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On the k-free divisor problem

by

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1. Introduction. Let $d(n)$ denote the divisor function. Dirichlet proved that the error term

$$\Delta(x) := \sum_{n\le x} d(n) - x\log x - (2\gamma-1)x, \qquad x\ge 2,$$

satisfies $\Delta(x) = O(x^{1/2})$. The exponent $1/2$ was improved by many authors. The latest result is due to Huxley [5], who proved that

$$\Delta(x) = (x^{131/416}(\log x)^{26957/8320}).$$

It is conjectured that

$$\Delta(x) = O(x^{1/4+\varepsilon}), \tag{1.1}$$

which is supported by the classical mean-square result

$$\int_1^T \Delta^2(x)\,dx = \frac{(\zeta(3/2))^4}{6\pi^2\zeta(3)}\,T^{3/2} + O(T\log^5 T) \tag{1.2}$$

proved by Tong [15].

Let $k\ge 2$ denote a fixed integer. An integer n is called *k-free* if p^k does not divide n for any prime p. Let $d^{(k)}(n)$ denote the number of k-free divisors of the positive integer n and define

$$D^{(k)}(x) := \sum_{n\le x} d^{(k)}(n).$$

Then the expected asymptotic formula for $D^{(k)}(x)$ is

$$D^{(k)}(x) = C_1^{(k)} x\log x + C_2^{(k)} x + \Delta^{(k)}(x),$$

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where $C_1^{(k)}, C_2^{(k)}$ are two constants, $\Delta^{(k)}(x)$ is the error term. In 1874 Mertens [10] proved that $\Delta^{(2)}(x) \ll x^{1/2}\log x$. In 1932 Hölder [4] proved that

$$\Delta^{(k)}(x) \ll \begin{cases} x^{1/2} & \text{if } k=2, \\ x^{1/3} & \text{if } k=3, \\ x^{33/100} & \text{if } k \ge 4. \end{cases}$$

For $k=2,3$, it is very difficult to improve the exponent $1/k$ in the bound $\Delta^{(k)}(x) \ll x^{1/k}$, unless we have substantial progress in the study of the zero-free region of $\zeta(s)$. Therefore it is reasonable to get better improvements by assuming the truth of the Riemann Hypothesis (RH). Such results were given in [1, 2, 9, 12, 13, 14]. For $k=2$, R. C. Baker proved in [2] that $\Delta^{(2)}(x) \ll x^{4/11+\varepsilon}$ (under RH). The exponent $4/11$ can be improved to $221/608$ by a very slight adaption of the argument in J. Wu [17]. For $k=3$, in [9] Kumchev proved $\Delta^{(3)}(x) \ll x^{27/85+\varepsilon}$ under RH. For $k \ge 4$, it is easy to show that if $\Delta(x) \ll x^{\alpha}$ is true, then the estimate $\Delta^{(k)}(x) \ll x^{\alpha}\log x$ follows.

We believe that the estimate

$$\Delta^{(k)}(x) = O(x^{1/4+\varepsilon}) \tag{1.3}$$

would be true for any $k \ge 2$, which is an analogue of (1.1). For $k \ge 4$ it is easily seen that if the conjecture (1.1) is true, then so is (1.3). For $k=2,3$, we cannot deduce the conjecture (1.3) from (1.1) directly; in this case we do not know the truth of (1.3) even if both (1.1) and RH are true. However, for any $k \ge 2$, the conjecture (1.3) cannot be proved by present methods.

In this paper we shall study the mean square of $\Delta^{(k)}(x)$ for $k \ge 4$, from which the truth of the conjecture (1.3) ($k \ge 4$) is supported partly. Our result is an analogue of (1.2).

THEOREM 1. *We have the asymptotic formula*

$$\int_1^T |\Delta^{(k)}(x)|^2\,dx = \frac{B_k}{6\pi^2}T^{3/2} + \begin{cases} O(T^{3/2}e^{-c\delta(T)}) & \textit{for } k=4, \\ O(T^{\delta_k+\varepsilon}) & \textit{for } k \ge 5, \end{cases}$$

where

$$B_k := \sum_{m=1}^{\infty} g_k^2(m)m^{-3/2}, \qquad g_k(m) := \sum_{m=nd^k} \mu(d)d(n)d^{k/2},$$

$$\delta(u) := (\log u)^{3/5}(\log\log u)^{-1/5},$$

$$\delta_5 := 29/20, \qquad \delta_k := 3/2 - 1/2k + 1/k^2 \quad (k \ge 6),$$

and where $c > 0$ is an absolute constant.

COROLLARY 1. *If $k \ge 4$, then*

$$\Delta^{(k)}(x) = \Omega(x^{1/4}).$$

By the same method we can study the mean square of $\Delta(1,1,k;x)$, which is defined by

$$\Delta(1,1,k;x) := \sum_{n\le x} d(1,1,k;n) - x\{\zeta(k)\log x + k\zeta'(k) + (2\gamma-1)\zeta(k)\} - \zeta^2(1/k)x^{1/k},$$

where $d(1,1,k;n) = \sum_{n=m_1m_2d^k} 1$ and γ is the Euler constant. This is a special three-dimensional divisor problem. From the formula (5.3) of Ivić [7] we have

$$\int_1^T \Delta^2(1,1,k;x)\,dx \ll T^{3/2+\varepsilon}. \tag{1.4}$$

From Krätzel [8] we know that

$$\Delta(1,1,k;x) = \Omega(x^{1/4}) \tag{1.5}$$

if $k\ge 5$.

Now we prove the following theorem, which improves (1.4).

THEOREM 2. *Suppose $k\ge 3$ is a fixed integer. Then*

$$\int_1^T \Delta^2(1,1,k;x)\,dx = \frac{C_k}{6\pi^2}T^{3/2} + \begin{cases} O(T^{53/36}\log^3 T) & \text{if } k=3,\\ O(T^{29/20}\log^{503} T) & \text{if } k=4,\\ O(T^{75/52}\log^{1000} T) & \text{if } k=5,\\ O(T^{3/2-1/2k+1/k^2+\varepsilon}) & \text{if } k\ge 6,\end{cases}$$

where

$$C_k := \sum_{m=1}^{\infty} f_k^2(m)m^{-3/2}, \qquad f_k(m) := \sum_{m=nd^k} d(n)d^{k/2}.$$

COROLLARY 2. *Formula* (1.5) *holds for $k=3,4$.*

NOTATIONS. For a real number u, $[u]$ denotes the integer part of u, $\{u\}$ denotes the fractional part of u, $\psi(u)=\{u\}-1/2$, $\|u\|$ denotes the distance from u to the integer nearest to u. We write $\mu(d)$ for the Möbius function. Let (m,n) denote the greatest common divisor of natural numbers m and n. Furthermore, $n\sim N$ means $N<n\le 2N$, and ε always denotes a sufficiently small positive constant which may be different at different places. Finally, $\mathrm{SC}(\sum)$ denotes the summation condition of the sum $\sum$.

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2. The expression for $\Delta^{(k)}(x)$. In order to prove Theorem 1, we shall give a simple expression of $\Delta^{(k)}(x)$ in this section.

LEMMA 2.1. *There exists an absolute constant $c_1 > 0$ such that the estimate*

$$M(u) := \sum_{n\le u} \mu(n) \ll ue^{-c_1\delta(u)}$$

holds for $u \ge 2$.

This is Theorem 12.7 of Ivić [6]. Now we prove the following

LEMMA 2.2. *Suppose $10 \le y \ll x^{1/k}$. Then*

$$\Delta^{(k)}(x) = \sum_{d\le y} \mu(d)\Delta(x/d^k) + O(xy^{1-k}e^{-c_1\delta(y)}\log x).$$

Proof. We have

$$\begin{aligned} D^{(k)}(x) &= \sum_{\substack{mn\le x\\ m:k\text{-free}}} 1 = \sum_{d^k mn\le x} \mu(d) = \sum_{d^k n\le x} \mu(d)d(n) \\ &= \sum_{d\le y} \mu(d)D(x/d^k) + \sum_{n\le x/y^k} d(n)M((x/n)^{1/k}) - D(x/y^k)M(y) \\ &= \sum\nolimits_1 + \sum\nolimits_2 - \sum\nolimits_3, \end{aligned}$$

say. From Lemma 2.1 and the estimate $D(u) \ll u\log u$ we directly get

$$\sum\nolimits_3 \ll xy^{1-k}e^{-c_1\delta(y)}\log x.$$

Similarly we get

$$\sum\nolimits_2 \ll xy^{1-k}e^{-c_1\delta(y)}\log x$$

if we note that $e^{-c_1\delta((x/n)^{1/k})} \le e^{-c_1\delta(y)}$ for all $n \le x/y^k$. By Lemma 2.1 and simple calculations we have

$$\begin{aligned} \sum\nolimits_1 &= \sum_{d\le y} \mu(d)\left\{\frac{x}{d^k}\log\frac{x}{d^k} + (2\gamma-1)\frac{x}{d^k}\right\} + \sum_{d\le y}\mu(d)\Delta\left(\frac{x}{d^k}\right) \\ &= (\text{main term}) + \sum_{d\le y}\mu(d)\Delta\left(\frac{x}{d^k}\right) + O(xy^{1-k}e^{-c_1\delta(y)}\log x). \end{aligned}$$

Hence Lemma 2.2 follows. ∎

3. Proof of Theorem 1 (beginning). Suppose $T \ge 10$ is large. It suffices to evaluate the integral $\int_T^{2T} |\Delta^{(k)}(x)|^2\,dx$.

Let $T^\varepsilon \ll y \ll T^{1/k-\varepsilon}$, $T^\varepsilon \ll z \ll T^{1-\varepsilon}$ be two parameters to be determined later. Let

$$\Delta_1(u) := \frac{u^{1/4}}{\pi\sqrt{2}} \sum_{n\le z} \frac{d(n)}{n^{3/4}} \cos(4\pi\sqrt{nu} - \pi/4), \qquad \Delta_2(u;z) := \Delta(u) - \Delta_1(u).$$

Then by Lemma 2.2 we can write

$$\Delta^{(k)}(x) = R_1^{(k)}(x) + R_2^{(k)}(x) + R_3^{(k)}(x), \tag{3.1}$$

where

$$R_1^{(k)}(x) := \frac{x^{1/4}}{\pi\sqrt{2}} \sum_{d\le y} \frac{\mu(d)}{d^{k/4}} \sum_{n\le z} \frac{d(n)}{n^{3/4}} \cos\left(4\pi\sqrt{\frac{nx}{d^k}} - \frac{\pi}{4}\right),$$

$$R_2^{(k)}(x) := \sum_{d\le y} \mu(d)\Delta_2(x/d^k; z),$$

$$R_3^{(k)}(x) := O(xy^{1-k}e^{-c_1\delta(y)}\log x).$$

LEMMA 3.1. *Suppose $A > 0$ is any fixed constant*, $T^\varepsilon \ll V \ll T^A$. *Then*

$$\int_V^{2V} \Delta_2^2(u;z)\,du \ll V^{3/2}z^{-1/2}\log^3 V + V\log^5 V.$$

Proof. Suppose $\min(z, V^{11}) < N \ll V^B$ is a large parameter, where $B > 0$ is a constant suitably large. By Lemma 3 of Meurman [11] we have

$$\Delta_2(u;z) = \frac{u^{1/4}}{\pi\sqrt{2}} \sum_{z<n\le N} \frac{d(n)}{n^{3/4}} \cos(4\pi\sqrt{nu} - \pi/4) + \Delta_2(u;N),$$

where $\Delta_2(u;N) \ll u^{-1/4}$ if $\|u\| \gg u^{5/2}N^{-1/2}$, and $\Delta_2(u;N) \ll u^\varepsilon$ otherwise. Thus

$$\int_V^{2V} \Delta_2^2(u;z)\,du \ll \int_1 + \int_2,$$

where

$$\int_1 = \int_V^{2V} \left| u^{1/4} \sum_{z<n\le N} \frac{d(n)}{n^{3/4}} \cos(4\pi\sqrt{nu} - \pi/4)\right|^2 du, \qquad \int_2 = \int_V^{2V} \Delta_2^2(u;N)\,du.$$

For $\int_1$ we have

$$\begin{aligned}
\int_1 &\ll \int_V^{2V} \left| u^{1/4} \sum_{z<n\le N} \frac{d(n)}{n^{3/4}}\, e(2\sqrt{nu})\right|^2 du \\
&\ll V^{3/2} \sum_{z<n\le N} \frac{d^2(n)}{n^{3/2}} + V \sum_{z<m<n\le N} \frac{d(n)d(m)}{(mn)^{3/4}(\sqrt{n}-\sqrt{m})} \\
&\ll \frac{V^{3/2}\log^3 V}{z^{1/2}} + V\log^5 V,
\end{aligned}$$

where we used the well known estimates

$$\sum_{n\le u} d^2(n) \ll u\log^3 u, \tag{3.2}$$
$$\sum_{z<m<n\le N} \frac{d(n)d(m)}{(mn)^{3/4}(\sqrt{n}-\sqrt{m})} \ll \log^5 N \ll \log^5 V.$$

For $\int_2$ we have

$$\int_2 \ll V(V^{5/2+\varepsilon}N^{-1/2}+V^{-1/4}) \ll V^{7/2+\varepsilon}N^{-1/2}+V^{3/4} \ll V.$$

Now Lemma 3.1 follows from the above estimates. ∎

By Cauchy's inequality and Lemma 3.1 we get

$$\begin{aligned}
\int_T^{2T} |R_2^{(k)}(x)|^2\,dx &= \int_T^{2T} \Big|\sum_{d\le y}\mu(d)d^{-1/2}d^{1/2}\Delta_2(x/d^k;z)\Big|^2\,dx \qquad (3.3)\\
&\ll \int_T^{2T}\Big(\sum_{d\le y} d^{-1}\Big)\Big(\sum_{d\le y} d|\Delta_2(x/d^k;z)|^2\Big)\,dx\\
&\ll \sum_{d\le y} d\int_T^{2T}|\Delta_2(x/d^k;z)|^2\,dx\log y \ll \sum_{d\le y} d^{k+1}\int_{T/d^k}^{2T/d^k}|\Delta_2(u;z)|^2\,du\log y\\
&\ll \sum_{d\le y} d^{k+1}((T/d^k)^{3/2}z^{-1/2}\log^3 T + Td^{-k}\log^5 T)\log y\\
&\ll T^{3/2}z^{-1/2}\sum_{d\le y} d^{1-k/2}\log^4 T + Ty^2\log^6 T\\
&\ll \begin{cases} T^{3/2}z^{-1/2}y^{1/2}\log^4 T + Ty^2\log^6 T & \text{if } k=3,\\ T^{3/2}z^{-1/2}\log^5 T + Ty^2\log^6 T & \text{if } k\ge 4.\end{cases}
\end{aligned}$$

If $k=4$, we take $y=T^{1/4}e^{-c_2\delta(T)}$, where $c_2=c_1/4^{8/5}$. It is easy to see that $R_3^{(k)}(x)\ll T^{1/4}e^{-c_3\delta(T)}$ holds for all $T\le x\le 2T$, where $0<c_3<c_1/4^{8/5}$ is an absolute constant. Hence

$$\int_T^{2T}|R_3^{(4)}(x)|^2\,dx \ll T^{3/2}e^{-2c_3\delta(T)}. \tag{3.4}$$

If $k\ge 5$, then

$$\int_T^{2T}|R_3^{(k)}(x)|^2\,dx \ll T^3y^{2-2k}. \tag{3.5}$$

Now we consider the mean square of $R_1^{(k)}(x)$. By the elementary formula

$$\cos u \cos v = \tfrac{1}{2}(\cos(u-v) + \cos(u+v))$$

we may write

$$|R_1^{(k)}(x)|^2 = \frac{x^{1/2}}{2\pi^2} \sum_{d_1,d_2 \le y} \frac{\mu(d_1)\mu(d_2)}{(d_1d_2)^{k/4}} \sum_{n_1,n_2 \le z} \frac{d(n_1)d(n_2)}{(n_1n_2)^{3/4}} \times \cos\biggl(4\pi\sqrt{\frac{n_1x}{d_1^k}} - \frac{\pi}{4}\biggr) \cos\biggl(4\pi\sqrt{\frac{n_2x}{d_2^k}} - \frac{\pi}{4}\biggr) = S_1(x) + S_2(x) + S_3(x), \tag{3.6}$$

where

$$S_1(x) = \frac{x^{1/2}}{4\pi^2} \sum_{\substack{d_1,d_2 \le y;\, n_1,n_2 \le z \\ n_1d_2^k = n_2d_1^k}} \frac{\mu(d_1)\mu(d_2)}{(d_1d_2)^{k/4}} \frac{d(n_1)d(n_2)}{(n_1n_2)^{3/4}},$$

$$S_2(x) = \frac{x^{1/2}}{4\pi^2} \sum_{\substack{d_1,d_2 \le y;\, n_1,n_2 \le z \\ n_1d_2^k \ne n_2d_1^k}} \frac{\mu(d_1)\mu(d_2)}{(d_1d_2)^{k/4}} \frac{d(n_1)d(n_2)}{(n_1n_2)^{3/4}} \times \cos\biggl(4\pi\sqrt{x}\biggl(\sqrt{\frac{n_1}{d_1^k}} - \sqrt{\frac{n_2}{d_2^k}}\biggr)\biggr),$$

$$S_3(x) = \frac{x^{1/2}}{4\pi^2} \sum_{d_1,d_2 \le y;\, n_1,n_2 \le z} \frac{\mu(d_1)\mu(d_2)}{(d_1d_2)^{k/4}} \frac{d(n_1)d(n_2)}{(n_1n_2)^{3/4}} \times \sin\biggl(4\pi\sqrt{x}\biggl(\sqrt{\frac{n_1}{d_1^k}} + \sqrt{\frac{n_2}{d_2^k}}\biggr)\biggr).$$

We have

$$\int_T^{2T} S_1(x)\,dx = \frac{B_k(y,z)}{4\pi^2} \int_T^{2T} x^{1/2}\,dx, \tag{3.7}$$

$$\text{where } B_k(y,z) := \sum_{\substack{d_1,d_2 \le y;\, n_1,n_2 \le z \\ n_1d_2^k = n_2d_1^k}} \frac{\mu(d_1)\mu(d_2)}{(d_1d_2)^{k/4}} \frac{d(n_1)d(n_2)}{(n_1n_2)^{3/4}}.$$

By the first derivative test we get

$$\int_T^{2T} S_2(x)\,dx \ll TE_k(y,z), \tag{3.8}$$

where

$$E_k(y,z) = \sum_{\substack{d_1,d_2 \le y;\, n_1,n_2 \le z \\ n_1d_2^k \ne n_2d_1^k}} \frac{d(n_1)d(n_2)}{(d_1d_2)^{k/4}(n_1n_2)^{3/4}} \min\biggl(T^{1/2}, \frac{1}{|\sqrt{n_1/d_1^k} - \sqrt{n_2/d_2^k}|}\biggr).$$

By the first derivative test again we get

$$\begin{aligned}(3.9)\qquad \int_T^{2T} S_3(x)\,dx &\ll \sum_{d_1,d_2\le y;\, n_1,n_2\le z} \frac{d(n_1)d(n_2)}{(d_1d_2)^{k/4}(n_1n_2)^{3/4}} \frac{1}{|\sqrt{n_1/d_1^k}+\sqrt{n_2/d_2^k}|}\\ &\ll \sum_{d_1,d_2\le y;\, n_1,n_2\le z} \frac{d(n_1)d(n_2)}{(d_1d_2)^{k/4}(n_1n_2)^{3/4}} \frac{1}{(\sqrt{n_1/d_1^k}\sqrt{n_2/d_2^k})^{1/2}}\\ &\ll \sum_{d_1,d_2\le y;\, n_1,n_2\le z} \frac{d(n_1)d(n_2)}{n_1n_2} \ll y^2\log^4 z,\end{aligned}$$

where the inequality $ab\ge 2\sqrt{ab}$ and the estimate $D(u)\ll u\log u$ were used. Now the problem is reduced to evaluating $B_k(y,z)$ and estimating $E_k(y,z)$.

4. Evaluation of $B_k(y,z)$. We have

$$\begin{aligned}B_k(y,z) &= \sum_{\substack{d_1,d_2\le y;\, n_1,n_2\le z\\ n_1d_2^k=n_2d_1^k}} \frac{\mu(d_1)\mu(d_2)d(n_1)d(n_2)(d_1d_2)^{k/2}}{(n_1d_2^kn_2d_1^k)^{3/4}}\\ &= \sum_{m\le zy^k} g_k^2(m;y,z)m^{-3/2},\end{aligned}$$

where

$$g_k(m;y,z) := \sum_{\substack{m=nd^k\\ n\le z,\, d\le y}} \mu(d)d(n)d^{k/2}.$$

Let

$$g_k(m)=\sum_{m=nd^k}\mu(d)d(n)d^{k/2},\qquad g_0(m)=f_k(m)=\sum_{m=nd^k}d(n)d^{k/2}.$$

Let $z_0:=\min(y,z)$. Obviously,

$$\begin{gathered}g_k(m;y,z)=g_k(m),\quad m\le z_0,\\ |g_k(m;y,z)|\le g_0(m),\quad |g_k(m)|\le g_0(m),\qquad m\ge 1.\end{gathered}$$

Thus

$$\begin{aligned}(4.1)\qquad B_k(y,z) &= \sum_{m\le z_0} g_k^2(m)m^{-3/2} + \sum_{z_0<m\le zy^k} g_k^2(m;y,z)m^{-3/2}\\ &= \sum_{m\le z_0} g_k^2(m)m^{-3/2} + O\Big(\sum_{z_0<m\le zy^k} |g_0^2(m)|m^{-3/2}\Big).\end{aligned}$$

For any $1<U<V<\infty$, we shall estimate the sum

$$W_k(U,V) := \sum_{U<m\le V} |g_0^2(m)|m^{-3/2}.$$

Obviously $g_0(m)$ is a multiplicative function. So for $m > 1$, we have

$$g_0(m) = \prod_{p^\alpha \| m} g_0(p^\alpha).$$

If $1 \le \alpha \le k-1$, then $g_0(p^\alpha) = \alpha + 1$, which implies that if n is k-free then $g_0(n) = d(n)$.

Now suppose $ek \le \alpha < (e+1)k$ for some integer $e \ge 1$. It can be easily seen that if we write p^α in the form $p^\alpha = nd^k$, then $n = p^{\alpha - jk}, d = p^j$, $j = 0, 1, \ldots, e$. Then we have

$$\begin{aligned} g_0(p^\alpha) &= \sum_{j=0}^{e} (\alpha - jk + 1) p^{jk/2} = p^{ek/2} \sum_{j=0}^{e} (\alpha - jk + 1) p^{-(e-j)k/2} \\ &\le (\alpha+1) p^{ek/2} \sum_{j=0}^{e} p^{-(e-j)k/2} = (\alpha+1) p^{ek/2} \sum_{j=0}^{e} p^{-jk/2} \\ &\le (\alpha+1) p^{ek/2} \sum_{j=0}^{\infty} 2^{-jk/2} \le 2(\alpha+1) p^{\alpha/2}, \end{aligned}$$

which implies that if l is k-full, then

$$g_0(l) \le \prod_{p^\alpha \| l} 2(\alpha+1) p^{\alpha/2} = 2^{\omega(l)} d(l) l^{1/2} \le d^2(l) l^{1/2}.$$

Let $\delta_{(k)}(n)$, $\delta^{(k)}(n)$ denote the characteristic function of k-free and k-full numbers, respectively. Each integer m can be uniquely written as $m = nl$, $(n,l) = 1$, $\delta_{(k)}(n) = 1$, $\delta^{(k)}(l) = 1$. Thus

$$W_k(U,V) = \sum_{\substack{U < nl \le V \\ (n,l)=1}} g_0^2(n) g_0^2(l) \delta_{(k)}(n) \delta^{(k)}(l) (nl)^{-3/2} \ll \sum\nolimits_4 + \sum\nolimits_5,$$

where

$$\begin{aligned} \sum\nolimits_4 &:= \sum_{l \le U/3,\, U < nl \le V} g_0^2(n) g_0^2(l) \delta_{(k)}(n) \delta^{(k)}(l) (nl)^{-3/2}, \\ \sum\nolimits_5 &:= \sum_{l > U/3,\, U < nl \le V} g_0^2(n) g_0^2(l) \delta_{(k)}(n) \delta^{(k)}(l) (nl)^{-3/2}. \end{aligned}$$

LEMMA 4.1. *We have the estimate*

$$\sum_{n \le u} d^4(n) \delta^{(k)}(n) \ll u^{1/k} \log^{(k+1)^4 - 1} u, \qquad u \ge 2. \tag{4.2}$$

Proof. For $\Re s > 1/k$, it is easy to show that

$$\sum_{n=1}^{\infty} d^4(n) \delta^{(k)}(n) n^{-s} = \zeta^{(k+1)^4}(ks) G_k(s),$$

where $G_k(s)$ is absolutely convergent for $\Re s > 1/(1+k)$. Hence (4.2) follows. ■

By (3.2), partial summation and Lemma 4.1 we have

$$\begin{aligned}\sum\nolimits_4 &\ll \sum_{l\le U/3} g_0^2(l)\delta^{(k)}(l)l^{-3/2} \sum_{U/l<n\le V/l} g_0^2(n)n^{-3/2}\\ &\ll \sum_{l\le U/3} g_0^2(l)\delta^{(k)}(l)l^{-3/2}(U/l)^{-1/2}\log^3 U \ll U^{-1/2}\log^3 U \sum_{l\le U/3} d^4(l)\delta^{(k)}(l)\\ &\ll U^{-1/2+1/k}\log^{(k+1)^4+2} U,\\ \sum\nolimits_5 &\ll \sum_{l>U/3} g_0^2(l)\delta^{(k)}(l)l^{-3/2}\sum_n g_0^2(n)n^{-3/2} \ll \sum_{l>U/3} d^4(l)\delta^{(k)}(l)l^{-1/2}\\ &\ll U^{-1/2+1/k}\log^{(k+1)^4+2} U.\end{aligned}$$

Thus

$$W_k(U,V) \ll U^{-1/2+1/k}\log^{(k+1)^4+2} U. \tag{4.3}$$

From (4.1) and (4.3) we immediately get

$$B_k(y,z) = \sum_{m=1}^{\infty} g_k^2(m)m^{-3/2} + O(z_0^{-1/2+1/k}\log^{(k+1)^4+2} z_0). \tag{4.4}$$

5. Estimation of $E_k(y,z)$. By a splitting argument, we have

$$E_k(y,z) \ll E_k(D_1,D_2,N_1,N_2)z^{\varepsilon}\log^2 y \tag{5.1}$$

for some (D_1,D_2,N_1,N_2) with $1 \ll D_j \ll y$, $1 \ll N_j \ll z$, $j=1,2$, where

$$\begin{aligned}&E_k(D_1,D_2,N_1,N_2)\\ &\qquad = \sum \frac{1}{(d_1d_2)^{k/4}(n_1n_2)^{3/4}}\min\biggl(T^{1/2}, \frac{1}{|\sqrt{n_1/d_1^k}-\sqrt{n_2/d_2^k}|}\biggr),\\ &\qquad \mathrm{SC}\Bigl(\sum\Bigr) : d_1\sim D_1,\ d_2\sim D_2,\ n_1\sim N_1,\ n_2\sim N_2,\ n_1d_2^k \ne n_2d_1^k.\end{aligned}$$

We write

$$\begin{aligned}&E_k(D_1,D_2,N_1,N_2)\\ &\quad = \sum\nolimits_6 \frac{1}{(d_1d_2)^{k/4}(n_1n_2)^{3/4}}\min\biggl(T^{1/2}, \frac{1}{|\sqrt{n_1/d_1^k}-\sqrt{n_2/d_2^k}|}\biggr)\\ &\qquad + \sum\nolimits_7 \frac{1}{(d_1d_2)^{k/4}(n_1n_2)^{3/4}}\min\biggl(T^{1/2}, \frac{1}{|\sqrt{n_1/d_1^k}-\sqrt{n_2/d_2^k}|}\biggr),\end{aligned}$$

where

$$\mathrm{SC}\Big(\sum\nolimits_6\Big): d_1 \sim D_1,\ d_2 \sim D_2,\ n_1 \sim N_1,\ n_2 \sim N_2,$$
$$|\sqrt{n_1/d_1^k} - \sqrt{n_2/d_2^k}| \ge (\sqrt{n_1/d_1^k}\sqrt{n_2/d_2^k})^{1/2}/10,$$
$$\mathrm{SC}\Big(\sum\nolimits_7\Big): d_1 \sim D_1,\ d_2 \sim D_2,\ n_1 \sim N_1,\ n_2 \sim N_2,$$
$$|\sqrt{n_1/d_1^k} - \sqrt{n_2/d_2^k}| < (\sqrt{n_1/d_1^k}\sqrt{n_2/d_2^k})^{1/2}/10.$$

Trivially we have

$$\sum\nolimits_6 \ll \sum_{\substack{d_j\sim D_j,\, n_j\sim N_j\\ j=1,2}} \frac{1}{(d_1d_2)^{k/4}(n_1n_2)^{3/4}}\bigg(\sqrt{\frac{n_1}{d_1^k}}\sqrt{\frac{n_2}{d_2^k}}\bigg)^{-1/2} \ll D_1D_2 \ll y^2. \tag{5.2}$$

Suppose $\delta > 0$, and let $\mathcal{A}(D_1, D_2, N_1, N_2; \delta)$ denote the number of solutions of inequality

$$|\sqrt{n_1/d_1^k} - \sqrt{n_2/d_2^k}| \le \delta, \quad d_1 \sim D_1, \quad d_2 \sim D_2, \quad n_1 \sim N_1, \quad n_2 \sim N_2. \tag{5.3}$$

In order to estimate $\sum_7$, we need an upper bound of $\mathcal{A}(D_1, D_2, N_1, N_2; \delta)$.

Lemma 5.1. *We have*

$$\mathcal{A}(D_1, D_2, N_1, N_2; \delta) \ll \delta(D_1D_2)^{1+k/4}(N_1N_2)^{3/4} + (D_1D_2N_1N_2)^{1/2}\log(2D_1D_2N_1N_2),$$

where the implied constant is absolute.

Proof. We shall use an idea of Fouvry and Iwaniec [3]. Suppose u and v are two positive integers and let $\mathcal{A}_{u,v}(D_1, D_2, N_1, N_2; \delta)$ denote the number of solutions of inequality (5.3) with $(n_1, n_2) = u$, $(d_1, d_2) = v$. Set $n_j = m_ju$, $d_j = l_jv$ $(j = 1, 2)$; then (m_1, m_2, l_1, l_2) satisfies

$$|\sqrt{m_1/m_2} - \sqrt{l_1^k/l_2^k}| \le 2^{k/2}\delta D_1^{k/2}N_2^{-1/2}, \tag{5.4}$$
$$|\sqrt{m_2/m_1} - \sqrt{l_2^k/l_1^k}| \le 2^{k/2}\delta D_2^{k/2}N_1^{-1/2}. \tag{5.5}$$

It is easy to show that $\sqrt{m_1/m_2}$ is $u^2N_2^{-3/2}N_1^{-1/2}$-spaced, so from (5.4) we get

$$\mathcal{A}_{u,v}(D_1, D_2, N_1, N_2; \delta) \ll \frac{D_1D_2}{v^2}\bigg(1 + \frac{\delta D_1^{k/2}N_2N_1^{1/2}}{u^2}\bigg) \ll \frac{D_1D_2}{v^2} + \frac{\delta D_1D_2D_1^{k/2}N_2N_1^{1/2}}{u^2v^2}.$$

Similarly, since $\sqrt{m_2/m_1}$ is $u^2 N_1^{-3/2} N_2^{-1/2}$-spaced, from (5.5) we get

$$\mathcal{A}_{u,v}(D_1, D_2, N_1, N_2; \delta) \ll \frac{D_1 D_2}{v^2} + \frac{\delta D_1 D_2 D_2^{k/2} N_1 N_2^{1/2}}{u^2 v^2}.$$

The above two estimates imply

$$\begin{aligned}(5.6)\qquad &\mathcal{A}_{u,v}(D_1, D_2, N_1, N_2; \delta)\\ &\ll \frac{D_1 D_2}{v^2} + \frac{\delta D_1 D_2}{u^2 v^2} \min(D_1^{k/2} N_2 N_1^{1/2}, D_2^{k/2} N_1 N_2^{1/2})\\ &\ll \frac{D_1 D_2}{v^2} + \frac{\delta (D_1 D_2)^{1+k/4} (N_1 N_2)^{3/4}}{u^2 v^2}\end{aligned}$$

if we note that $\min(a, b) \le a^{1/2} b^{1/2}$.

It is easy to show that $(l_1/l_2)^{k/2}$ is $v^2 D_2^{-2} (D_1/D_2)^{k/2-1}$-spaced, and so from (5.4) we get

$$\begin{aligned}\mathcal{A}_{u,v}(D_1, D_2, N_1, N_2; \delta) &\ll \frac{N_1 N_2}{u^2} (1 + \delta D_1^{k/2} N_2^{-1/2} v^{-2} D_2^2 (D_1/D_2)^{-k/2+1})\\ &\ll \frac{N_1 N_2}{u^2} + \frac{\delta D_1 D_2 D_2^{k/2} N_1 N_2^{1/2}}{u^2 v^2}.\end{aligned}$$

Similarly from (5.5) we get

$$\mathcal{A}_{u,v}(D_1, D_2, N_1, N_2; \delta) \ll \frac{N_1 N_2}{u^2} + \frac{\delta D_1 D_2 D_1^{k/2} N_2 N_1^{1/2}}{u^2 v^2}.$$

From the above two estimates we have

$$\mathcal{A}_{u,v}(D_1, D_2, N_1, N_2; \delta) \ll \frac{N_1 N_2}{u^2} + \frac{\delta (D_1 D_2)^{1+k/4} (N_1 N_2)^{3/4}}{u^2 v^2},$$

which combined with (5.6) gives

$$\mathcal{A}_{u,v}(D_1, D_2, N_1, N_2; \delta) \ll \frac{\delta (D_1 D_2)^{1+k/4} (N_1 N_2)^{3/4}}{u^2 v^2} + \min\left(\frac{N_1 N_2}{u^2}, \frac{D_1 D_2}{v^2}\right).$$

Summing over u and v completes the proof of Lemma 5.1. ∎

Now we estimate $\sum_7$. Let $\Omega = \sqrt{n_1/d_1^k} - \sqrt{n_2/d_2^k}$. By Lemma 5.1 the contribution of $T^{1/2}$ is (note that $|\Omega| \le T^{-1/2}$)

$$\begin{aligned}&\ll \frac{T^{1/2}}{(D_1 D_2)^{k/4} (N_1 N_2)^{3/4}} \mathcal{A}_{u,v}(D_1, D_2, N_1, N_2; T^{-1/2})\\ &\ll \frac{T^{1/2} \log T}{(D_1 D_2)^{k/4-1/2} (N_1 N_2)^{1/4}} + D_1 D_2.\end{aligned}$$

Divide the remaining range into $O(\log T)$ intervals of the form $T^{-1/2} < \delta <$

$|\Omega| \le 2\delta$. By Lemma 5.1 again we find that the contribution of $1/|\Omega|$ is

$$\ll \log T \max_{\delta > T^{-1/2}} \frac{\mathcal{A}_{u,v}(D_1, D_2, N_1, N_2; 2\delta)}{(D_1D_2)^{k/4}(N_1N_2)^{3/4}\delta}$$
$$\ll \frac{T^{1/2}\log^2 T}{(D_1D_2)^{k/4-1/2}(N_1N_2)^{1/4}} + D_1D_2\log T.$$

From the above two estimates we get

$$\sum\nolimits_7 \ll \frac{T^{1/2}\log^2 T}{(D_1D_2)^{k/4-1/2}(N_1N_2)^{1/4}} + y^2 \log T. \tag{5.7}$$

Now we give another estimate of $\sum_7$. By noting that $\sqrt{n_1/d_1^k} \asymp \sqrt{n_2/d_2^k}$ we get

$$\frac{1}{|\Omega|} = \frac{\sqrt{n_1/d_1^k} + \sqrt{n_2/d_2^k}}{|n_1/d_1^k - n_2/d_2^k|} \ll \frac{(d_1d_2)^k(\sqrt{n_1/d_1^k} + \sqrt{n_2/d_2^k})}{|n_1d_2^k - n_2d_1^k|}$$
$$\ll (d_1d_2)^k(\sqrt{n_1/d_1^k}\sqrt{n_2/d_2^k})^{1/2} \ll (d_1d_2)^{3k/4}(n_1n_2)^{1/4}$$
$$\ll (D_1D_2)^{3k/4}(N_1N_2)^{1/4}.$$

The range of Ω can be divided into $O(\log T)$ intervals of the form

$$(D_1D_2)^{-3k/4}(N_1N_2)^{-1/4} \ll \delta \le |\Omega| \le 2\delta.$$

By Lemma 5.1 we have

$$\begin{aligned} \sum\nolimits_7 &\ll \frac{1}{(D_1D_2)^{k/4}(N_1N_2)^{3/4}} \sum_{\Omega} \frac{1}{|\Omega|} \\ &\ll \frac{\log T}{(D_1D_2)^{k/4}(N_1N_2)^{3/4}} \max_{\delta} \frac{\mathcal{A}_{u,v}(D_1, D_2, N_1, N_2; \delta)}{\delta} \\ &\ll (D_1D_2)^{(k+1)/2}\log^2 T \end{aligned} \tag{5.8}$$

if we note that $\delta \gg (D_1D_2)^{-3k/4}(N_1N_2)^{-1/4}$.

From (5.7) and (5.8) we get

$$\begin{aligned} \sum\nolimits_7 \ll{} & y^2 \log T \\ & + \min\left(\frac{T^{1/2}}{(D_1D_2)^{k/4-1/2}(N_1N_2)^{1/4}}, (D_1D_2)^{(k+1)/2}\right)\log^2 T \\ \ll{} & y^2 \log T \\ & + \left(\frac{T^{1/2}}{(D_1D_2)^{k/4-1/2}(N_1N_2)^{1/4}}\right)^{(2k+2)/3k} ((D_1D_2)^{(k+1)/2})^{(k-2)/3k}\log^2 T \\ \ll{} & y^2\log T + T^{(k+1)/3k}\log^2 T. \end{aligned} \tag{5.9}$$

Finally, from (5.1), (5.2) and (5.9) we have

$$E_k(y,z) \ll y^2 z^\varepsilon \log^4 T + T^{(k+1)/3k} z^\varepsilon \log^4 T. \tag{5.10}$$

6. Proof of Theorem 1 (completion). First consider the case $k=4$. Take $z = e^{10c_3\delta(T)}$, where c_3 was the constant in (3.4). From (3.3) and (3.4) we get

$$\int_T^{2T} |R_2^{(4)}(x) + R_3^{(4)}(x)|^2\,dx \ll T^{3/2} e^{-2c_3\delta(T)}.$$

From (3.6)–(3.9), (4.4) and (5.10) we get

$$\begin{aligned}\int_T^{2T} |R_1^{(4)}(x)|^2\,dx &= \frac{B_4}{4\pi^2}\int_T^{2T} x^{1/2}\,dx + O(T^{3/2} z_0^{-1/4}\log^{627} T)\\ &\quad + O(Ty^2 z^\varepsilon \log^5 T + T^{17/12} z^\varepsilon \log^6 T)\\ &= \frac{B_4}{4\pi^2}\int_T^{2T} x^{1/2}\,dx + O(T^{3/2} e^{-2c_3\delta(T)}).\end{aligned}$$

The above two estimates and Cauchy's inequality yield

$$\int_T^{2T} R_1^{(4)}(x)(R_2^{(4)}(x) + R_3^{(4)}(x))\,dx \ll T^{3/2} e^{-c_3\delta(T)}.$$

From the above three estimates we obtain

$$\begin{aligned}\int_T^{2T} |\Delta^{(4)}(x)|^2\,dx \\ &= \int_T^{2T} |R_1^{(4)}(x)|^2\,dx + 2\int_T^{2T} R_1^{(4)}(x)(R_2^{(4)}(x)+R_3^{(4)}(x))\,dx\\ &\quad + \int_T^{2T} |R_2^{(4)}(x)+R_3^{(4)}(x)|^2\,dx\\ &= \frac{B_4}{4\pi^2}\int_T^{2T} x^{1/2}\,dx + O(T^{3/2}e^{-c_3\delta(T)}),\end{aligned} \tag{6.1}$$

which implies the case $k=4$ of Theorem 1.

Now suppose $k \ge 5$. Take $z = T^{1-\varepsilon}$. From (3.3) and (3.5) we get

$$\int_T^{2T} |R_2^{(k)}(x) + R_3^{(k)}(x)|^2\,dx \ll T^{1+\varepsilon} y^2 + T^3 y^{2-2k}.$$

From (3.6)–(3.9), (4.4) and (5.10) we get

$$\int_T^{2T} |R_1^{(k)}(x)|^2\,dx = \frac{B_k}{4\pi^2}\int_T^{2T} x^{1/2}\,dx + O(T^{3/2+\varepsilon}y^{1/k-1/2}) + O(T^{1+\varepsilon}y^2 + T^{1+(k+1)/3k+\varepsilon}).$$

The above two estimates imply

$$\int_T^{2T} R_1^{(k)}(x)(R_2^{(k)}(x) + R_3^{(k)}(x))\,dx \ll T^{5/4+\varepsilon}y + T^{9/4}y^{1-k}.$$

From the above three estimates we deduce

$$\int_T^{2T} |\Delta^{(k)}(x)|^2\,dx = \frac{B_k}{4\pi^2}\int_T^{2T} x^{1/2}\,dx + O(T^{1+(k+1)/3k+\varepsilon}) + O(T^{5/4+\varepsilon}y + T^{9/4}y^{1-k} + T^{3/2+\varepsilon}y^{1/k-1/2}).$$

Now on taking $y = T^{1/5}$ if $k = 5$ and $y = T^{1/k-\varepsilon}$ if $k \geq 6$, we get

$$\int_T^{2T} |\Delta^{(k)}(x)|^2\,dx = \frac{B_k}{4\pi^2}\int_T^{2T} x^{1/2}\,dx + O(T^{\delta_k+\varepsilon}), \tag{6.2}$$

where δ_k was defined in Section 1. The case $k \geq 5$ of Theorem 1 now follows from equation (6.2).

7. An expression for $\Delta(1,1,k;x)$. In order to prove Theorem 2, we give an expression for $\Delta(1,1,k;x)$. We write

$$\begin{aligned} D(1,1,k;x) &= \sum_{nd^k\leq x} d(n) \\ &= \sum_{d\leq y} D(x/d^k) + \sum_{n\leq x/y^k} d(n)[(x/n)^{1/k}] - D(x/y^k)[y] \\ &= \sum\nolimits_8 + \sum\nolimits_9 - \sum\nolimits_{10}, \end{aligned} \tag{7.1}$$

say, where $x^\varepsilon \ll y \ll x^{1/k-\varepsilon}$ is a parameter.

We write $\sum_8$ as

$$\begin{aligned} \sum\nolimits_8 &= \sum_{d\leq y}\left(\frac{x}{d^k}\log\frac{x}{d^k} + (2\gamma-1)\frac{x}{d^k} + \Delta\left(\frac{x}{d^k}\right)\right) \\ &= x\log x\sum_{d\leq y}\frac{1}{d^k} - kx\sum_{d\leq y}\frac{\log d}{d^k} + (2\gamma-1)x\sum_{d\leq y}\frac{1}{d^k} + \sum_{d\leq y}\Delta\left(\frac{x}{d^k}\right). \end{aligned}$$

By the well known Euler–Maclaurin formula we have

$$\sum_{d\le y}\frac{1}{d^k}=\zeta(k)-\sum_{d>y}\frac{1}{d^k}=\zeta(k)-\frac{y^{1-k}}{k-1}-\psi(y)y^{-k}+O(y^{-k-1}),$$

$$\begin{aligned}\sum_{d\le y}\frac{\log d}{d^k}&=-\zeta'(k)-\sum_{d>y}\frac{\log d}{d^k}\\&=-\zeta'(k)+\frac{y^{1-k}\log y}{1-k}-\frac{y^{1-k}}{(k-1)^2}-\frac{\psi(y)\log y}{y^k}+O(y^{-k-1}\log y).\end{aligned}$$

From the above three formulas we get

$$\begin{aligned}\textstyle\sum_8&=\zeta(k)x\log x-\frac{xy^{1-k}\log x}{k-1}-\psi(y)xy^{-k}\log x\\&\quad+k\zeta'(k)x-\frac{kxy^{1-k}\log y}{1-k}+\frac{kxy^{1-k}}{(k-1)^2}+\frac{kx\psi(y)\log y}{y^k}\\&\quad+(2\gamma-1)\zeta(k)x-(2\gamma-1)\,\frac{xy^{1-k}}{k-1}-(2\gamma-1)\psi(y)xy^{-k}\\&\quad+\sum_{d\le y}\Delta\left(\frac{x}{d^k}\right)+O(xy^{-k-1}\log x).\end{aligned}\tag{7.2}$$

We write

$$\begin{aligned}\textstyle\sum_9&=\sum_{n\le x/y^k}d(n)((x/n)^{1/k}-1/2-\psi((x/n)^{1/k}))\\&=x^{1/k}\sum_{n\le x/y^k}d(n)n^{-1/k}-\frac{1}{2}\,D(xy^{-k})-\sum_{n\le x/y^k}d(n)\psi((x/n)^{1/k}).\end{aligned}$$

By partial summation we get (with $M=xy^{-k}$)

$$\begin{aligned}\sum_{n\le M}d(n)n^{-1/k}&=\int_{1^-}^{M}\frac{dD(u)}{u^{1/k}}=\int_{1^-}^{M}\frac{d(u\log u+(2\gamma-1)u)}{u^{1/k}}+\int_{1^-}^{M}\frac{d\Delta(u)}{u^{1/k}}\\&=\int_{1}^{M}\frac{\log u+1+2\gamma-1}{u^{1/k}}\,du+\frac{\Delta(M)}{M^{1/k}}+\frac{1}{k}\int_{1}^{M}\frac{\Delta(u)}{u^{1+1/k}}\,du\\&=\zeta^2(1/k)+\frac{M^{1-1/k}\log M}{1-1/k}-\frac{M^{1-1/k}}{(1-1/k)^2}+\frac{M^{1-1/k}}{1-1/k}+(2\gamma-1)\,\frac{M^{1-1/k}}{1-1/k}\\&\quad+\Delta(M)M^{-1/k}+O(M^{-1/k}),\end{aligned}$$

where we used the estimate

$$\int_M^{\infty} \frac{\Delta(u)}{u^{1+1/k}}\, du \ll M^{-1/k},$$

which follows from the well known estimate $\int_1^t \Delta(u)\, du \ll t$.

From the above two formulas we get

$$\sum\nolimits_9 = \zeta^2(1/k)x^{1/k} + \frac{xy^{1-k}\log xy^{-k}}{1-1/k} - \frac{xy^{1-k}}{(1-1/k)^2} + \frac{xy^{1-k}}{1-1/k} + (2\gamma-1)\,\frac{xy^{1-k}}{1-1/k} + y\Delta(xy^{-k}) - \frac{1}{2}\, D(xy^{-k}) - \sum_{n\le x/y^k} d(n)\psi((x/n)^{1/k}) + O(y). \tag{7.3}$$

For $\sum_{10}$ we have

$$-\sum\nolimits_{10} = \psi(y)xy^{-k}\log xy^{-k} + (2\gamma-1)\psi(y)xy^{-k} + \psi(y)\Delta(xy^{-k}) + \frac{1}{2}\, D(xy^{-k}) - xy^{1-k}\log xy^{-k} - (2\gamma-1)xy^{1-k} - y\Delta(xy^{-k}). \tag{7.4}$$

From (7.1)–(7.4) we get

$$\Delta(1,1,k;x) = \sum_{d\le y}\Delta(x/y^k) - \sum_{n\le x/y^k} d(n)\psi((x/n)^{1/k}) + O(y) + O(xy^{-k-1}\log x) + O(|\Delta(xy^{-k})|).$$

From $\Delta(u) \ll u^{1/3}$ we obtain

$$|\Delta(xy^{-k})| \ll x^{1/3}y^{-k/3} \ll y + xy^{-k-1}.$$

Thus we deduce the following lemma.

Lemma 7.1. *Suppose $x^{\varepsilon} \ll y \ll x^{1/k-\varepsilon}$. Then*

$$\Delta(1,1,k;x) = \sum_{d\le y}\Delta(x/y^k) - \sum_{n\le x/y^k} d(n)\psi((x/n)^{1/k}) + O(xy^{-k-1}\log x) + O(y).$$

8. Proof of Theorem 2. It suffices to evaluate $\int_T^{2T} \Delta^2(1,1,k;x)\, dx$ for large T. Suppose $T^{\varepsilon} \ll y \ll T^{1/k-\varepsilon}$ is a parameter to be determined later and $z = T^{1-\varepsilon}$. For simplicity, we write $\mathcal{L} = \log T$ in this section. Similar to (3.1), by Lemma 7.1 we may write

$$\Delta(1,1,k;x) = R_{1,k}(x) + R_{2,k}(x) - R_{3,k}(x), \tag{8.1}$$

where

$$R_{1,k}(x) := \frac{x^{1/4}}{\pi\sqrt{2}} \sum_{d\le y} \frac{1}{d^{k/4}} \sum_{n\le z} \frac{d(n)}{n^{3/4}} \cos\Big(4\pi\sqrt{\frac{nx}{d^k}} - \frac{\pi}{4}\Big),$$

$$R_{2,k}(x) := \sum_{d\le y} \Delta_2(x/d^k; z),$$

$$R_{3,k}(x) := \sum_{n\le x/y^k} d(n)\psi((x/n)^{1/k}) + O(xy^{-k-1}\log x) + O(y).$$

Similar to the mean square of $R_1^{(k)}(x)$, we can prove that

$$\int_T^{2T} |R_{1,k}(x)|^2\,dx = \frac{C_k}{4\pi^2}\int_T^{2T} x^{1/2}\,dx + O(T^{3/2+\varepsilon}y^{1/k-1/2}) + O(T^{1+\varepsilon}y^2+T^{1+(k+1)/3k+\varepsilon}). \tag{8.2}$$

From (3.3) we have

$$\int_T^{2T} |R_{2,k}(x)|^2\,dx \ll Ty^2\mathcal{L}^6. \tag{8.3}$$

Now we study the mean square of

$$S(x) = \sum_{n\le x/y^k} d(n)\psi((x/n)^{1/k}).$$

Let $J = [\log^{-1} 2\log(Ty^{-k}\mathcal{L}^{-1})]$. Then $J \ll \mathcal{L}$ and we may write

$$S(x) = \sum_{j=0}^{J} S_j(x) + O(\mathcal{L}^2), \qquad S_j(x) := \sum_{x2^{-j-1}y^{-k}<n\le x2^{-j}y^{-k}} d(n)\psi((x/n)^{1/k}).$$

Let $1/T \ll \eta < 1/10$ be a real number and let $\eta T = N$. Let

$$M(x,\eta) := \sum_{\eta x<n\le 2\eta x} d(n)\psi((x/n)^{1/k}).$$

Then $S_j(x) = M(x, 2^{-j-1}y^{-k})$, $j = 0, 1, \ldots, J$. We shall study the integral $\int_T^{2T} M^2(x,\eta)\,dx$.

According to Vaaler [16], we may write

$$\psi(t) = \sum_{1\le|h|\le N} a(h)e(ht) + O\Big(\sum_{|h|\le N} b(h)e(ht)\Big)$$

with $a(h) \ll 1/|h|$, $b(h) \ll 1/N$. Thus

$$\begin{aligned} M(x,\eta) &= \sum_{1\le|h|\le N} a(h) \sum_{\eta x<n\le 2\eta x} d(n)e(h(x/n)^{1/k}) \\ &\quad + O\Big(\sum_{|h|\le N} b(h) \sum_{\eta x<n\le 2\eta x} d(n)e(h(x/n)^{1/k})\Big) \\ &\ll 1 + \sum_{1\le h\le N} h^{-1/2}h^{-1/2}\Big|\sum_{\eta x<n\le 2\eta x} d(n)e(h(x/n)^{1/k})\Big|. \end{aligned}$$

By Cauchy's inequality we get

$$M^2(x,\eta) \ll 1 + \sum_{1\le h\le N} \frac{\mathcal{L}}{h}\Big|\sum_{\eta x<n\le 2\eta x} d(n)e(h(x/n)^{1/k})\Big|^2.$$

Integrating, squaring out and then applying the first derivative test we get

$$\begin{aligned} \int_T^{2T} M^2(x,\eta)\,dx &\ll T + \sum_{1\le h\le N} \frac{\mathcal{L}}{h}\int_T^{2T}\Big|\sum_{\eta x<n\le 2\eta x} d(n)e(h(x/n)^{1/k})\Big|^2\,dx \\ &= T + \sum_{1\le h\le N} \frac{\mathcal{L}}{h}\int_T^{2T} \sum_{\eta x<n\le 2\eta x} d^2(n)\,dx \\ &\quad + \sum_{1\le h\le N} \frac{\mathcal{L}}{h}\int_T^{2T} \sum_{\substack{\eta x<n,m\le 2\eta x\\ m\ne n}} d(m)d(n)e(hx^{1/k}(m^{-1/k}-n^{-1/k}))\,dx \\ &= O(TN\mathcal{L}^5) \\ &\quad + \sum_{1\le h\le N} \frac{\mathcal{L}}{h} \sum_{\substack{N<n,m\le 4N\\ m\ne n}} d(m)d(n)\int_{I(m,n)} e(hx^{1/k}(m^{-1/k}-n^{-1/k}))\,dx \\ &\ll TN\mathcal{L}^5 + \sum_{1\le h\le N} \frac{\mathcal{L}}{h} \sum_{\substack{N<n,m\le 4N\\ m\ne n}} \frac{T^{1-1/k}d(n)d(m)}{h|m^{-1/k}-n^{-1/k}|} \\ &\ll TN\mathcal{L}^5 + \sum_{1\le h\le N} \frac{\mathcal{L}}{h} \sum_{\substack{N<n,m\le 4N\\ m\ne n}} \frac{T^{1-1/k}N^{1+1/k}d(n)d(m)}{h|m-n|} \\ &\ll TN\mathcal{L}^5 + T^{1-1/k}N^{2+1/k}\mathcal{L}^5, \end{aligned}$$

where $I(m,n)$ is a subinterval of $[T,2T]$.

From Cauchy's inequality and the above estimate we get

$$\int_T^{2T} S^2(x)\,dx \ll \int_T^{2T} \Big|\sum_{j=0}^{J} S_j(x)\Big|^2\,dx + T\mathcal{L}^2 \ll \mathcal{L}\sum_{j=0}^{J}\int_T^{2T} |S_j(x)|^2\,dx + T\mathcal{L}^2$$
$$\ll (T^2y^{-k} + T^3y^{-2k-1})\mathcal{L}^6,$$

which implies that

$$\int_T^{2T} R_{3,k}^2(x)\,dx \ll (T^2y^{-k} + T^3y^{-2k-1})\mathcal{L}^6 + Ty^2. \tag{8.4}$$

From (8.2)–(8.4) and Cauchy's inequality we get

$$\int_T^{2T} R_{1,k}(x)(R_{2,k}(x) + R_{3,k}(x))\,dx \ll T^{5/4}y\mathcal{L}^3 + T^{7/4}y^{-k/2}\mathcal{L}^3 + T^{9/4}y^{-k-1/2}\mathcal{L}^3. \tag{8.5}$$

From (8.1)–(8.5) we get

$$\int_T^{2T} \Delta^2(1,1,k;x)\,dx = \frac{C_k}{4\pi^2}\int_T^{2T} x^{1/2}\,dx + O(T^{5/4}y\mathcal{L}^3 + T^{7/4}y^{-k/2}\mathcal{L}^3 + T^{9/4}y^{-k-1/2}\mathcal{L}^3) + O(T^{3/2}y^{1/k-1/2}\mathcal{L}^{(k+1)^4+2} + T^{1+(k+1)/3k+\varepsilon}).$$

Now on taking

$$y = \begin{cases} T^{2/9} & \text{if } k = 3,\\ T^{1/5}\mathcal{L}^{2496/5} & \text{if } k = 4,\\ T^{5/26}\mathcal{L}^{10(6^4-1)/13} & \text{if } k = 5,\\ T^{1/k-\varepsilon} & \text{if } k \ge 6 \end{cases}$$

we get

$$\int_T^{2T} \Delta^2(1,1,k;x)\,dx = \frac{C_k}{4\pi^2}\int_T^{2T} x^{1/2}\,dx + \begin{cases} O(T^{53/36}\mathcal{L}^3) & \text{if } k = 3,\\ O(T^{29/20}\mathcal{L}^{503}) & \text{if } k = 4,\\ O(T^{75/52}\mathcal{L}^{1000}) & \text{if } k = 5,\\ O(T^{3/2-1/2k+1/k^2+\varepsilon}) & \text{if } k \ge 6. \end{cases} \tag{8.6}$$

Theorem 2 follows from (8.6) immediately.

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