On the Koolen–Park inequality and Terwilliger graphs

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Abstract

J.H. Koolen and J. Park proved a lower bound for the intersection number c_2 of a distance-regular graph Γ. Moreover, they showed that a graph Γ, for which equality is attained in this bound, is a Terwilliger graph. We prove that Γ is the icosahedron, the Doro graph or the Conway–Smith graph if equality is attained and $c_2 \geqslant 2$.

1 Introduction

Let Γ be a distance-regular graph with degree k and diameter at least 2. Let c be maximal such that, for each vertex $x \in \Gamma$ and every pair of nonadjacent vertices y, z of $\Gamma_1(x)$, there exists a c-coclique in $\Gamma_1(x)$ containing y, z. In [1], J.H. Koolen and J. Park showed that the following bound holds:

$$c_2 - 1 \geqslant \max\{\frac{c'(a_1+1)-k}{\binom{c'}{2}} \mid 2 \leqslant c' \leqslant c\}, \tag{1}$$

and equality implies that Γ is a Terwilliger graph. (For definitions see Sections 2 and 3.)

A similar inequality for a distance-regular graph with a c-claw was proved by C.D. Godsil, see [2]. J.H. Koolen and J. Park [1] noted that the bound (1) is met for the three known examples of Terwilliger graphs with $c_2 \geqslant 2$. We recall that only three examples of distance-regular Terwilliger graphs with $c_2 \geqslant 2$ are known: the icosahedron, the Doro graph and the Conway–Smith graph.

In this paper, we will show that a distance-regular graph Γ with $c_2 \geqslant 2$, for which equality is attained in (1), is a known Terwilliger graph.

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2 Definitions and preliminaries

We consider only finite undirected graphs without loops or multiple edges. Let Γ be a connected graph. The *distance* $\mathrm{d}(u, w)$ between any two vertices u and w of Γ is the length of a shortest path from u to w in Γ. The *diameter* $\mathrm{diam}(\Gamma)$ of Γ is the maximal distance occurring in Γ.

For a subset A of the vertex set of Γ, we will also write A for the subgraph of Γ induced by A. For a vertex u of Γ, define $\Gamma_i(u)$ to be the set of vertices that are at distance i from u ($0 \leqslant i \leqslant \mathrm{diam}(\Gamma)$). The subgraph $\Gamma_1(u)$ is called the *local graph* of a vertex u and the *degree* of u is the number of neighbors of u, i.e., $|\Gamma_1(u)|$.

For two vertices $u, w \in \Gamma$ with $\mathrm{d}(u, w) = 2$, the subgraph $\Gamma_1(u) \cap \Gamma_1(w)$ is called the *μ-subgraph* of vertices u, w. We say that the number $\mu(\Gamma)$ is *well-defined* if each μ-subgraph occurring in Γ contains the same number of vertices and this number is equal to $\mu(\Gamma)$.

Let Δ be a graph. A graph Γ is *locally* Δ if, for all $u \in \Gamma$, the subgraph $\Gamma_1(u)$ is isomorphic to Δ. A graph is *regular* with degree k if the degree of each of its vertices is k.

A connected graph Γ with diameter $d = \mathrm{diam}(\Gamma)$ is *distance-regular* if there are integers b_i, c_i ($0 \leqslant i \leqslant d$) such that, for any two vertices $u, w \in \Gamma$ with $\mathrm{d}(u, w) = i$, there are exactly c_i neighbors of w in $\Gamma_{i-1}(u)$ and b_i neighbors of w in $\Gamma_{i+1}(u)$ (we assume that $\Gamma_{-1}(u)$ and $\Gamma_{d+1}(u)$ are empty sets). In particular, a distance-regular graph Γ is regular with degree b_0, $c_1 = 1$ and $c_2 = \mu(\Gamma)$. For each vertex $u \in \Gamma$ and $0 \leqslant i \leqslant d$, the subgraph $\Gamma_i(u)$ is regular with degree $a_i = b_0 - b_i - c_i$. The numbers a_i, b_i, c_i ($0 \leqslant i \leqslant d$) are called the *intersection numbers* and the array $\{b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d\}$, is called the *intersection array* of the distance-regular graph Γ.

A graph Γ is *amply regular* with parameters (v, k, λ, μ) if Γ has v vertices, is regular with degree k and satisfies the following two conditions:

i) for each pair of adjacent vertices $u, w \in \Gamma$, the subgraph $\Gamma_1(u) \cap \Gamma_1(w)$ contains exactly λ vertices;

ii) $\mu = \mu(\Gamma)$ is well-defined.

An amply regular graph with diameter 2 is called a *strongly regular* graph and is a distance-regular graph. A distance-regular graph is an amply regular graph with parameters $k = b_0$, $\lambda = b_0 - b_1 - 1$ and $\mu = c_2$.

A *c-clique* C of Γ is a complete subgraph (i.e., every two vertices of C are adjacent) of Γ with exactly c vertices. We say that C is a clique if it is a c-clique for certain c. A coclique C of Γ is an induced subgraph of Γ with empty edge set. We say a coclique is a *c-coclique* if it has exactly c vertices.

Let Γ be a strongly regular graph with parameters $(v, k, \lambda, 1)$. There are integers r and s such that the local graph of each vertex of Γ is the disjoint union of r copies of the s-clique. Furthermore, $v = 1 + rs + s^2 r(r-1)$, $k = rs$ and $\lambda = s - 1$. The set of strongly regular graph with parameters $(1 + rs + s^2 r(r-1), rs, s-1, 1)$ is denoted by $\mathcal{F}(s, r)$.

Any graph of $\mathcal{F}(1,r)$, i.e., a strongly regular graph with $\lambda = 0$ and $\mu = 1$, is called a *Moore* strongly regular graph. It is well known (see Ch. 1 [3]) that any Moore strongly regular graph has degree 2, 3, 7 or possibly 57. The graphs with degree 2, 3 and 7 are the pentagon, the Petersen graph and the Hoffman–Singleton graph, respectively. It is still unknown whether there exists a Moore graph with degree 57.

Lemma 2.1 *If $\mathcal{F}(s,r)$ is a nonempty set of graphs, then $s+1 \leqslant r$.*

Proof. Let Γ be a graph of $\mathcal{F}(s,r)$. We can choose vertices u and w from Γ with $\mathrm{d}(u,w) = 2$. Let x be a vertex of $\Gamma_1(u) \cap \Gamma_1(w)$. Then the subgraph $\Gamma_1(w) - (\Gamma_1(x) \cup \{x\})$ contains a coclique of size at most $r-1$. Let us consider an s-clique of $\Gamma_1(u) - \Gamma_1(w)$ on vertices $y_1, y_2, .., y_s$. The subgraph $\Gamma_1(w) \cap \Gamma_1(y_i)$ $(1 \leqslant i \leqslant s)$ contains a single vertex z_i. The vertices $z_1, z_2, .., z_s$ are mutually nonadjacent and distinct. Hence, $s \leqslant r-1$. The lemma is proved. ∎

3 Terwilliger graphs

In this section we give a definition of Terwilliger graphs and some useful facts concerning them.

A *Terwilliger graph* is a connected non-complete graph Γ such that $\mu(\Gamma)$ is well-defined and each μ-subgraph occurring in Γ is a complete graph (hence, there are no induced quadrangles in Γ). If $\mu(\Gamma) > 1$, then, for each vertex $u \in \Gamma$, the local graph of u is also a Terwilliger graph with diameter 2 and $\mu(\Gamma_1(u)) = \mu(\Gamma) - 1$.

For an integer $\alpha \geqslant 1$, the α-clique extension of a graph $\bar{\Gamma}$ is the graph Γ obtained from $\bar{\Gamma}$ by replacing each vertex $\bar{u} \in \bar{\Gamma}$ by a clique U with α vertices, where, for any $\bar{u}, \bar{w} \in \bar{\Gamma}$, $u \in U$ and $w \in W$, $\bar{u}$ and $\bar{w}$ are adjacent if and only if u and w are adjacent.

Lemma 3.1 *Let Γ be an amply regular Terwilliger graph with parameters (v,k,λ,μ), where $\mu > 1$. Then there is a number α such that the local graph of each vertex of Γ is the α-clique extension of a strongly regular Terwilliger graph with parameters $(\bar{v}, \bar{k}, \bar{\lambda}, \bar{\mu})$, where*

$$\bar{v} = k/\alpha,\ \bar{k} = (\lambda - \alpha + 1)/\alpha,\ \bar{\mu} = (\mu - 1)/\alpha,$$

and $\alpha \leqslant \bar{\lambda} + 1$. In particular, if $\bar{\lambda} = 0$, then $\alpha = 1$.

Proof. The result follows from [3, Theorem 1.16.3]. ∎

There are only three amply regular Terwilliger graphs known with $\mu \geqslant 2$. All of them are distance-regular and are characterized by theirs intersection arrays. The three examples are:

(1) the icosahedron with intersection array $\{5,2,1;1,2,5\}$ is locally pentagon graph;

(2) the Doro graph with intersection array $\{10,6,4;1,2,5\}$ is locally Petersen graph;

(3) the Conway–Smith graph with intersection array $\{10,6,4,1;1,2,6,10\}$ is locally Petersen graph.

In [4], A. Gavrilyuk and A. Makhnev showed that a distance-regular locally Hoffman–Singleton graph has intersection array $\{50, 42, 9; 1, 2, 42\}$ or $\{50, 42, 1; 1, 2, 50\}$ and hence it is a Terwilliger graph. Whether there exist graphs with these intersection arrays is an open question.

Lemma 3.2 *Let Γ be a Terwilliger graph. Suppose that, for an integer $\alpha \geqslant 1$, the local graph of each vertex of Γ is the α-clique extension of a Moore strongly regular graph Δ. Then $\alpha = 1$ and one of the following holds*:

(1) Δ *is the pentagon and* Γ *is the icosahedron*;

(2) Δ *is the Petersen graph and* Γ *is the Doro graph or the Conway–Smith graph*;

(3) Δ *is the Hoffman–Singleton graph or a Moore graph with degree* 57; *in both cases, the diameter of* Γ *is at least* 3.

Proof. It is easy to see that the graph Γ is amply regular. By Lemma 3.1, we have $\alpha = 1$. Statements (1) and (2) follow from [3, Proposition 1.1.4] and [3, Theorem 1.16.5], respectively.

If the graph Δ is the Hoffman–Singleton graph and the diameter of Γ is 2, then Γ is strongly regular with parameters (v, k, λ, μ), where $k = 50$, $\lambda = 7$ and $\mu = 2$. By [3, Theorem 1.3.1], the eigenvalues of Γ are k and the roots of the quadratic equation $x^2 + (\mu - \lambda)x + (\mu - k) = 0$. The roots of the equation $x^2 - 5x - 48 = 0$ are not integers, a contradiction. In the remaining case, when Δ is regular with degree 57, we get the same contradiction. The lemma is proved. ∎

The next lemma will be used in the proof of Theorem 4.2 (see Section 4).

Lemma 3.3 *Let Γ be a strongly regular Terwilliger graph with parameters (v, k, λ, μ). Suppose that, for an integer $\alpha \geqslant 1$, the local graph of each vertex of Γ is the α-clique extension of a strongly regular graph with parameters $(\bar{v}, \bar{k}, \bar{\lambda}, \bar{\mu})$. Then the inequality $\bar{k} - \bar{\lambda} - \bar{\mu} > 1$ implies that $k - \lambda - \mu > 1$.*

Proof. We have $k = \alpha(1+\bar{k}+\bar{k}(\bar{k}-\bar{\lambda}-1)/\bar{\mu})$, $\lambda = \alpha\bar{k}+\alpha-1$ and $\mu = \alpha\bar{\mu}+1$. If $\bar{k}-\bar{\lambda}-\bar{\mu} > 1$, then $\bar{k}(\bar{k} - \bar{\lambda} - 1)/\bar{\mu} > \bar{k}$ and this implies that $k - \lambda - \mu = \alpha(\bar{k}(\bar{k} - \bar{\lambda} - 1)/\bar{\mu} - \bar{\mu}) > \alpha(\bar{k} - \bar{\mu}) > \alpha(\bar{\lambda} + 1) \geqslant 1$. ∎

4 The Koolen–Park inequality

In this section, we consider bound (1) and classify distance-regular graphs with $c_2 \geqslant 2$, for which this bound is attained.

The next statement is a slight generalization of Proposition 3 from [1], which was formulated by J.H. Koolen and J. Park for distance-regular graphs. We generalize it to amply regular graphs. (Our proof is similar to the proof in [1], but we give it for the convenience of the reader.)

Proposition 4.1 *Let Γ be an amply regular graph with parameters (v, k, λ, μ), and let $c \geqslant 2$ be maximal such that, for each vertex $x \in \Gamma$ and every pair of nonadjacent vertices y, z of $\Gamma_1(x)$, there exists a c-coclique in $\Gamma_1(x)$ containing y, z. Then*

$$\mu - 1 \geqslant \max\{\frac{c'(\lambda+1)-k}{\binom{c'}{2}} \mid 2 \leqslant c' \leqslant c\},$$

and, if equality is attained, then Γ is a Terwilliger graph.

Proof. Let $\Gamma_1(x)$ contain a coclique C' on vertices $y_1, y_2, \ldots, y_{c'}$, $c' \geqslant 2$. Since $\mathrm{d}(y_i, y_j) = 2$, it follows that $|\Gamma_1(x) \cap \Gamma_1(y_i) \cap \Gamma_1(y_j)| \leqslant \mu - 1$ holds for all $i \neq j$. Then, by the inclusion–exclusion principle,

$$k = |\Gamma_1(x)| \geqslant | \cup_{i=1}^{c'} (\Gamma_1(x) \cap (\Gamma_1(y_i) \cup \{y_i\}))|$$

$$\geqslant \sum_{i=1}^{c'} |\Gamma_1(x) \cap (\Gamma_1(y_i) \cup \{y_i\})| - \sum_{1 \leqslant i < j \leqslant c'} |\Gamma_1(x) \cap \Gamma_1(y_i) \cap \Gamma_1(y_j)|$$

$$\geqslant c'(\lambda+1) - \binom{c'}{2}(\mu - 1).$$

So,

$$\mu - 1 \geqslant \frac{c'(\lambda+1)-k}{\binom{c'}{2}}. \tag{2}$$

Note that equality in (2) implies that the inclusion $\Gamma_1(x) \subseteq \cup_{i=1}^{c'}(\Gamma_1(y_i) \cup \{y_i\})$ holds and we have $|\Gamma_1(x) \cap \Gamma_1(y_i) \cap \Gamma_1(y_j)| = \mu - 1$ for all $i \neq j$.

Let c be the maximal number satisfying the condition of Proposition 4.1. Then

$$\mu - 1 \geqslant \max\{\frac{c'(\lambda+1)-k}{\binom{c'}{2}} \mid 2 \leqslant c' \leqslant c\}. \tag{3}$$

We may assume that for an integer c'', where $2 \leqslant c'' \leqslant c$, (3) turns into equality, i.e.,

$$\mu - 1 = \frac{c''(\lambda+1)-k}{\binom{c''}{2}} = \max\{\frac{c'(\lambda+1)-k}{\binom{c'}{2}} \mid 2 \leqslant c' \leqslant c\}. \tag{4}$$

We will show that $c = c''$. For a vertex $x \in \Gamma$ and nonadjacent vertices $y, z \in \Gamma_1(x)$, there exists a c-coclique C in $\Gamma_1(x)$ containing y, z. Equality (4) implies that, for any subset of vertices $\{y_1, y_2, \ldots, y_{c''}\} \subseteq C$, we have $\Gamma_1(x) \subseteq \cup_{i=1}^{c''}(\Gamma_1(y_i) \cup \{y_i\})$. However, if $c'' < c$, then $C \not\subseteq \cup_{i=1}^{c''}(\Gamma_1(y_i) \cup \{y_i\})$, a contradiction.

Hence, $c = c''$ and we have $|\Gamma_1(x) \cap \Gamma_1(y) \cap \Gamma_1(z)| = \mu - 1$ for every pair of nonadjacent vertices $y, z \in \Gamma_1(x)$ and for all $x \in \Gamma$. This implies that each μ-subgraph in Γ is a clique of size μ and Γ is a Terwilliger graph. ∎

We call inequality (3) the *μ-bound.*

It is easy to check that the three known Terwillger graphs with $\mu \geqslant 2$ (see Section 3) have equality in the μ-bound.

Our main theorem is to show that the only Terwilliger graphs with $\mu \geqslant 2$ and equality in the μ-bound are the three known examples (of Section 3).

Theorem 4.2 *Let Γ be an amply regular graph with parameters (v, k, λ, μ), and let $\mu > 1$. If the μ-bound is attained, then $\mu = 2$ and Γ is the icosahedron, the Doro graph or the Conway–Smith graph.*

Proof. By Proposition 4.1, the graph Γ is a Terwilliger graph and, by Lemma 3.1, there is an integer $\alpha \geqslant 1$ such that the local graph of each vertex of Γ is the α-clique extension of a strongly regular Terwilliger graph with parameters $(\bar{v}, \bar{k}, \bar{\lambda}, \bar{\mu})$. By Lemma 3.1, we have $k = \alpha\bar{v}$, $\lambda = \alpha\bar{k} + (\alpha - 1)$ and $\mu = \alpha\bar{\mu} + 1$.

By the assumption on Γ, for a vertex $u \in \Gamma$, the local graph of u contains a c-coclique, for which equality is attained in the μ-bound, i.e.,

$$\mu - 1 = \alpha\bar{\mu} = \frac{c(\lambda + 1) - k}{\binom{c}{2}} = \frac{c(\alpha\bar{k} + (\alpha - 1) + 1) - \alpha\bar{v}}{\binom{c}{2}} = \alpha\frac{c(\bar{k} + 1) - \bar{v}}{\binom{c}{2}}$$

and

$$\bar{\mu} = \frac{c(\bar{k} + 1) - \bar{v}}{\binom{c}{2}}.$$

Hence, c satisfies the following quadratic equation:

$$c^2\bar{\mu} - c(\bar{\mu} + 2(\bar{k} + 1)) + 2\bar{v} = 0,$$

in other words,

$$c = \frac{(\bar{\mu} + 2(\bar{k} + 1)) \pm \sqrt{(\bar{\mu} + 2(\bar{k} + 1))^2 - 8\bar{v}\bar{\mu}}}{2\bar{\mu}}.$$

This implies that

$$(\bar{\mu} + 2(\bar{k} + 1))^2 \geqslant 8\bar{v}\bar{\mu}.$$

Let the subgraph $\Gamma_1(u)$ be isomorphic to the α-clique extension of a strongly regular Terwilliger graph with parameters $(\bar{v}, \bar{k}, \bar{\lambda}, \bar{\mu})$, say Δ. The cardinality of the vertex set of Δ is $\bar{v} = 1 + \bar{k} + \bar{k}(\bar{k} - \bar{\lambda} - 1)/\bar{\mu}$, hence

$$(\bar{\mu} + 2(\bar{k} + 1))^2 \geqslant 8(\bar{\mu} + \bar{k}\bar{\mu} + \bar{k}(\bar{k} - \bar{\lambda} - 1)),$$

$$\bar{\mu}^2 + 4 \geqslant 4\bar{\mu} + 4\bar{k}\bar{\mu} + 4\bar{k}^2 - 8\bar{k}\bar{\lambda} - 16\bar{k}.$$

Further,

$$(\bar{\mu}/2)^2 + 1 \geqslant \bar{\mu} + \bar{k}\bar{\mu} + \bar{k}^2 - 2\bar{k}\bar{\lambda} - 4\bar{k},$$

$$((\bar{\mu}/2) - (\bar{k} + 1))^2 \geqslant 2\bar{k}(\bar{k} - \bar{\lambda} - 1). \qquad (5)$$

Let us first consider the case $\bar{\mu} = 1$. There are integers s, r such that $\Delta \in \mathcal{F}(s, r)$ and $\bar{k} = rs$, $\bar{\lambda} = s - 1$. If $\bar{k} - \bar{\lambda} - 1 \geqslant \bar{k}/2 + 1$, then $2\bar{k}(\bar{k} - \bar{\lambda} - 1) \geqslant 2\bar{k}(\bar{k}/2 + 1) = \bar{k}^2 + 2\bar{k}$. It follows from (5) that $(\bar{k} + 1/2)^2 \geqslant \bar{k}^2 + 2\bar{k}$ and hence $1/4 \geqslant \bar{k}$, which is impossible. Therefore, $\bar{k} - \bar{\lambda} - 1 < \bar{k}/2 + 1$, i.e., $\bar{k} < 2(\bar{\lambda} + 2)$. Substituting the expressions for $\bar{k}$ and $\bar{\lambda}$ into the previous inequality, we get $rs < 2(s + 1)$. By Lemma 2.1, we have $s + 1 \leqslant r$. Hence, $s + 1 \leqslant r < 2(s + 1)/s$ and it follows that $s = 1$, $r \in \{2, 3\}$ and Δ is the pentagon or the Petersen graph. As we already checked that the three examples in Lemma 3.2 (i) and (ii) satisfy equality in the μ-bound, Theorem 4.2 follows in this case from Lemma 3.2.

Now we may assume $\bar{\mu} > 1$. Since $\bar{\mu} < \bar{k}$, the left-hand side of (5) is at most $\bar{k}^2$. On the other hand, if $\bar{k} - \bar{\lambda} - 1 > \bar{k}/2$, then the right-hand side of (5) is greater than $2\bar{k}\bar{k}/2 = \bar{k}^2$, which is impossible. Hence, we have $\bar{k} - \bar{\lambda} - 1 \leqslant \bar{k}/2$, i.e., $\bar{k} \leqslant 2(\bar{\lambda} + 1)$.

Since $\bar{\mu} > 1$, there is an integer $\alpha_1 \geqslant 1$ such that, for a vertex $w \in \Delta$, the subgraph $\Delta_1(w)$ is the α_1-clique extension of a strongly regular Terwilliger graph, say Σ, with parameters $(v_1, k_1, \lambda_1, \mu_1)$, where $v_1 = \dfrac{\bar{k}}{\alpha_1}$, $k_1 = \dfrac{\bar{\lambda} - (\alpha_1 - 1)}{\alpha_1}$, $\mu_1 = \dfrac{\bar{\mu} - 1}{\alpha_1}$. Then the inequality $\bar{k} \leqslant 2(\bar{\lambda} + 1)$ is equivalent to the inequality $v_1 \leqslant 2(k_1 + 1)$ and the cardinality of the vertex set of Σ is

$$v_1 = 1 + k_1 + k_1\frac{(k_1 - \lambda_1 - 1)}{\mu_1}.$$

Further, $v_1 \leqslant 2(k_1 + 1)$ implies that

$$\frac{k_1(k_1 - \lambda_1 - 1)}{\mu_1} \leqslant k_1 + 1,$$

so

$$k_1 - \lambda_1 - 1 \leqslant \mu_1(1 + 1/k_1) < \mu_1 + 1$$

and

$$k_1 < \lambda_1 + \mu_1 + 2. \tag{6}$$

If $\mu_1 = 1$, then, for certain s_1, r_1, we have $k_1 = r_1 s_1$ and $\lambda_1 = s_1 - 1$. It follows from (6) that $r_1 s_1 < s_1 - 1 + 1 + 2 = s_1 + 2$, $r_1 < 1 + 2/s_1$ and $s_1 = 1$, $r_1 = 2$. Hence, the graph $\Delta_1(w)$ is the α_1-clique extension of the pentagon. By Lemma 3.2, the graph Δ is the icosahedron and the diameter of $\Gamma_1(u)$ is 3, which is impossible because Γ is a Terwilliger graph.

Hence, $\mu_1 > 1$. Let us consider a sequence of strongly regular graphs $\Sigma_1 = \Sigma$, $\Sigma_2, \ldots, \Sigma_h$, $h \geqslant 2$, such that, for an integer $\alpha_{i+1} \geqslant 1$, the local graph of a vertex in Σ_i is the α_{i+1}-clique extension of a strongly regular Terwilliger graph Σ_{i+1} with parameters $(v_{i+1}, k_{i+1}, \lambda_{i+1}, \mu_{i+1})$, $1 \leqslant i < h$ and $\mu(\Sigma_h) = 1$, i.e., $\Sigma_h \in \mathcal{F}(s_h, r_h)$ for certain s_h, r_h. Such a sequence exists by Lemma 3.1.

Assuming $s_h > 1$, we get $k_h - \lambda_h - \mu_h = r_h s_h - (s_h - 1) - 1 = s_h(r_h - 1) > 1$. According to Lemma 3.3, we have $k_i - \lambda_i - \mu_i > 1$ for all $1 \leqslant i \leqslant h - 1$, which contradicts (6). Hence, $s_h = 1$ and Σ_h is a Moore strongly regular graph. By Lemma 3.2, the diameter of Σ_{h-1} is at least 3, and this contradiction completes the proof. ■

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