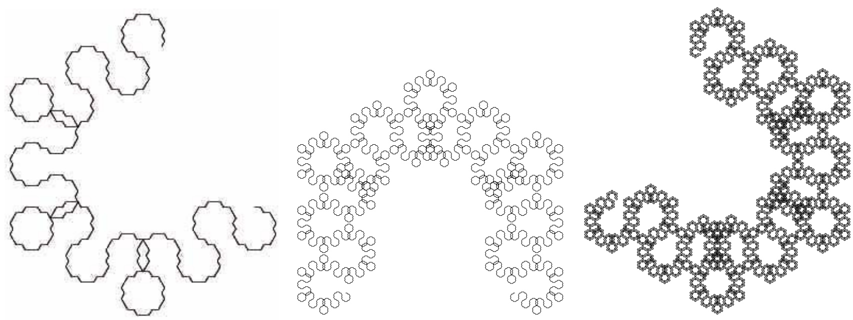


Figure 2: Curves $\mathcal{F}_{4,4}, \mathcal{F}_{4,6}, \mathcal{F}_{4,8}$ with $\theta = 60^\circ$

*In Figure 3 $\mathcal{F}_{5,3}$ looks similar to $\mathcal{F}_{5,6}$.*

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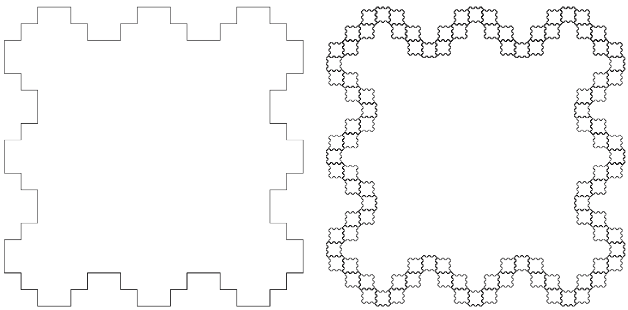

Figure 3: Curves $\mathcal{F}_{5,3}, \mathcal{F}_{5,6}$ with $\theta = 60°$

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