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ON THE KOEBE-MASKIT THEOREM

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Introduction. Let G be a finitely generated Kleinian group and denote by $\{\Delta_i\}$ the set of all components of G. In [3], Maskit showed the convergence of the series $\sum \mathrm{DIA}^4(\Delta_i)$, where $\mathrm{DIA}(\Delta_i)$ denotes the spherical diameter of Δ_i and $\sum$ is taken over all components of G. We shall call this result the Koebe-Maskit theorem. In his proof it is shown that if there is no parabolic element in G, then $\sum \mathrm{DIA}^2(\Delta_i) < \infty$, but it is not necessarily true if G is infinitely generated. So he asks whether or not $\sum \mathrm{DIA}^2(\Delta_i)$ converges if G is finitely generated. In this article we shall fill in the gap between the above question and the Koebe-Maskit theorem by showing the following.

THEOREM 1. *Let G be a finitely generated Kleinian group and let $\{\Delta_i\}$ be the set of all components of G. If $\alpha > 2$, then*

$$\sum \mathrm{DIA}^{\alpha}(\Delta_i) < \infty ,$$

where $\sum$ is taken over all components of G.

The proof of Theorem 1 is given in § 4. In some sense, our proof is a prolongation of Maskit's method, so we restate it in § 1. There are finitely generated Kleinian groups containing parabolic elements for which the sum of the squares of the spherical diameters of the components converges. That web groups containing parabolic elements are those is shown by Kuroda, Mori and Takahashi in [2]. There is also a result of Yamamoto showing the square-convergence for some class [5]. In § 3, we shall show the square-convergence for some class which contains geometrically finite groups. § 2 contains an observation on some free abelian groups of rank 2. Some results scattered in § 1 ~ § 3 are gathered together and used fully in § 4.

1. Preliminaries and Maskit's method. First we shall recall some properties of Kleinian groups. Let G be a finitely generated Kleinian group and denote by $\Omega(G)$ and $\Lambda(G)$ the region of discontinuity and the limit set of G, respectively. A component Δ of $\Omega(G)$ is called a component of G. Two components Δ_1 and Δ_2 of G are called equivalent if there is an element g of G such that $g(\Delta_1)=\Delta_2$. The number of inequivalent components of G is finite. The component subgroup G_Δ corresponding to Δ is the maximal subgroup of G consisting of all elements of G which keep Δ invariant. Each component subgroup of G is finitely generated. Let Δ' be a component of G_Δ different from Δ. The component subgroup $(G_\Delta)_{\Delta'}$ of G_Δ corresponding to Δ' is a quasi-Fuchsian group of the first kind, so the boundary $\partial\Delta'$ of Δ' is a quasi-circle contained in $\Lambda(G)$ and is called a separator for G. A parabolic element γ of $(G_\Delta)_{\Delta'}$ is called primitive if γ is not a power of any element of $(G_\Delta)_{\Delta'}\setminus\{\gamma,\gamma^{-1}\}$. A primitive parabolic element γ of $(G_\Delta)_{\Delta'}$ has an open disc D in Δ' which is precisely invariant under γ in G_Δ. The quotient $R=D/\langle\gamma\rangle$, where $\langle\gamma\rangle$ denotes the cyclic group generated by γ, is called a cusped region. In the case where the fixed point of γ is not ∞, the conjugation of G by a rotation after a translation shows that R is bounded by three circles

$$C_1=\{z\,|\,|z-ai|=a\},\ C_2=\{z\,|\,|z+ai|=a\}\quad\text{and}\quad C_3=\{z\,|\,|z+b|=b\}\,,$$

where a and b are positive numbers. We shall call this R the normalized cusped region.

Next we recall the inversions with respect to the isometric circles. Let $g(z)=(az+b)/(cz+d)$, $ad-bc=1$ and $c\neq 0$. The isometric circle of g is the circle with center $c(g)=-d/c$ and radius $r(g)=|c|^{-1}$. Let z be a point in the extended complex plane. The image of z under the inversion with respect to the isometric circle of g, which we shall denote by $I_g(z)$, is given by

$$\overline{I_g(z)}=r(g)^2/(z-c(g))+\overline{c(g)}=|c|^{-2}c/(cz+d)-\overline{(d/c)}\,,$$

where $\overline{}$ means the complex conjugation. Let z' be a point in the extended complex plane different from z. The Euclidean distance between $I_g(z)$ and $I_g(z')$ is given by

$$|I_g(z)-I_g(z')|=|z-z'|/|cz+d||cz'+d|\,. \tag{1}$$

In particular, in the case $z'=0$ we have

$$|I_g(0)-I_g(z)|=|d|^{-2}|c/d+1/z|^{-1}\,. \tag{2}$$

Now we restate the first part of Maskit's method developed in [3]. Let G be a finitely generated Kleinian group and let Δ be a component of G. Since there are only a finite number of inequivalent components, it suffices to show the convergence for each equivalence class of components. We denote by $\{\Delta_i\}$ the set of equivalent components of Δ and assume that ∞ lies in Δ. This normalization shows that the convergence of the series $\sum \mathrm{dia}^\alpha(\Delta_i)$, $(\alpha > 0)$ for the Euclidean diameter $\mathrm{dia}(\Delta_i)$ of Δ_i implies the convergence of $\sum \mathrm{DIA}^\alpha(\Delta_i)$. Hence, hereafter, we always make this normalization and use the Euclidean diameter. Let G_Δ be the component subgroup corresponding to Δ and let $G = \sum g_i G_\Delta$ be a coset decomposition. Since, for each coset, the set of centers of the isometric circles of its elements is invariant under G_Δ, we choose the coset representative g_i so that $c(g_i)$ lies in a particular fundamental set for G_Δ. We choose and fix a fundamental set for the action of G_Δ on $\Omega(G_\Delta)\setminus\Delta$ which is relatively compact in $\Omega(G_\Delta)\setminus\Delta$ except for a finite number of cusped regions. Let $g_i \in G\setminus G_\Delta$. Then the diameter of $\Delta_i = g_i(\Delta)$ is estimated by inverting Δ with respect to the isometric circle of g_i. Writing

$$g_i(z) = (a_i z + b_i)/(c_i z + d_i)\ , \quad a_i d_i - b_i c_i = 1\ ,$$

and denoting by $d(c(g_i), \Delta)$ the Euclidean distance from $c(g_i)$ to Δ, we have

$$\mathrm{dia}\,(\Delta_i) \leqq 2|c_i|^{-2} d(c(g_i),\ \Delta)^{-1}\ .$$

Using the well-known fact $\sum |c_i|^{-4} < \infty$, we have $\sum \mathrm{dia}^2(\Delta_i) < \infty$, whenever the set $\{c(g_i)\}$ is bounded away from Δ. This implies that if we show $\sum' \mathrm{dia}^\alpha(\Delta_i) < \infty$ with an $\alpha \geqq 2$, then we have $\sum \mathrm{dia}^\alpha(\Delta_i) < \infty$, where $\sum'$ is taken over all Δ_i for which $c(g_i)$ lies in arbitrarily small cusped regions. Assuming that R is a normalized cusped region containing $c(g_i)$ and putting

$$R_n = R \cap \{z \,|\, |z + b/n| < b/n\}\ ,$$

we formulate this fact in the following form, which is a starting point for our discussion.

PROPOSITION 2 ([3]). *Let $\alpha \geqq 2$. If, for each normalized cusped region R, there is an integer n such that*

$$\sum_{c(g_i) \in R_n} \mathrm{dia}^\alpha(\Delta_i) < \infty\ ,$$

then

$$\sum \mathrm{DIA}^\alpha(\Delta_i) < \infty\ .$$

2. Free abelian parabolic subgroups of rank 2. First we recall some properties of parabolic subgroups of a Kleinian group G. Let ζ be the fixed point of a parabolic element of G. Let M_ζ denote the maximal subgroup consisting of the identity together with all those parabolic elements of G which keep ζ fixed. Clearly M_ζ is either infinite cyclic or free abelian of rank 2. Let G, Δ, R and γ be as in §1. We assume that M_0 is free abelian of rank 2. Then there is a parabolic element δ in G of the form $\delta(z) = z/(\tau z + 1)$ with $\mathrm{Re}\,\tau > 0$, so that M_0 is generated by γ and δ.

LEMMA 3. *For each positive integer m and an element g of G with $c(g) \in R \cap \delta^{-m}(\mathrm{int}\, C_3)$, there are an element h of G with $c(h) \in R \cap \{\delta^{-m+1}(\mathrm{int}\, C_3) \setminus \delta^{-m}(\mathrm{int}\, C_3)\}$ and a positive integer n such that*

$$g(\Delta) = h\delta^n(\Delta)\ ,$$

where $\mathrm{int}\, C_3$ *means the open disc bounded by* C_3.

PROOF. Let n be the maximal positive integer such that $c(g) = g^{-1}(\infty) \in R \cap \delta^{-m+1-n}(\mathrm{int}\, C_3)$. Put $h^* = g\delta^{-n}$. Then we see that $(h^*)^{-1}(\infty) = \delta^n g^{-1}(\infty) \in \{\delta^{-m+1}(\mathrm{int}\, C_3) \setminus \delta^{-m}(\mathrm{int}\, C_3)\}$. Since $\{\delta^{-m+1}(\mathrm{int}\, C_3) \setminus \delta^{-m}(\mathrm{int}\, C_3)\}$ is invariant under γ, there is an integer l such that $\gamma^l(h^*)^{-1}(\infty) \in R$. Putting $h = h^*\gamma^{-l}$, we have

$$c(h) = h^{-1}(\infty) \in R \cap \{\delta^{-m+1}(\mathrm{int}\, C_3) \setminus \delta^{-m}(\mathrm{int}\, C_3)\}\ .$$

Since $g = h\gamma^l\delta^n$ and since $\delta^n(\Delta)$ is invariant under γ, we have $g(\Delta) = h\delta^n(\Delta)$.

Now we prove the following.

PROPOSITION 4. *If M_0 is a free abelian subgroup of rank 2, then*

$$\sum_{c(g_i) \in R} \mathrm{dia}^2(\Delta_i) < \infty\ .$$

PROOF. Let B be the complement of the component of G_Δ containing R and let $z \in B \setminus \{0\}$. Since z lies outside C_3, we see that $\mathrm{Re}\,(1/z) > -1/2b$. On the other hand, for $c(g_i) \in R_2$, we see that $\mathrm{Re}\,(c_i/d_i) \geqq 1/b$. Hence, putting

$$A_i = \mathrm{Re}\,(c_i/d_i - 1/2b)/\mathrm{Re}\,\tau > 0\ ,$$

we have by (2), for $z \in B$ and $c(g_i) \in R_2$,

$$\begin{aligned} |I_{g_i}(0) - I_{g_i}(\delta^j(z))| &= |d_i|^{-2}|j\tau + c_i/d_i + 1/z|^{-1} \\ &\leqq |d_i|^{-2}(\mathrm{Re}\,\tau)^{-1}|j + (\mathrm{Re}\,\tau)^{-1}(\mathrm{Re}\,(c_i/d_i) + \mathrm{Re}\,(1/z))|^{-1} \\ &< (\mathrm{Re}\,\tau)^{-1}|d_i|^{-2}(j + A_i)^{-1}\ . \end{aligned}$$

Using this inequality, we estimate the diameter of $g_i\delta^j(\Delta)$ which equals that of $I_{g_i}(\delta^j(\Delta))$. Let x and y be points on the boundary of Δ such that

$$\mathrm{dia}\,(I_{g_i}(\delta^j(\Delta))) = |I_{g_i}(\delta^j(x)) - I_{g_i}(\delta^j(y))|\ .$$

Since both x and y lie on B, we have

$$\begin{aligned}\operatorname{dia}(g_i\delta^j(\Delta)) &= |I_{g_i}(\delta^j(x)) - I_{g_i}(\delta^j(y))| \\ &\leqq |I_{g_i}(0) - I_{g_i}(\delta^j(x))| + |I_{g_i}(0) - I_{g_i}(\delta^j(y))| \\ &< 2(\operatorname{Re}\tau)^{-1}|d_i|^{-2}(j + A_i)^{-1}\,.\end{aligned}$$

Hence we have

$$(3)\qquad \sum_{j=1}^{\infty}\operatorname{dia}^2(g_i\delta^j(\Delta)) < 4(\operatorname{Re}\tau)^{-2}|d_i|^{-4}\sum_{j=1}^{\infty}(j + A_i)^{-2} < 4(\operatorname{Re}\tau)^{-2}|d_i|^{-4}A_i^{-1}\,.$$

To estimate the diameter of $g_i(\Delta)$ from below, we take z on the boundary of B such that 0, $c(g_i)$ and z lie on a straight line in this order. Then by (2) and $2\operatorname{Re}(c_i/d_i - 1/2b) \geqq \operatorname{Re}(c_i/d_i + 1/z)$, we have

$$(4)\qquad \begin{aligned}\operatorname{dia}(g_i(\Delta)) &\geqq |I_{g_i}(0) - I_{g_i}(z)| = |d_i|^{-2}|c_i/d_i + 1/z|^{-1} \\ &= |d_i|^{-2}(\operatorname{Re}(c_i/d_i + 1/z))^{-1}\cos(\arg(c_i/d_i + 1/z)) \\ &\geqq 2^{-1}|d_i|^{-2}(\operatorname{Re}\tau)^{-1}A_i^{-1}\cos(\arg(-c(g_i)))\,.\end{aligned}$$

(See Figure 1.) Thus by (3) and (4) we have

$$(5)\qquad \sum_{j=1}^{\infty}\operatorname{dia}^2(g_i\delta^j(\Delta)) < 16A_i\sec^2\arg(-c(g_i))\cdot\operatorname{dia}^2(g_i(\Delta))\,.$$

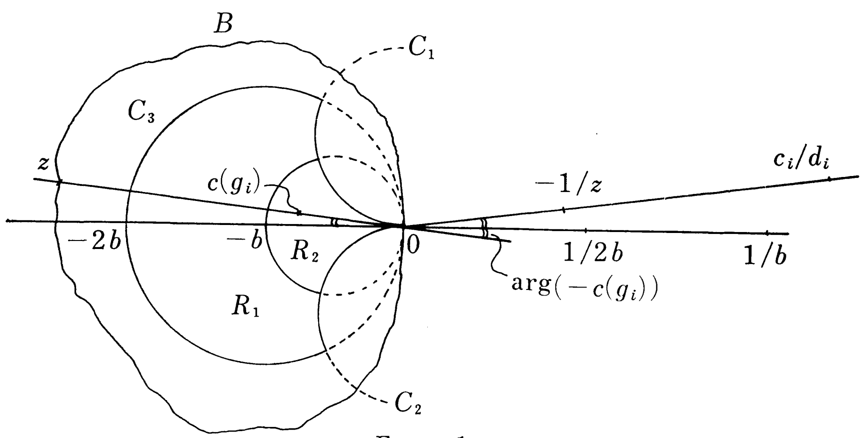


FIGURE 1

Now to get the conclusion we shall divide $\sum$ into three parts:

$$\sum_{c(g_i)\in R}\operatorname{dia}^2(\Delta_i) = \sum\nolimits_1\operatorname{dia}^2(\Delta_i) + \sum\nolimits_2\operatorname{dia}^2(\Delta_i) + \sum\nolimits_3\operatorname{dia}^2(\Delta_i)\,,$$

where $\sum_1$, $\sum_2$ and $\sum_3$ mean the summations over those g_i's for which $c(g_i)$ lies in $R\setminus R\cap\delta^{-m+1}(\operatorname{int}C_3)$, in $R\cap\{\delta^{-m+1}(\operatorname{int}C_3)\setminus\delta^{-m}(\operatorname{int}C_3)\}$ and in $R\cap\delta^{-m}(\operatorname{int}C_3)$, respectively. Taking m so large that $R\cap\delta^{-m+1}(\operatorname{int}C_3)\subset R_2$, we see by Lemma 3 and (5) that

$$\sum_{c(g_i)\in R}\operatorname{dia}^2(\Delta_i) \leqq \sum\nolimits_1\operatorname{dia}^2(\Delta_i) + \sum\nolimits_2(1 + 16A_i\sec^2\arg(-c(g_i)))\operatorname{dia}^2(\Delta_i)\,.$$

In the summation $\sum_2$, we see that the set $\{A_i\}$ is bounded and the set $\{|\arg(-c(g_i))|\}$ is bounded away from $\pi/2$ so that the set $\{(1+16A_i \sec^2 \arg(-c(g_i)))\}$ is bounded. Therefore by Proposition 2 we have

$$\sum_{c(g_i)\in R} \mathrm{dia}^2(\Delta_i) < \infty .$$

3. Extensions of Kuroda-Mori-Takahashi's theorem. A finitely generated Kleinian group G is called a web group if each component subgroup of G is quasi-Fuchsian. In their study of web groups, Kuroda, Mori and Takahashi showed the following.

THEOREM 5 ([2]). *If G is a web group, then the sum of the squares of the spherical diameters of the components of G converges.*

For later use we extend Theorem 5 in the following form, which follows easily from the argument in [2]. Before stating the theorem we recall the definition of auxiliary domains. Let G be a finitely generated Kleinian group with more than one component. Let Δ be a component of G and let E be an arbitrary set lying in a complementary component of the closure of Δ. The auxiliary domain for Δ relative to E is the complement of the closure of the component of G_Δ containing E, and is denoted by $D(\Delta, E)$. For example, B in § 2 is identical with the closure of $D(\Delta, R)$.

THEOREM 6. *Let G be a finitely generated Kleinian group. Assume that ∞ is not a limit point of G. Let $S=\{\Delta_i\}$ be a subset of components of G not containing ∞ such that $D(\Delta_i, \infty)\cap D(\Delta_j, \infty)=\emptyset$ whenever $i\neq j$. Then*

$$\sum_{\Delta_i\in S} \mathrm{dia}^2(\Delta_i) < \infty .$$

PROOF. Since the boundary of $D(\Delta_i, \infty)$ is a separator for G, the argument in [2] shows that there is a constant $K(G)$ depending only on G such that

$$\mathrm{dia}^2(D(\Delta_i, \infty)) < K(G)\,\mathrm{Area}\,(D(\Delta_i, \infty)) ,$$

where $\mathrm{Area}\,(D(\Delta_i, \infty))$ means the Euclidean area of $D(\Delta_i, \infty)$. Since $\bigcup_i D(\Delta_i, \infty)$ is bounded and since each $D(\Delta_i, \infty)$ lies exterior to each other, we have

$$\sum_{\Delta_i\in S} \mathrm{Area}\,(D(\Delta_i, \infty)) < \infty .$$

Since $\mathrm{dia}\,(\Delta_i)=\mathrm{dia}\,(D(\Delta_i, \infty))$, the theorem follows.

Now we prove the following which is an extension of Theorem 5 and a result of Yamamoto in [5].

THEOREM 7. *Let G be a finitely generated Kleinian group. If, for each component Δ of G and for each parabolic fixed point ζ on the boundary of Δ, either M_ζ is free abelian of rank 2 or there is an open disc in $\Omega(G)$ whose boundary passes through ζ, then the sum of the squares of the spherical diameters of the components of G converges.*

PROOF. Without loss of generality we may assume that $\infty \in \Delta$ and $\zeta = 0$. If M_0 is free abelian of rank 2, then the assertion follows from Proposition 4. Let D be an open disc in $\Omega(G)$ whose boundary passes through 0 and let R be the normalized cusped region. If $R \cap D \neq \emptyset$, then $\{c(g_i)\} \cap R$ is a finite set so that the assertion follows from Proposition 2. Next we assume that $D \subset B$ and assert that $g_i(D) \cap g_j(D) = \emptyset$ for $i \neq j$. If $g_i(D) \cap g_j(D) \neq \emptyset$, then $g_j^{-1}g_i \in G_{\Delta^*}$, where Δ^* is the component of G containing D. Note that $D \subset \Delta^* \subset B$. Since $g_j^{-1}g_i \notin G_\Delta$ and since either $g_j^{-1}g_i(D(\Delta, R)) \supsetneqq D(\Delta, R)$ or $g_i^{-1}g_j(D(\Delta, R)) \supsetneqq D(\Delta, R)$, we see that $g_j^{-1}g_i$ is loxodromic and that one of the fixed points of $g_j^{-1}g_i$ lies in the complement of B. This contradicts $g_j^{-1}g_i \in G_{\Delta^*}$. Thus we have our assertion. Since the set $\bigcup_i g_i(D)$ is bounded and each $g_i(D)$ lies exterior to each other, we see that

$$\sum_{c(g_i) \in R} \operatorname{dia}^2(g_i(D)) < \infty .$$

Hence, to complete the proof of the theorem, we have only to show the following.

LEMMA 8. *There is a number n such that if $c(g_i) \in R_n$, then*

$$\operatorname{dia}(\Delta_i) < 2 \cdot \operatorname{dia}(g_i(D)) .$$

PROOF. Since Δ lies outside C_3, we see that

$$\operatorname{dia}(\Delta_i) < \operatorname{dia}(I_{g_i}(C_3)) . \tag{6}$$

We choose n so large that

$$\operatorname{Re}(c_i/d_i) > \max(5/\operatorname{dia}(D),\ 5|\operatorname{Im}(c_i/d_i)|,\ (2 + 2^{1/2})/2b) , \tag{7}$$

whenever $c(g_i) \in R_n$. Since 0 and $\operatorname{dia}(D)$ are points on the boundary of D, we have, by (2) and (7),

$$\begin{aligned} \operatorname{dia}(I_{g_i}(D)) &\geqq |I_{g_i}(0) - I_{g_i}(\operatorname{dia}(D))| = |d_i|^{-2}|c_i/d_i + 1/\operatorname{dia}(D)|^{-1} \\ &\geqq |d_i|^{-2}|\operatorname{Re}(c_i/d_i) + |\operatorname{Im}(c_i/d_i)| + 1/\operatorname{dia}(D)|^{-1} \\ &> |d_i|^{-2}2^{-1/2}(\operatorname{Re}(c_i/d_i))^{-1} . \end{aligned} \tag{8}$$

On the other hand, for $z \in C_3$, we have, by (2) and (7),

$$\begin{aligned} |I_{g_i}(0) - I_{g_i}(z)| &= |d_i|^{-2}|c_i/d_i + 1/z|^{-1} \leqq |d_i|^{-2}|\operatorname{Re}(c_i/d_i) + \operatorname{Re}(1/z)|^{-1} \\ &= |d_i|^{-2}|\operatorname{Re}(c_i/d_i) - 1/2b|^{-1} < 2^{1/2}|d_i|^{-2}(\operatorname{Re}(c_i/d_i))^{-1} , \end{aligned} \tag{9}$$

so that

$$\text{dia}\,(I_{g_i}(C_3)) < 2^{1/2}|d_i|^{-2}(\text{Re}\,(c_i/d_i))^{-1}\,. \tag{10}$$

Hence we have, by (6), (8) and (10),

$$\text{dia}\,(\Delta_i) < \text{dia}\,(I_{g_i}(C_3)) < 2\cdot\text{dia}\,(I_{g_i}(D)) = 2\cdot\text{dia}\,(g_i(D))\,. \qquad \text{q.e.d.}$$

Thus we have Lemma 8 and also completed the proof of Theorem 7.

COROLLARY 9. *Let G be a finitely generated Kleinian group and let $\{\Delta_i\}$ be the set of all components of G. If G is geometrically finite, then*

$$\sum \text{DIA}^2\,(\Delta_i) < \infty\,.$$

PROOF. In [1] it is shown that G is geometrically finite if and only if each limit point of G is either a cusped parabolic fixed point or a point of approximation. It is also shown that the fixed point of any parabolic element of G is not a point of approximation. A fixed point ζ of a parabolic element of G is called cusped if M_ζ is free abelian of rank 2 or if there is a set A which is the disjoint union of two open circular discs (or half planes) such that A is precisely invariant under M_ζ. If there is such an A, then we can choose a half of A for the disc in Theorem 7. Thus Corollary 9 follows.

4. **Proof of Theorem 1.** Without loss of generality we may assume as in § 1 that $\infty \in \Delta$, $\{\Delta_i\}$ is the set of equivalent components of Δ, $0 \in \partial\Delta$, 0 is the fixed point of a parabolic element γ of G and R is the normalized cusped region. By Proposition 2 we may only consider the summation with respect to a small R_n. By Proposition 4 we may assume that M_0 is infinite cyclic. Then it is shown in [4] that there are only a finite number of components of G on whose boundaries 0 lies.

Before proving Theorem 1, we give the following lemma.

LEMMA 10. *For any number $\alpha > 2$, there is a positive integer n such that if $c(g_i)$ and $c(g_j)$ are in R_n and if $\Delta_i \subset D(\Delta_j, \infty)$ and $g_j(0) \notin \partial\Delta_i$, then*

$$\text{dia}\,(\Delta_j) > 8^{1/(\alpha-2)}\,\text{dia}\,(\Delta_i)\,.$$

PROOF. First we observe by (2) that

$$\text{dia}\,(\Delta_j) > |I_{g_j}(0) - I_{g_j}(\infty)| = |c_j d_j|^{-1}\,. \tag{11}$$

Let $x \neq 0$ and $y \neq 0$ be the points of intersection of C_3 with C_1 and C_2, respectively. Then by (1)

$$|I_{g_j}(x) - I_{g_j}(y)| = |c_j|^{-2}|x - y||x + d_j/c_j|^{-1}|y + d_j/c_j|^{-1}\,.$$

Hence there is an integer n_1 such that

$$2|d_j/c_j| < |x| = |y| \qquad \text{for } c(g_j) \in R_n,\ n \geqq n_1$$

so that

$$|I_{g_j}(x) - I_{g_j}(y)| < 8|c_j|^{-2}|x|^{-1} \qquad \text{for } c(g_j) \in R_n,\ n \geqq n_1\,. \tag{12}$$

Therefore, we have, by (11) and (12),

$$|I_{g_j}(x) - I_{g_j}(y)| < 8|x|^{-1}|d_j/c_j|\,\mathrm{dia}\,(\Delta_j) \tag{13}$$

for $c(g_j) \in R_n$, $n \geqq n_1$. Let D be an open disc lying in B and attaching to 0. Since the fixed point of $g_j\gamma g_j^{-1}$ does not lie on $g_i(B)$, we see $g_j\gamma g_j^{-1}(g_i(B)) \cap g_i(B) = \varnothing$ so that $g_j\gamma g_j^{-1}(g_i(D)) \cap g_i(D) = \varnothing$. Note that x is equivalent to y under γ and that $I_{g_j}(x)$, $I_{g_j}(y)$ and $I_{g_j}(0)$ lie on $I_{g_j}(C_3)$. Since by (2) and (9) we have

$$|I_{g_j}(x) - I_{g_j}(0)|/\mathrm{dia}\,(I_{g_j}(C_3)) > |\mathrm{Re}\,(c_j/d_j) - 1/2b|/|c_j/d_j + 1/x|\,,$$

which is close to 1 when $c(g_j) = -d_j/c_j$ lies in a small R_n, there is an integer n_2 such that

$$\mathrm{dia}\,(g_i(D)) \leqq |I_{g_j}(x) - I_{g_j}(y)| \quad \text{for } c(g_j) \in R_n,\ n \geqq n_2\,.$$

(See Figure 2.) Hence we have

$$\mathrm{dia}\,(g_i(D)) < 8|x|^{-1}|d_j/c_j|\,\mathrm{dia}\,(\Delta_j) \tag{14}$$

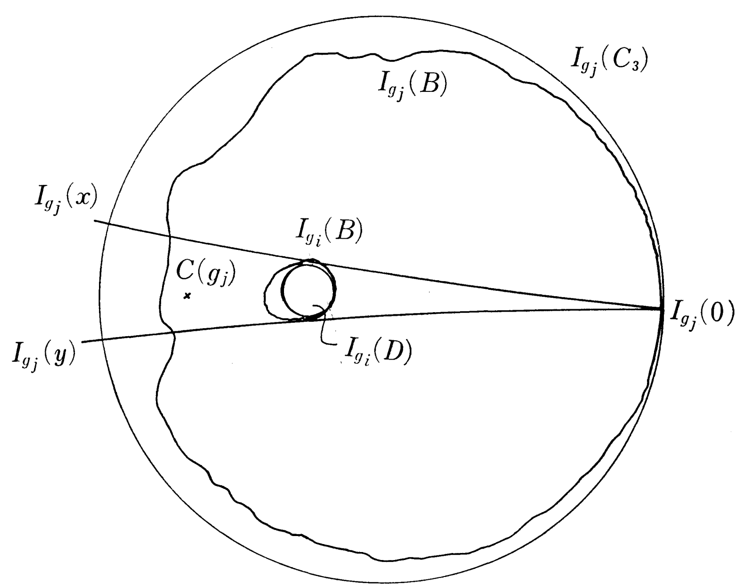


FIGURE 2

for $c(g_j) \in R_n$, $n \geqq \max(n_1, n_2)$. Lemma 8 implies that there is an integer n_3 such that

$$\text{(15)} \qquad \operatorname{dia}(\Delta_i) < 2\cdot\operatorname{dia}(g_i(D)) \quad \text{for } c(g_i) \in R_n,\ n \geqq n_3 .$$

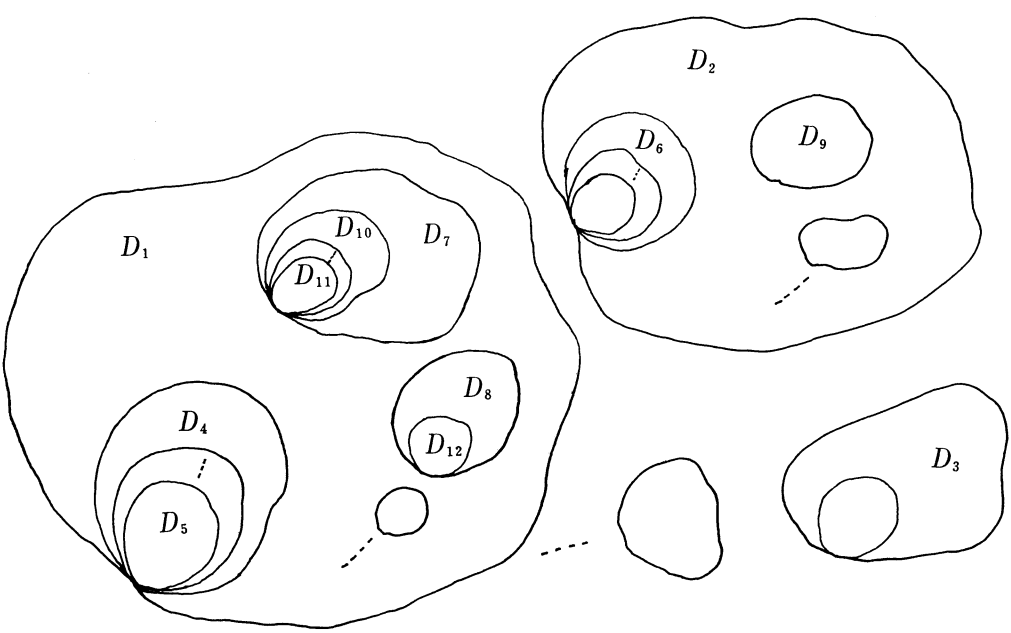
Finally, let n_4 be an integer such that

$$\text{(16)} \qquad 16|x|^{-1}|d_j/c_j| < 8^{-1/(\alpha-2)} \quad \text{for } c(g_j) \in R_n,\ n \geqq n_4 .$$

Putting $n = \max(n_1, n_2, n_3, n_4)$, we have, by (13), (14), (15) and (16),

$$\operatorname{dia}(\Delta_j) > 8^{1/(\alpha-2)} \operatorname{dia}(\Delta_i) \quad \text{for } c(g_i) \text{ and } c(g_j) \in R_n . \qquad \text{q.e.d.}$$

Now let us begin the proof of Theorem 1. Let S_n be the subset of $\{\Delta_i\}$ consisting of those Δ_i for which $c(g_i) \in R_n$. Let $S'_{n,1}$ be the subset of S_n consisting of those Δ_i for which $D(\Delta_i, \infty)$ is not contained in any other $D(\Delta_j, \infty)$. Let $S''_{n,1}$ be the subset of $S_n \setminus S'_{n,1}$ consisting of such Δ_i that there is an element Δ_j of $S'_{n,1}$ with $\partial\Delta_i \cap \partial\Delta_j \ni g_j(0)$. Put $S_{n,1} = S'_{n,1} \cup S''_{n,1}$. Inductively, let $S'_{n,m}$ be the subset of $S_n \setminus \bigcup_{k=1}^{m-1} S_{n,k}$ consisting of those Δ_i for which $D(\Delta_i, \infty)$ is not contained in any other $D(\Delta_j, \infty)$, $\Delta_j \in S_n \setminus \bigcup_{k=1}^{m-1} S_{n,k}$, and let $S''_{n,m}$ be the subset of $S_n \setminus (\bigcup_{k=1}^{m-1} S_{n,k} \cup S'_{n,m})$



$D_i = D(\Delta_i, \infty)$ for some Δ_j of S_n and $\{D_1, D_2, D_3\}$, $\{D_4, D_5, D_6\}$, $\{D_7, D_8, D_9\}$ and $\{D_{10}, D_{11}, D_{12}\}$ correspond to elements of $S'_{n,m-1}$, $S''_{n,m-1}$, $S'_{n,m}$ and $S''_{n,m}$, respectively.

FIGURE 3

consisting of such Δ_i that there is an element Δ_j of $S'_{n,m}$ with $\partial\Delta_i \cap \partial\Delta_j \ni g_j(0)$, and put $S_{n,m} = S'_{n,m} \cup S''_{n,m}$. (See Figure 3.) Observe that, as we noted just above Lemma 10, there are only a finite number, say $N-1$, of components in $S''_{n,m}$ corresponding to each component of $S'_{n,m}$. Now we rewrite

$$\sum_{c(g_i)\in R_n} \mathrm{dia}^\alpha(\Delta_i) = \sum_{m=1}^{\infty}\Big(\sum_{\Delta_i\in S_{n,m}} \mathrm{dia}^\alpha(\Delta_i)\Big) \leqq N\sum_{m=1}^{\infty}\Big(\sum_{\Delta_i\in S'_{n,m}} \mathrm{dia}^\alpha(\Delta_i)\Big) .$$

Rewriting

$$\sum_{\Delta_i\in S'_{n,m}} \mathrm{dia}^\alpha(\Delta_i) = \sum_{\Delta_j\in S'_{n,m-1}} \Big(\sum{}^* \mathrm{dia}^\alpha(\Delta_i)\Big) = \sum_j\sum_i \mathrm{dia}^\alpha(\Delta_i) ,$$

where $\sum^*$ is taken over all $\Delta_i \in S'_{n,m}$ which lie in $D(\Delta_j, \infty)$, we have by Lemmas 8 and 10 that, for a large n,

$$\begin{aligned}\sum_{\Delta_i\in S'_{n,m}} \mathrm{dia}^\alpha(\Delta_i) &= \sum_j\sum_i \mathrm{dia}^{\alpha-2}(\Delta_i)\,\mathrm{dia}^2(\Delta_i) < \sum_j (\mathrm{dia}(\Delta_j)/8^{1/(\alpha-2)})^{\alpha-2}\sum_i \mathrm{dia}^2(\Delta_i)\\ &< 8^{-1}\sum_j \mathrm{dia}^{\alpha-2}(\Delta_j)\sum_i 4\cdot\mathrm{dia}^2(g_i(D)) < 2^{-1}\sum_j \mathrm{dia}^{\alpha-2}(\Delta_j)\,\mathrm{dia}^2(\Delta_j)\\ &= 2^{-1}\Big(\sum_{\Delta_i\in S'_{n,m-1}} \mathrm{dia}^\alpha(\Delta_i)\Big) .\end{aligned}$$

Hence we have

$$\sum_{c(g_i)\in R_n} \mathrm{dia}^\alpha(\Delta_i) < 2N\Big(\sum_{\Delta_i\in S'_{n,1}} \mathrm{dia}^\alpha(\Delta_i)\Big) .$$

By Theorem 6 we conclude

$$\sum_{c(g_i)\in R_n} \mathrm{dia}^\alpha(\Delta_i) < \infty . \qquad \text{q.e.d.}$$

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