Advances in Differential Equations Volume 1, Number 1, January 1996, pp. 1 – 20

ON THE KORTEWEG-DE VRIES-KURAMOTO-SIVASHINSKY EQUATION

H.A. Biagioni
Departamento de Matemática, Universidade Estadual de Campinas, Campinas S.P. Brasil

J.L. Bona[1,3]
Department of Mathematics, University of Texas, Austin, Texas 78712

R.J. Iorio, jr.[2]
Instituto de Matemática Pura e Aplicada, IMPA/CNPq, Rio de Janeiro, RJ, Brasil

M. Scialom[3]
Departamento de Matemática, Universidade Estadual de Campinas, Campinas S.P. Brasil

Abstract. Considered herein is the Korteweg-de Vries equation with a Kuramoto-Sivashinsky dissipative term appended. This evolution equation, which arises as a model for a number of interesting physical phenomena, has been extensively investigated in a recent paper of Ercolani, McLaughlin and Roitner. The numerical simulations of the initial-value problem reported in the just-mentioned study showed solutions to possess a more complex range of behavior than the unadorned Korteweg-de Vries equation. The present work contributes some basic analytical facts relevant to the initial-value problem and to some of the conclusions drawn by Ercolani *et al.* In addition to showing the initial-value problem is well posed, we determine the limiting behavior of solutions as the dissipative or the dispersive parameter tends to zero.

1. Introduction. In this article, attention is focussed upon real-valued solutions of the Cauchy problem for the generalization

$$v_t + \tfrac{1}{2}(v_x)^2 + \delta v_{xxx} + \beta(v_{xx} + v_{xxxx}) = 0, \quad v(\cdot,0) = \phi(\cdot), \tag{1.1}$$

of the Kuramoto-Sivashinsky equation and also on solutions of the "derivative equation"

$$w_t + ww_x + \delta w_{xxx} + \beta(w_{xx} + w_{xxxx}) = 0, \quad w(\cdot,0) = \psi(\cdot). \tag{1.2}$$

These partial differential equations combine characteristics of the Korteweg-de Vries equation (KdV-equation henceforth) and the Kuramoto-Sivashinsky equation (KS-equation hereafter), and it is in the combined effect of these traits that we are ultimately interested.

Received December 1994, in revised form April 1995.
[1,3]Centre de Mathématiques et de Leurs Applications, Ecole Normale Supérieure de Cachan, 94235 Cachan Cedex, France.
[2]Instituto Politécnico, Universidade Estadual do Rio de Janeiro, RJ, Brasil.
[3]Department of Mathematics, The Pennsylvania State University, University Park, PA 16802.
AMS Subject Classifications: 35B30, 35B45, 35B65, 76B15, 76D33, 76E30.

The KdV-equation

$$u_t + uu_x + u_{xxx} = 0$$

and the KS-equation

$$u_t + \frac{1}{2}(u_x)^2 + \beta(u_{xx} + u_{xxxx}) = 0$$

have been studied by many scientists. We point especially to the works [1], [2], [5], [6], [8], [9], [10], [11], [13], [15], not because they represent a complete listing, but because these are papers from which we draw ideas in the present work.

The combined KdV-KS equations featured in (1.1) and (1.2) arise in interesting physical situations, for example as a model for long waves on a viscous fluid flowing down an inclined plane (see [17]) and to describe drift waves in a plasma (cf. [4]). Our interest in this model was piqued by the extensive study of Roitner ([14]—see also [5]). These works cover a wide range of issues connected with the initial-value problems (1.1) and (1.2). We point especially to the numerical results showing travelling-wave attractors in the situation where dispersion is dominant ($\delta \gg 1$, or, by rescaling, $\delta > 0$ fixed and $\beta \ll 1$) and the theoretical study of travelling-wave solutions when $\beta \ll 1$. Interest was also focussed on the dynamics when β is held fixed and $\delta \ll 1$. In this latter situation, it was observed that even quite small values of δ served to regularize the chaotic regime that obtains for the KS-equation itself (the case $\delta = 0$).

Our purpose here is two-fold. First, a firm foundation is provided for the initial-value problem for (1.2) posed on the entire real line $\mathbb{R}$. For fixed $\beta > 0$ and $\delta \neq 0$, the initial-value problem (1.2) is shown to have globally defined, unique solutions corresponding to smooth initial data. Moreover, it will be seen that solutions depend continuously on variations of the initial data within reasonable function classes, thus completing the proof that the initial-value problem is globally well posed.

These preliminary results set the stage for an investigation of the limiting behavior of solutions as δ or β tends to zero. It will be shown that both of these limits are nonsingular, and that solutions converge smoothly, and uniformly on compact temporal intervals to the solutions of the initial-value problem for (1.1) with $\delta = 0$ or with $\beta = 0$, respectively. Thus in the presence of even a small amount of the KS-dissipation, the zero-dispersion limit of the KdV-equation or its integrated form ((1.1) with $\beta = 0$) remains globally smooth and continuously dependent upon the initial data. In a similar vein, the travelling-wave attractor that was observed in the numerical simulations carried out in [5] and [14] for $\beta > 0$ would necessarily appear at later and later times as β becomes smaller, and would cease to exist altogether in the limit as β tends to zero.

It deserves remark that while the results presented here pertain to the pure initial-value problem, posed on $\mathbb{R}$ with initial data that decays at least weakly to zero at $\pm\infty$, the theory goes over in every respect to two other interesting problems. If instead of posing initial data ϕ in L_2-based function classes, it is assumed that ϕ is a periodic function of period L, say, then solutions of (1.1) or (1.2) corresponding to ϕ are also spatially periodic of period L. The entire corpus of our development goes over essentially unchanged except that, because of the Poincaré inequality that obtains on a bounded domain, certain estimates are a little easier. In the periodic context, our results considerably improve upon those obtained via standard semigroup methods in [5, Appendix A]. The other context which is generally in range of the analysis to be set forth here is that pertaining to bore-like data, in which the initial state and the resulting solution, both defined on all of $\mathbb{R}$, do not have the same asymptotic state at $+\infty$ as they do at $-\infty$. Solutions that feature a transition from one state to another arise in various contexts. Such solutions can be analyzed using the present techniques coupled with the ideas put forward by Bona, Schonbek and Rajopadhye ([3]) in their study of bore propagation on the surface of water.

The notation to be used is more or less standard. If X is any Banach space, its norm is written $\|\cdot\|_X$. For $1 \leq p \leq \infty$, the usual class of p^{th}-power Lebesgue-integrable (essentially bounded if $p = \infty$), real-valued functions defined on the real line $\mathbb{R}$ is written $L_p = L_p(\mathbb{R})$. The class of L_p-functions whose derivatives up to order s also lie in L_p is connoted W_p^s. Appearing frequently is the value $p = 2$, in which case W_2^s is usually written H^s. One of the standard norms

$$\|f\|_{H^s}^2 = \int_{-\infty}^{\infty} \Big[\sum_{0 \leq j < s} y^{2j} + (y^2)^s \Big] |\widehat{f}(y)|^2 dy$$

on H^s is abbreviated $\|f\|_s$. Here the circumflex surmounting a function is meant to denote that function's Fourier transform. Because it arises very often in our analysis, the L_2-norm, which is also the H^0-norm, is written unadorned as simply

$$\|f\|^2 = \int_{-\infty}^{\infty} |f(x)|^2 dx.$$

If $T > 0$ and X is a Banach space, $C(0,T;X)$ is the class of all continuous maps $u : [0,T] \to X$ equipped with the norm

$$\|u\|_{C(0,T;X)} = \sup_{0 \leq t \leq T} \|u(t)\|_X.$$

The value $T = +\infty$ is allowed, and in this case it is demanded that $u(t)$ be bounded in X, independently of $t \geq 0$. Similarly $C^1(0,T;X)$ is the linear subspace of $C(0,T;X)$ consisting of those functions whose (distributional) derivative with respect to t lies also in $C(0,T;X)$. This space, too, is a Banach space if it is endowed with the norm

$$\|u\|_{C^1(0,T;X)} = \|u\|_{C(0,T;X)} + \|u'\|_{C(0,T;X)}.$$

This scheme generalizes straightforwardly to provide a definition of $C^k(0,T;X)$ for any non-negative integer k. If X and Y are Banach spaces, $B(X,Y)$ connotes the Banach space of bounded linear mappings of X into Y with the usual operator norm.

The paper is organized as follows. In Section 2, results are presented about the linear initial-value problem associated with (1.1) and (1.2), namely

$$v_t + \delta v_{xxx} + \beta(v_{xx} + v_{xxxx}) = 0. \tag{1.3}$$

In Section 3, we establish local well-posedness in H^s of the nonlinear problems (1.1) and (1.2), where $s \geq 1$, $\delta \geq 0$, $\beta > 0$, while global solutions are constructed in Section 5 using the *a priori* estimates derived in Section 4 together with the technical results in the Appendix. Sections 6 and 7 are dedicated to the study of the limiting forms of solutions as δ or β tend separately to zero. These limits are studied in H^1 and H^2, respectively. In taking the limit as β tends to zero, use is made of the theory for the KdV-equation due to Kenig, Ponce and Vega in [11]. In the process of analysing this limit, results of Tadmor ([15]) on the KS-equation are extended.

2. The linear equation. In this section, consideration is given to the Cauchy problem associated with the linear part of (1.1) and (1.2), namely to find a function $v = v(x,t)$ that solves (1.3) with initial value $v(\cdot,0) = \phi(\cdot)$, where $\phi \in H^s$ for suitable values of s. Here, the parameters δ and β are taken to be nonnegative. The solution of this initial-value problem can be obtained explicitly by taking the Fourier transform of the equation with respect to the spatial variable x, solving the resulting ordinary differential equation in the temporal variable, and then recovering v by taking the inverse Fourier transform. For $t \geq 0$ and $\xi \in \mathbb{R}$, let

$$F_{\delta,\beta}(t,\xi) = \exp(t(i\delta\xi^3 - \beta(\xi^4 - \xi^2))) \tag{2.1}$$

and define the semigroup $\{E_{\delta,\beta}(t)\}_{t\geq 0}$ of operators on L_2 by

$$E_{\delta,\beta}(t)f = \mathcal{F}^{-1}(F_{\delta,\beta}(t,\cdot)\widehat{f}(\cdot)) \tag{2.2}$$

where $\mathcal{F}^{-1}$ connotes the inverse Fourier transform.

We begin the analysis with several technical results.

Lemma 2.1. *Let there be given $\lambda \geq 0$, $\beta > 0$ and $t > 0$. Then*

$$\xi^{2\lambda} e^{-2t\beta(\xi^4-\xi^2)} \leq C_\lambda \big[1 + \frac{1}{(t\beta)^{\frac{\lambda}{2}}}\big] e^{\frac{t\beta}{4}(1+\sqrt{1+\frac{4\lambda}{t\beta}})} \tag{2.3}$$

for all $\xi \in \mathbb{R}$, where $C_\lambda \geq 0$ is a constant depending only on λ. Moreover, it follows from (2.3) *that*

$$\int_{-\infty}^{\infty} (1+\xi^2)^\lambda e^{-2t\beta(\xi^4-\xi^2)} d\xi \leq 2^{\frac{3}{2}} 3^\lambda e^{\frac{t\beta}{2}} + 2^{\lambda-1}\Gamma(\frac{2\lambda+1}{4})(t\beta)^{-\frac{2\lambda+1}{4}}, \tag{2.4}$$

where Γ denotes the gamma function.

A proof of this may be found in [15, Lemma 3.1]. Next, the following detailed estimates are established.

Proposition 2.2. *Let λ, β be as in Lemma 2.1. Then, the following are valid.*

(1) $E_{\delta,\beta}(t) \in B(H^s(\mathbb{R}), H^{s+\lambda}(\mathbb{R}))$ *for all $t > 0$ and $s \geq 0$ and satisfies*

$$\|E_{\delta,\beta}(t)\phi\|_{s+\lambda} \leq C_\lambda\big[e^{\frac{t\beta}{4}} + [1 + (t\beta)^{-\frac{\lambda}{4}}]e^{\frac{t\beta}{8}(1+\sqrt{1+\frac{4\lambda}{t\beta}})}\big]\|\phi\|_s \tag{2.5}$$

for all $\phi \in H^s(\mathbb{R})$, where C_λ is a constant depending only on λ. Moreover, the map $t \in (0, \infty) \mapsto E_{\delta,\beta}(t)\phi$ is continuous with respect to the topology of $H^{s+\lambda}(\mathbb{R})$.

(2) $E_{\delta,\beta}(t) \in B(L_1(\mathbb{R}), H^s(\mathbb{R}))$ *for all $t > 0$ and $s \geq 0$ and satisfies*

$$\|E_{\delta,\beta}(t)\eta\|_s \leq C_s[e^{\frac{t\beta}{4}} + (t\beta)^{-\frac{2s+1}{8}}]\|\eta\|_{L_1} \tag{2.6}$$

for all $\eta \in L_1(\mathbb{R})$, where C_s is a constant depending only on s. The map $t \in (0, \infty) \mapsto E_{\delta,\beta}(t)\eta$ is continuous with respect to the topology of $H^s(\mathbb{R})$.

Proof. By the definition of $E_{\delta,\beta}$, we may write

$$\begin{aligned}
\|E_{\delta,\beta}(t)\phi\|^2_{s+\lambda} &= \int_{-\infty}^{\infty}(1+\xi^2)^{s+\lambda}|F_{\delta,\beta}(t,\xi)\hat{\phi}(\xi)|^2 d\xi \\
&= \int_{-\infty}^{\infty}(1+\xi^2)^{\lambda}e^{-2t\beta(\xi^4-\xi^2)}(1+\xi^2)^s|\hat{\phi}(\xi)|^2 d\xi \qquad (2.7)\\
&\leq \sup_\xi[(1+\xi^2)^\lambda e^{-2t\beta(\xi^4-\xi^2)}]\|\phi\|_s^2 \leq C[e^{\frac{t\beta}{2}} + \xi^{2\lambda}e^{-2t\beta(\xi^4-\xi^2)}]\|\phi\|_s^2.
\end{aligned}$$

The inequality (2.5) follows from (2.3) and (2.7). To prove the continuity result, assume, without loss of generality, that $t > \tau$ and apply the dominated convergence theorem to ascertain that

$$\begin{aligned}
\|E_{\delta,\beta}(t)\phi - E_{\delta,\beta}(\tau)\phi\|^2_{s+\lambda} &= \int_{-\infty}^{\infty}(1+\xi^2)^{s+\lambda}\big[e^{-t\beta(\xi^4-\xi^2)} - e^{-\tau\beta(\xi^4-\xi^2)}\big]^2|\hat{\phi}(\xi)|^2 d\xi \\
&= \int_{-\infty}^{\infty}(1+\xi^2)^{s+\lambda}e^{-2t\beta(\xi^4-\xi^2)}\big[1 - e^{(t-\tau)\beta(\xi^4-\xi^2)}\big]^2|\hat{\phi}(\xi)|^2 d\xi
\end{aligned}$$

tends to zero as $t \to \tau$. In view of (2.4), one has

$$\begin{aligned}
\|E_{\delta,\beta}(t)\eta\|^2_s &= \int_{-\infty}^{\infty}(1+\xi^2)^s|F_{\delta,\beta}(t,\xi)\hat{\eta}(\xi)|^2 d\xi \\
&\leq (2\pi)^{-1}\|\eta\|^2_{L_1}\int_{-\infty}^{\infty}(1+\xi^2)^s e^{-2t\beta(\xi^4-\xi^2)}d\xi \\
&\leq (2\pi)^{-1}\|\eta\|^2_{L_1}[2^{\frac{3}{2}}3^s e^{\frac{t\beta}{2}} + 2^{s-1}\Gamma(\frac{2s+1}{4})(t\beta)^{-\frac{2s+1}{4}}],
\end{aligned}$$

from which (2.6) follows. Continuity is then a consequence of an argument similar to the one used in the first part of the proof.

Proposition 2.3. *Let $\beta, \delta \geq 0$ be given. Then for any $s \in \mathbb{R}$, the map $t \in [0,\infty) \mapsto E_{\delta,\beta}(t)$ defines a C^0-semigroup in $H^s(\mathbb{R})$ satisfying*

$$\|E_{\delta,\beta}(t)\| \leq e^{\frac{t\beta}{4}}, \tag{2.8}$$

where the norm is that of $B(H^s, H^s)$. In particular, if $\phi \in H^s(\mathbb{R})$, then $v(\cdot,t) = E_{\delta,\beta}(t)\phi$ is the unique solution of (1.3) in the class

$$C(0,\infty; H^s(\mathbb{R})) \cap C^1(0,\infty; H^{s-4}(\mathbb{R})).$$

Proof. If $\beta = 0$, we obtain the unitary group associated with the KdV equation. In this case $(s-4)$ may be replaced by $(s-3)$ and (2.8) can be replaced by $\|E_{\delta,0}(t)\| = 1$ for all t. For positive β, it is easy to verify that the function $g(\xi) = -\beta(\xi^4 - \xi^2)$ is uniformly bounded above by $\beta/4$ so that (2.8) holds. The semigroup property is obvious and the continuity follows from the definition of $E_{\delta,\beta}(t)$, Parseval's identity and the dominated convergence theorem. The last assertion is an easy consequence of the previous statements. The result is proved.

3. Local theory in $\mathbf{H^s}(\mathbb{R})$, $\mathbf{s \geq 1}$, $\beta > \mathbf{0}$. The nonlinear problems (1.1) and (1.2) are now analysed using the linear results formulated in Section 2. A satisfactory theory of local well-posedness is the outcome.

Theorem 3.1. *Let $\delta \geq 0$, $\beta > 0$ be fixed and suppose $\phi \in H^s(\mathbb{R})$ to be given, where $s \geq 1$. Then there exists $T_s > 0$ depending on s, $\|\phi\|_s$ and β (but independent of δ) and a unique function $v = v_{\delta,\beta} \in C(0, T_s; H^s(\mathbb{R}))$ satisfying the integral equation*

$$v(\cdot,t) = E_{\delta,\beta}(t)\phi(\cdot) - \tfrac{1}{2}\int_0^t E_{\delta,\beta}(t-t')(\partial_x v)^2(\cdot,t')\,dt', \tag{3.1}$$

where $E_{\delta,\beta}(t)$ is defined in (2.2).

Proof. Consider first the range $1 \leq s < \frac{7}{2}$ and let

$$(Af)(t) = E_{\delta,\beta}(t)\phi - \tfrac{1}{2}\int_0^t E_{\delta,\beta}(t-t')(\partial_x f)^2(t')\,dt' \tag{3.2}$$

be defined in the complete metric space

$$\chi_s(T) = \{f \in C(0,T;H^s) : \sup_{0\leq t\leq T} \|f(t) - E_{\delta,\beta}(t)\phi\|_s \leq M\}, \tag{3.3}$$

where $T > 0$ and the topology of $\chi_s(T)$ is that induced by $C(0,T;H^s)$. We will show, by taking $T = T_s$ sufficiently small, that the map (3.2) is a contraction in $\chi_s(T)$. Once this is established, standard uniqueness arguments (cf. [9]) show that this is in fact the only possible solution in $C(0,T;H^s(\mathbb{R}))$.

To the just-stated end, first combine the estimates and continuity results of Proposition 2.2 with the dominated convergence theorem to verify that $Af \in C(0,T;H^s(\mathbb{R}))$ for all $f \in \chi_s(T)$, $s \geq 1$, $T > 0$. Next it is proved that for $T_1 > 0$ small enough, $A(\chi_s(T_1)) \subset \chi_s(T_1)$. If $u \in \chi_s(T)$, it follows that

$$\|u(t)\|_s \leq \|E_{\delta,\beta}(t)\phi\|_s + M \leq e^{\frac{t\beta}{4}}\|\phi\|_s + M,$$

whence (2.6) implies

$$\begin{aligned} \|Au(t) - E_{\delta,\beta}(t)\phi\|_s &\leq \tfrac{1}{2}\int_0^t \|E_{\delta,\beta}(t-t')(\partial_x u)^2(t')\|_s \, dt' \\ &\leq C \sup_{0\leq t\leq T} \|u(t)\|_1^2 \big[\frac{4}{\beta}(e^{\frac{T\beta}{4}} - 1) + \frac{8\beta^{\frac{-2s+1}{8}}}{-2s+7} T^{\frac{-2s+7}{8}}\big] \\ &\leq C'(M^2 + e^{\frac{t\beta}{2}}\|\phi\|_s^2)\big[\frac{4}{\beta}(e^{\frac{T\beta}{4}} - 1) + \frac{8\beta^{\frac{-2s+1}{8}}}{-2s+7} T^{\frac{-2s+7}{8}}\big]. \end{aligned} \tag{3.4}$$

Since the term on the right-hand side of (3.4) inside square brackets, which we temporarily denote by $g(T)$, tends to 0 as $T \to 0$, we can choose $T_1 > 0$ such that the right-hand side of (3.4) is less than M.

Finally, it is shown that there exists $T_2,\ 0 < T_2 \leq T_1$, such that A is a contraction in $\chi_s(T_2)$. In fact, for $0 \leq t \leq T_1$, one has

$$\begin{aligned} \|Au(t) - Av(t)\|_s &\leq \tfrac{1}{2} \sup_{0\leq t\leq T_1} \|(\partial_x u)^2(t) - (\partial_x v)^2(t)\|_{L_1} g(T_1) \\ &\leq \tfrac{1}{2} \sup_{0\leq t\leq T_1} \|u(t) - v(t)\|_1 (\|u(t)\|_1 + \|v(t)\|_1) g(T_1) \\ &\leq \sup_{0\leq t\leq T_1} \|u(t) - v(t)\|_s \ (M + e^{\frac{T_1\beta}{4}}\|\phi\|_s) g(T_1). \end{aligned}$$

Let $T_2 \leq T_1$ be such that $(M + e^{\frac{T_2\beta}{8}}\|\phi\|_s)g(T_2) < 1$. It follows that, if $1 \leq s < \frac{7}{2}$, then the operator A has a unique fixed point in $\chi_s(T_2)$ which satisfies (3.1) and where T_1 and T_2 depend on s, $\|\phi\|_s$ and β. If $s \geq \frac{7}{2}$, an easy bootstrapping argument using the integral equation (3.1) satisfied by $v(t)$ (in H^3 say) implies that $v \in C(0,T_3;H^s)$ where T_3 is the H^3-existence time. For any $s \geq 1$, (2.5) and a further bootstrapping argument implies that for any $\epsilon > 0$, $v \in C(\epsilon, T_3; H^\infty)$ where H^∞ is endowed with its usual Fréchet-space topology. Moreover, $\partial_t v \in C(0,T;H^{s-4}(\mathbb{R}))$ and v satisfies the equation (1.1).

4. A priori estimates. In this section, global *a priori* estimates are obtained that will enable the local solutions in Section 3 to be extended to the entire temporal half-line $[0,\infty)$.

Lemma 4.1. *Consider the initial-value problem* (1.1) *with* $\phi \in H^k$ *for some integer* $k \geq 3$. *Let* v *be a solution of* (1.1) *in* $C(0,T;H^k)$ *for some* $T > 0$. *Then there exists a constant* $c > 0$ *independent of* β *and* δ *such that the following estimates are valid:*

$$\|v\| \leq c[\|\phi\| + Te^{cT\beta}(\|\phi'\|^{\frac{5}{3}} + \beta^{\frac{1}{2}}\|\phi'\|)]e^{cT}, \tag{4.1}$$

$$\|v_x\| \leq \|\phi'\|e^{\frac{T\beta}{4}}, \tag{4.2}$$

$$\|v_{xx}\|^2 \leq \{e^{cT\beta}[P_1(\|\phi\|_2,\delta) + (1+\beta T)P_2(\|\phi\|_2,\delta)] + \frac{\beta T}{2\delta^2}\}e^{cT\beta}, \tag{4.3}$$

$$\|v_{xxx}\|^2 \leq [Q_3 + e^{cT\beta}Q_4 + \beta Te^{cT\beta}Q_1]\exp(\beta e^{cT\beta}Q_2T), \tag{4.4}$$

$$\|v_x\|_j^2 \leq \|\phi'\|_j^2 \exp[(c\|v_x\|_2 + \frac{\beta}{2})T], \tag{4.5}$$

for $0 \leq t \leq T$, *where* $j \geq \max\{3, k-1\}$, *the unadorned norm is that of* L_2 *and* $P_\ell(\cdot,\delta)$, $\ell = 1,2$, *are nondecreasing functions of their first argument. Moreover, for* $\beta > 0$, *it is the case that*

$$\|v_{xx}\| \leq \|\phi''\|\exp[cT(\beta^{-\frac{3}{5}}\|\phi'\|^{\frac{8}{5}}e^{cT\beta} + \beta)], \tag{4.6}$$

$$\|v_{xxx}\| \leq \|\phi'''\|\exp[cT(\beta^{-\frac{3}{5}}\|\phi'\|^{\frac{8}{5}}e^{cT\beta} + \beta)]. \tag{4.7}$$

Before proceeding, it should be noted that some of the inequalities used below are established in the Appendix. Note also that (4.6) and (4.7) will not be used until Section 7.

Proof. We begin by considering the L_2-norm of $w = v_x$. Differentiating (1.1) with respect to x, there appears the initial-value problem (1.2) for w, namely

$$w_t + ww_x + \delta w_{xxx} + \beta(w_{xx} + w_{xxxx}) = 0, \quad w(\cdot,0) = \phi'(\cdot). \tag{4.8}$$

Multiply this equation by w and integrate over $\mathbb{R}$ to obtain

$$\tfrac{1}{2}\frac{d}{dt}\|w\|^2 + \beta(w, w_{xx}) + \beta(w, w_{xxxx}) = 0,$$

where the inner product is that of L_2. Integration by parts and the Cauchy-Schwarz inequality then imply

$$\frac{d}{dt}\|w\|^2 \leq 2\beta\|w\|\|w_{xx}\| - 2\beta\|w_{xx}\|^2 \leq \frac{\beta}{2}\|w\|^2.$$

Integrating the last relation over $[0,t]$, where $0 \leq t \leq T$ and applying Gronwall's lemma gives (4.2).

To prove (4.1), use will be made of (4.2). Multiplying (1.1) by v and integrating over $\mathbb{R}$ leads to

$$\begin{aligned}\tfrac{1}{2}\frac{d}{dt}\|v\|^2 &= -\tfrac{1}{2}(v,(v_x)^2) - \beta(v, v_{xx} + v_{xxxx}) \leq \tfrac{1}{2}\|v\|_{L_\infty}\|v_x\|^2 + \beta\|v_x\|^2 - \beta\|v_{xx}\|^2 \\ &\leq \|v\|^{\frac{1}{2}}\|v_x\|^{\frac{5}{2}} + \beta\|v_x\|^2 \leq (\|v\|^2 + \|v_x\|^{\frac{10}{3}}) + \beta\|v_x\|^2 \\ &\leq \|v\|^2 + (\|\phi'\|^{\frac{10}{3}} + \beta\|\phi'\|^2)e^{cT\beta},\end{aligned}$$

so that (4.1) follows from Gronwall's lemma.

Differentiating (4.8) with respect to x and defining $u = w_x = v_{xx}$, it is seen that u satisfies

$$u_t + (wu_x + u^2) + \delta u_{xxx} + \beta(u_{xx} + u_{xxxx}) = 0. \tag{4.9}$$

Multiplying (4.9) by u and integrating over $\mathbb{R}$, it appears that, for any $\epsilon > 0$,

$$\begin{aligned}
\tfrac{1}{2}\frac{d}{dt}\|u\|^2 &= -(u, wu_x) - (u, u^2) - \beta(u, u_{xx}) - \beta(u, u_{xxxx}) \\
&= (u, wu_x) - \beta(u, u_{xx}) - \beta(u, u_{xxxx}) \\
&\le \|w\|\|u\|\|u_x\|_{L_\infty} + \beta\|u\|\|u_{xx}\| - \beta\|u_{xx}\|^2 \\
&\le \|w\|\|u\|\|u_x\|^{\frac{1}{2}}\|u_{xx}\|^{\frac{1}{2}} + \beta\|u\|\|u_{xx}\| - \beta\|u_{xx}\|^2 \\
&\le \|w\|(\varepsilon\|u\|^2 + \varepsilon^{-\frac{5}{3}}\|u_{xx}\|^2) + \beta\|u\|\|u_{xx}\| - \beta\|u_{xx}\|^2 \\
&\le \varepsilon\|\phi'\|e^{\frac{T\beta}{4}}\|u\|^2 + (\varepsilon^{-\frac{5}{3}}\|\phi'\|e^{\frac{T\beta}{4}} - \beta)\|u_{xx}\|^2 + \beta\|u\|\|u_{xx}\|.
\end{aligned} \tag{4.10}$$

Upon choosing $\varepsilon = (\frac{2\|\phi'\|e^{T\beta/4}}{\beta})^{\frac{3}{5}}$, there obtains from (4.10) the differential inequality

$$\frac{d}{dt}\|u\|^2 \le c\beta^{-\frac{3}{5}}\|\phi'\|^{\frac{8}{5}}e^{cT\beta}\|u\|^2 + \beta\|u\|\|u_{xx}\| - \frac{\beta}{2}\|u_{xx}\|^2 \le (c\beta^{-\frac{3}{5}}\|\phi'\|^{\frac{8}{5}}e^{cT\beta} + \beta)\|u\|^2.$$

Gronwall's lemma is now seen to imply (4.6).

To prove (4.3), use is made of the quantity

$$\Phi_2(w) = \tfrac{1}{2}\int_{-\infty}^{\infty}[\tfrac{1}{3}w^3 - \delta(w_x)^2]\,dx, \tag{4.11}$$

which is a Hamiltonian for the KdV-equation, and is therefore conserved by the KdV-flow. It follows that (4.8) can be written as

$$w_t = -\partial_x(\Phi_2'(w)) - \beta(w_{xx} + w_{xxxx}), \tag{4.12}$$

where

$$\Phi_2'(w) = \tfrac{1}{2}w^2 + \delta w_{xx} \tag{4.13}$$

is the directional (i.e., Gateaux) derivative of Φ_2. Multiplying (4.12) by $\Phi_2'(w)$, integrating over $\mathbb{R}$ and using the fact that $(\Phi_2'(w), -\partial_x\Phi_2'(w)) = 0$, we obtain

$$\partial_t\Phi_2(w) = (\Phi_2'(w), w_t) = -\beta(\Phi_2'(w), w_{xx} + w_{xxxx}). \tag{4.14}$$

Combining formulas (4.11), (4.13) and (4.14) gives

$$\begin{aligned}
&\tfrac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty}(\tfrac{1}{3}w^3 - \delta(w_x)^2)\,dx \\
&\quad = \beta\delta(w_x, w_{xxx}) + \beta\delta\|w_{xxx}\|^2 - \tfrac{1}{2}\beta(w^2, w_{xx}) - \tfrac{1}{2}\beta(w^2, w_{xxxx}).
\end{aligned}$$

Integrating over $[0,t]$, we see that

$$\delta\|w_x\|^2 = \int_{-\infty}^{\infty}[\tfrac{1}{3}w^3 - \tfrac{1}{3}(\phi')^3 + \delta(\phi'')^2]\,dx - 2\beta\delta\int_0^t[(w_x, w_{xxx}) + \|w_{xxx}\|^2]\,d\tau$$
$$+ \beta\int_0^t[(w^2, w_{xx}) + (w^2, w_{xxxx})]\,d\tau. \tag{4.15}$$

Substituting inequalities (8.5)–(8.8) in the Appendix into (4.15) leads to

$$(\delta - c\varepsilon_1)\|w_x\|^2 \le c\varepsilon_1^{-\frac{1}{3}}\|w\|^{\frac{10}{3}} + c\|\phi\|_2^3 + \delta\|\phi\|_2^2 - 2\beta\delta\int_0^t\|w_{xxx}\|^2\,d\tau \tag{4.16}$$
$$+ c\beta\int_0^t[(\delta\varepsilon_2\|w_x\|^2 + \varepsilon_3^{-5}\|w\|^{13} + \varepsilon_4^{-3}\|w\|^6) + \varepsilon_3\|w_{xxx}\| + (\delta\varepsilon_2^{-1} + \varepsilon_4)\|w_{xxx}\|^2]\,d\tau,$$

valid for any positive constants ϵ_1, ϵ_2, ϵ_3 and ϵ_4. Choosing $\varepsilon_1 = \frac{\delta}{2c}$, $\varepsilon_2 = 2c$, $\varepsilon_3 = \frac{1}{c}$, $\varepsilon_4 = \frac{\delta}{2c}$, the inequality (4.16) becomes

$$\frac{\delta}{2}\|w_x\|^2 \le \frac{c}{\delta^{\frac{1}{3}}}\|w\|^{\frac{10}{3}} + c\|\phi\|_2^3 + \delta\|\phi\|_2^2 \tag{4.17}$$
$$+ \beta\int_0^t[-\delta\|w_{xxx}\|^2 + \|w_{xxx}\| + c(\|w\|^{13} + \frac{1}{\delta^3}\|w\|^6 + \delta\|w_x\|^2)]\,d\tau.$$

Upon substituting (4.2) in (4.17), there obtains

$$\|w_x\|^2 \le \frac{c}{\delta^{\frac{4}{3}}}e^{cT\beta}\|\phi'\|^{\frac{10}{3}} + \frac{c}{\delta}\|\phi\|_2^3 + 2\|\phi\|_2^2 + \frac{2\beta T}{\delta}[\frac{1}{4\delta} + c(\|\phi'\|^{13} + \frac{1}{\delta^3}\|\phi'\|^6)e^{cT\beta}]$$
$$+ c\beta\int_0^t\|w_x\|^2\,d\tau. \tag{4.18}$$

It follows that

$$\|w_x\|^2 \le e^{cT\beta}P_1(\|\phi\|_2,\delta) + \widetilde{P}_2(\|\phi\|_2,\delta) + \frac{\beta T}{2\delta^2} + \beta Te^{cT\beta}\widetilde{P}_3(\|\phi\|_2,\delta) + c\beta\int_0^t\|w_x\|^2\,d\tau,$$

where

$$P_1(\|\phi\|_2,\delta) = c\delta^{-\frac{4}{3}}\|\phi'\|^{\frac{10}{3}}, \qquad \widetilde{P}_2(\|\phi\|_2,\delta) = c\delta^{-1}\|\phi\|_2^3 + 2\|\phi\|_2^2,$$
$$\widetilde{P}_3(\|\phi\|_2,\delta) = c\delta^{-1}(\|\phi'\|^{13} + \delta^{-3}\|\phi'\|^6).$$

Another application of Gronwall's lemma gives (4.3) with $P_2 = \widetilde{P}_2 + \widetilde{P}_3$, say.

Let $r = u_x = w_{xx}$. Differentiating (4.9) with respect to x gives

$$r_t + (3ur + wr_x) + \delta r_{xxx} + \beta(r_{xx} + r_{xxxx}) = 0,$$

so that multiplication by r followed by integration over $\mathbb{R}$ implies

$$\begin{aligned}\tfrac{1}{2}\frac{d}{dt}\|r\|^2 &= -3(ur,r) - (w, rr_x) - \beta(r_{xx}, r) - \beta(r_{xxxx}, r)\\ &\le 5\|w\|\|r\|\|r_x\|_{L_\infty} + \beta\|r\|\|r_{xx}\| - \beta\|r_{xx}\|^2,\end{aligned}$$

where the last inequality was obtained via integration by parts and the Cauchy-Schwarz inequality. Proceeding as in (4.10), with u replaced by r, leads directly to (4.7).

To establish (4.4), attention is given to the next conserved quantity for the KdV-equation, namely

$$\Phi_4(w) = \int_{-\infty}^{\infty} (\tfrac{5}{12}w^4 - 5\delta w w_x^2 + 3\delta^2 w_{xx}^2)\,dx. \tag{4.19}$$

Multiplying (4.12) by

$$\Phi_4'(w) = \tfrac{5}{3}w^3 + 5\delta w_x^2 + 10\delta w w_{xx} + 6\delta^2 w_{xxxx} \tag{4.20}$$

and integrating over $\mathbb{R}$, we come to

$$\begin{aligned}\partial_t\Phi_4(w) &= -\beta(\Phi_4'(w), w_{xx} + w_{xxxx}) \\ &= -\beta(\tfrac{5}{3}w^3 + 5\delta w_x^2 + 10\delta w w_{xx} + 6\delta^2 w_{xxxx}, w_{xx} + w_{xxxx}).\end{aligned} \tag{4.21}$$

Now substitute inequalities (8.9)–(8.15) from the Appendix into (4.21) to derive

$$\begin{aligned}\partial_t\Phi_4(w) \le\ & c\beta\big[\|w\|^6 + \varepsilon_1^{-\frac{5}{3}}\|w\|^{\frac{22}{3}} + \delta\varepsilon_2^{-\frac{13}{3}}\|w\|^{\frac{22}{3}} + \delta\varepsilon_3^{-\frac{9}{7}}\|w\|^{\frac{30}{7}} + \delta\varepsilon_4^{-\frac{13}{3}}\|w\|^{\frac{22}{3}}\big]\\ &+ c\beta[1 + \|w\|^2 + \delta^2\varepsilon_5^{-1}]\|w_{xx}\|^2 \\ &+ [c\beta\varepsilon_1 + c\beta\delta\varepsilon_2 + c\beta\delta\varepsilon_3 + c\beta\delta\varepsilon_4 + 6\beta\delta^2\varepsilon_5 - 6\beta\delta^2]\|w_{xxxx}\|^2,\end{aligned} \tag{4.22}$$

valid for any positive constants $\varepsilon_1, \ldots \varepsilon_5$. Taking $\varepsilon_1 = \frac{\delta^2}{c}$, $\varepsilon_2 = \varepsilon_3 = \varepsilon_4 = \frac{\delta}{c}$, $\varepsilon_5 = \frac{1}{6}$, in (4.22) and using (4.2) again, we obtain

$$\partial_t\Phi_4(w) \le \beta e^{cT\beta}Q_1(\|\phi'\|, \delta) + \|w_{xx}\|^2\beta e^{cT\beta}Q_2(\|\phi'\|, \delta). \tag{4.23}$$

Upon integrating (4.23) over $[0, t]$, it follows that

$$\begin{aligned}\int_{-\infty}^{\infty}[\tfrac{5}{12}w^4 - 5\delta w w_x^2 + 3\delta^2 w_{xx}^2]\,dx \le &\int_{-\infty}^{\infty}[\tfrac{5}{12}(\phi')^4 - 5\delta\phi'(\phi'')^2 + 3\delta^2(\phi''')^2]\,dx\\ &+ \beta T e^{cT\beta}Q_1(\|\phi'\|, \delta) + \beta e^{cT\beta}Q_2(\|\phi'\|, \delta)\int_0^t \|w_{xx}\|^2\,d\tau.\end{aligned} \tag{4.24}$$

Substituting (8.16) and (8.17) from the Appendix into (4.24) leads to the inequality

$$\delta^2\|w_{xx}\|^2 \le c\big[\|\phi\|_2^4 + \delta\|\phi\|_3^3 + \delta^2\|\phi\|_3^2 + \varepsilon_1\|w_{xx}\|^2 + \varepsilon_1^{-\frac{1}{3}}\|w\|^{\frac{14}{3}} + \delta(\varepsilon_2\|w_{xx}\|^2$$
$$+ \varepsilon_2^{-\frac{5}{3}}\|w\|^{\frac{14}{3}})\big] + \beta T e^{cT\beta}Q_1(\|\phi'\|,\delta) + \beta e^{cT\beta}Q_2(\|\phi'\|,\delta)\int_0^t \|w_{xx}\|^2\,d\tau,$$

valid for any positive constants ε_1 and ε_2. Choosing $\varepsilon_1 = \frac{\delta^2}{4c}$ and $\varepsilon_2 = \frac{\delta}{4c}$ gives

$$\tfrac{1}{2}\delta^2\|w_{xx}\|^2 \le Q_3(\|\phi\|_3,\delta) + e^{cT\beta}Q_4(\|\phi\|_1,\delta) + \beta T e^{cT\beta}Q_1(\|\phi\|_1,\delta)$$
$$+ \beta e^{cT\beta}Q_2(\|\phi\|_1,\delta)\int_0^t \|w_{xx}\|^2\,d\tau,$$

and consequently Gronwall's lemma implies (4.4).

We turn to the estimation of $\|w\|_j$, for $j > 2$. Integration by parts together with the formulas

$$(w,w_t)_j + (w,ww_x)_j + \delta(w,w_{xxx})_j + \beta(w,w_{xx})_j + \beta(w,w_{xxxx})_j = 0,$$

for $j = 0,1,2,\ldots$, where the inner product is that of H^j,

$$(u,v)_j = \sum_{\ell=0}^{j}(\partial_x^\ell u, \partial_x^\ell v),$$

shows that

$$\tfrac{1}{2}\frac{d}{dt}\|w\|_j^2 \le |(w,ww_x)_j| + \beta\sum_{\ell=0}^{j}[\|\partial_x^\ell w\|\|\partial_x^{\ell+2}w\| - \|\partial_x^{\ell+2}w\|^2]$$
$$\le c\|w\|_2\|w\|_j^2 + \frac{\beta}{4}\sum_{\ell=0}^{j}\|\partial_x^\ell w\|^2 = (c\|w\|_2 + \frac{\beta}{4})\|w\|_j^2.$$

An inequality due to Kato (see (8.4)) has been used to estimate the inner product containing the nonlinear term. Integration over $[0,t]$ and Gronwall's lemma imply the desired result (4.5).

5. Global well posedness in H^s, $s \ge 1$, $\beta > 0$. We will now show that problem (1.1) is globally well posed in $H^s(\mathbb{R})$, for $s \ge 1$. If s is a positive integer, the result follows immediately from the local theory and the *a priori* bounds obtained in the previous section. To handle noninteger values of s, nonlinear interpolation theory is applied ([1], [16]). In what follows we adopt the notation used in [1]: let $k \ge 2$ be an integer, $k-1 < s < k$, $B_0^1 = L_2$, $B_0^2 = C(0,T;L_2)$, $B_1^1 = H^k$, $B_2^1 = C(0,T;H^k)$, $\lambda = \frac{k-1}{k}$, $\theta = \frac{s}{k}$. Then

$$B_{\lambda,2}^1 = [B_0, H^k]_{\lambda,2} \approx H^{k-1}, \quad B_{\theta,2}^1 = [B_0, H^k]_{\theta,2} \approx H^s,$$

where the symbol $\approx$ connotes equality as linear spaces and equivalence of the interpolated norm with the standard norm for the space on the right-hand side.

Theorem 5.1. *Assume that $\beta > 0$. Then problem* (1.1) *is globally well posed in $H^s(\mathbb{R})$, for any $s \geq 1$.*

Proof. Let A be the map which takes the initial data $\phi \in H^k$ into the unique solution $v \in C(0,T;H^k)$ of (1.1) obtained in Theorem 3.1. From (4.1), (4.2), (4.6), (4.7) and (4.5), it follows that

$$\|A\phi\|_k \leq c_1(\|\phi\|_{k-1})\|\phi\|_k$$

for all $\phi \in H^k$, where $c_1 : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous, nondecreasing function.

We now prove the continuity of A and the inequality

$$\|A(\phi) - A(\psi)\|_{C(0,T;L_2)} \leq c_0(\|\phi\|_{k-1} + \|\psi\|_{k-1})\|\phi - \psi\| \tag{5.1}$$

for all $\phi, \psi \in H^{k-1}$. Let $\phi, \psi \in H^{k-1}$, $u = A(\phi)$, $v = A(\psi)$ and $w = u - v$. It follows readily that

$$\begin{aligned}
\tfrac{1}{2}\frac{d}{dt}\|w\|^2 &= -\tfrac{1}{2}((u_x + v_x)w_x, w) - \beta(w, w_{xx} + w_{xxxx}) \\
&\leq \tfrac{1}{2}\|w\|_{L_\infty}(\|u_x\| + \|v_x\|)\|w_x\| + \beta\|w\|\|w_{xx}\| - \beta\|w_{xx}\|^2 \\
&\leq c\|w\|^{\frac{1}{2}}\|w_x\|^{\frac{3}{2}}(\|u_x\| + \|v_x\|) + \beta\|w\|\|w_{xx}\| - \beta\|w_{xx}\|^2 \\
&\leq c\|w\|^{\frac{5}{4}}\|w_{xx}\|^{\frac{3}{4}}(\|u_x\| + \|v_x\|) + \beta\|w\|\|w_{xx}\| - \beta\|w_{xx}\|^2 \\
&\leq c(\varepsilon\|w\|^2 + \varepsilon^{-\frac{5}{3}}\|w_{xx}\|^2)(\|u_x\| + \|v_x\|) + \beta\|w\|\|w_{xx}\| - \beta\|w_{xx}\|^2,
\end{aligned}$$

valid for any $\varepsilon > 0$ on account of Young's inequality. Take $\varepsilon = (\frac{2c(\|u_x\|+\|v_x\|)}{\beta})^{\frac{3}{5}}$ so that the last inequality reads

$$\begin{aligned}
\frac{d}{dt}\|w\|^2 &\leq \|w\|^2 c\beta^{-\frac{3}{5}}(\|u_x\| + \|v_x\|)^{\frac{8}{5}} + 2\beta\|w\|\|w_{xx}\| - \beta\|w_{xx}\|^2 \\
&\leq \|w\|^2(c\beta^{-\frac{3}{5}}(\|u_x\| + \|v_x\|)^{\frac{8}{5}} + \beta).
\end{aligned}$$

Integrating over $[0,t]$, combining Gronwall's lemma with (4.2), and taking the supremum over t in $[0,T]$, we obtain

$$\|w\|^2 \leq \|\phi - \psi\|^2 \exp[T(c\beta^{-\frac{3}{5}}(\|\phi'\|^{\frac{8}{5}} + \|\psi'\|^{\frac{8}{5}}) + \beta)], \tag{5.2}$$

and (5.1) follows.

For $\phi, \psi \in H^k$, the formula (3.1) implies

$$w(\cdot,t) = E_{\delta,\beta}(t)(\phi - \psi)(\cdot) - \tfrac{1}{2}\int_0^t E_{\delta,\beta}(t-t')(u_x^2 - v_x^2)(\cdot,t')dt'.$$

From (2.5) with $\lambda = 0$, $s = k$ and $\lambda = 1$, $s = k-1$, it follows that

$$\begin{aligned}\|w(\cdot,t)\|_k &\le e^{\frac{t\beta}{4}}\|\phi-\psi\|_k + \tfrac{1}{2}\int_0^t \|u_x^2 - v_x^2\|_{k-1}[e^{\frac{(t-t')\beta}{4}} \\ &\qquad + [1+((t-t')\beta)^{-\frac{1}{4}}]e^{\frac{(t-t')\beta}{8}(1+\sqrt{1+\frac{4}{(t-t')\beta}})}dt' \\ &\le e^{\frac{T\beta}{4}}\|\phi-\psi\|_k + \sup_t(\|u\|_k + \|v\|_k) \\ &\qquad \times [\frac{e^{\frac{T\beta}{4}}[(T\beta)^{\frac{1}{4}}+1]e^{\frac{T\beta}{8}+\frac{1}{8}\sqrt{(T\beta)^2+4T\beta}}}{2\beta^{\frac{1}{4}}}]\int_0^t \frac{\|w(\cdot,t')\|_k}{(t-t')^{\frac{1}{4}}}dt'.\end{aligned}$$

Using a generalization of Gronwall's lemma (see [6, Lemma 7.1.1]), there obtains

$$\|w(\cdot,t)\|_k \le e^{\frac{T\beta}{4}}\|\phi-\psi\|_k E_{\frac{3}{4}}(\gamma t), \tag{5.3}$$

with $\gamma = \left[\sup_{0\le t\le T}(\|u(\cdot,t)\|_k + \|v(\cdot,t)\|_k)F(T,\beta)\Gamma(\frac{3}{4})\right]^{\frac{4}{3}}$,

$$F(T,\beta) = \frac{e^{\frac{T\beta}{4}}[(T\beta)^{\frac{1}{4}}+1]e^{\frac{T\beta}{8}+\frac{1}{8}\sqrt{(T\beta)^2+4T\beta}}}{2\beta^{\frac{1}{4}}}$$

and

$$E_\alpha(z) = \sum_{n=0}^{\infty}\frac{z^{\alpha n}}{\Gamma(\alpha n+1)}. \tag{5.4}$$

From (4.2), (4.1), (4.6), (4.7) and (4.5), it is seen that for $0 \le t \le T$,

$$\|u(\cdot,t)\|_k \le \|\phi\|_k P_1(\beta,\delta,T,\|\phi\|_{k-1}), \quad \|v(\cdot,t)\|_k \le \|\psi\|_k P_2(\beta,\delta,T,\|\psi\|_{k-1}),$$

and thus, from (5.3), continuity of A for $k \ge 2$ follows. According to Theorems 1 and 2 of [1], A is therefore a continuous map from H^s into $C(0,T;H^s)$ and the proof is finished.

Remark. The case $\beta = 0$ will be treated in Section 7.

6. Convergence of solutions of KdV-KS to solutions of KS. Attention is now turned to the situation wherein $\beta > 0$ is fixed, but δ tends to zero. The not-surprising outcome of the analysis to follow is that the limiting behavior is simply to provide solutions of the KS equation. Thus, in the presence of the KS-dissipation, the small dispersion limit is not in any way singular. This contrasts strongly with the case $\beta = 0$ for which the small-dispersion limit is singular in a very complex way (cf. [12]).

Proposition 6.1. *Let $\beta > 0$ be fixed. If $v_\delta \in C(0,T;H^s)$ is the solution of* (1.1) *corresponding to a given $\phi \in H^s$ obtained in Theorem* 5.1, *where $s \ge 1$ and $\delta > 0$,* *then the limit* $v_0 = \lim_{\delta\to 0} v_\delta$ *exists in* $C(0,T;H^s)$ *and defines a solution of* (1.1) *with* $\delta = 0$.

Proof. Consider $\phi \in H^s$ and let $v^{(1)}$, $v^{(2)}$ be two solutions of (1.1) corresponding to the values of δ equal to δ_1 and δ_2, respectively. Then $v = v^{(1)} - v^{(2)}$ satisfies the integral equation

$$\begin{aligned} v(\cdot,t) &= (E_{\delta_1,\beta}(t) - E_{\delta_2,\beta}(t))(\phi)(\cdot) \\ &\quad - \tfrac{1}{2}\int_0^t [E_{\delta_1,\beta}(t-t')(v_x^{(1)})^2(\cdot,t') - E_{\delta_2,\beta}(t-t')(v_x^{(2)})^2(\cdot,t')]dt'. \end{aligned} \tag{6.1}$$

Computing the H^s-norm of both sides of (6.1) leads to the inequality

$$\begin{aligned} \|v(\cdot,t)\|_s &\le \|(E_{\delta_1,\beta}(t) - E_{\delta_2,\beta}(t))(\phi)\|_s \\ &\quad + \tfrac{1}{2}\int_0^t [\|(E_{\delta_1,\beta}(t-t') - E_{\delta_2,\beta}(t-t'))(v_x^{(1)})^2\|_s \\ &\quad + \|E_{\delta_2,\beta}(t-t')((v_x^{(1)})^2 - (v_x^{(2)})^2)\|_s]dt'. \end{aligned} \tag{6.2}$$

Each term on the right-hand side of (6.2) will be estimated separately. We begin with a straightforward case:

$$\begin{aligned} \|(E_{\delta_1,\beta}(t) - E_{\delta_2,\beta}(t))(\phi)\|_s^2 &= \int_{-\infty}^{\infty} (1+\xi^2)^s e^{-2t\beta(\xi^4-\xi^2)}(e^{it\delta_1\xi^3} - e^{it\delta_2\xi^3})^2|\hat{\phi}(\xi)|^2\,d\xi \\ &\le |\delta_1-\delta_2|^2 t^2 \int_{-\infty}^{\infty} (1+\xi^2)^s e^{-2t\beta(\xi^4-\xi^2)}\xi^6|\hat{\phi}(\xi)|^2\,d\xi \\ &\le |\delta_1-\delta_2|^2 t^2 \|\phi\|_s^2 \sup_\xi(\xi^6 e^{-2t\beta(\xi^4-\xi^2)}) \\ &\le |\delta_1-\delta_2|^2 t^2\|\phi\|_s^2 C[1 + \frac{1}{(t\beta)^{\frac{3}{2}}}]e^{\frac{t\beta}{4}(1+\sqrt{1+\frac{12}{t\beta}})} \le |\delta_1-\delta_2|^2\|\phi\|_s^2 C(T,\beta). \end{aligned} \tag{6.3}$$

Here we have used the mean value theorem and inequality (2.3) with $\lambda = 3$. The constant $C(T,\beta)$ is given by

$$C(T,\beta) = CT^{\frac{1}{2}}\frac{(T\beta)^{\frac{3}{2}}+1}{\beta^{\frac{3}{2}}}e^{\frac{1}{4}(T\beta+\sqrt{(T\beta)^2+12T\beta})}.$$

Inequality (2.4) then implies that

$$\begin{aligned} &\|(E_{\delta_1,\beta}(t-t') - E_{\delta_2,\beta}(t-t'))(v_x^{(1)})^2(t')\|_s^2 \\ &\quad\le |t-t'|^2|\delta_1-\delta_2|^2\int_{-\infty}^{\infty}(1+\xi^2)^s e^{-2(t-t')\beta(\xi^4-\xi^2)}\xi^6|\widehat{(v_x^{(1)})^2}(\xi)|^2\,d\xi \\ &\quad\le (2\pi)^{-1}|t-t'|^2|\delta_1-\delta_2|^2\|(v_x^{(1)})^2\|_{L_1}^2\int_{-\infty}^{\infty}(1+\xi^2)^{s+3}e^{-2(t-t')\beta(\xi^4-\xi^2)}\,d\xi \qquad (6.4) \\ &\le (2\pi)^{-1}|t-t'|^2|\delta_1-\delta_2|^2\|v_x^{(1)}\|^4[2^{\frac{3}{2}}3^{s+3}e^{\frac{(t-t')\beta}{2}} + 2^{s+2}\Gamma(\frac{2s+7}{4})[\beta(t-t')]^{-\frac{2s+7}{4}}] \\ &\quad\le C|\delta_1-\delta_2|^2\|\phi'\|^4 e^{T\beta}[C(T,\beta) + (t-t')^{-\frac{2s-1}{4}}]. \end{aligned}$$

Finally, the last term in the integral on the right-hand side of (6.2) can be estimated as follows:

$$\begin{aligned}
&\|E_{\delta_2,\beta}(t-t')((v_x^{(1)})^2-(v_x^{(2)})^2)(t')\|_s \\
&\le C_s[e^{\frac{(t-t')\beta}{4}}+((t-t')\beta)^{-\frac{2s+1}{8}}]\|[(v_x^{(1)})^2-(v_x^{(2)})^2](\cdot,t')\|_{L_1} \\
&\le C_s[e^{\frac{(t-t')\beta}{4}}+((t-t')\beta)^{-\frac{2s+1}{8}}]\|(v_x^{(1)}-v_x^{(2)})(\cdot,t')\|\|(v_x^{(1)}+v_x^{(2)})(\cdot,t')\| \\
&\le C(T,\beta)\|\phi'\|(t-t')^{-\frac{2s+1}{8}}\|v(\cdot,t')\|_1, \qquad (6.5)
\end{aligned}$$

where (2.6) and (4.2) have been used to obtain the last inequality. Now substitute (6.3), (6.4) and (6.5) in (6.2) to get, for $1 \le s \le \frac{5}{2}$ and $0 \le t \le T$,

$$\|v(\cdot,t)\|_s \le |\delta_1-\delta_2|C(T,\beta)(\|\phi\|_s+\|\phi\|_1^2)+C(T,\beta)\|\phi\|_1\int_0^t (t-t')^{-\frac{2s+1}{8}}\|v(\cdot,t')\|_s\,dt'. \tag{6.6}$$

Gronwall's lemma then implies that for $0 \le t \le T$,

$$\|v(\cdot,t)\|_1 \le |\delta_1-\delta_2|C(T,\beta,\|\phi\|_s)E_{\frac{7-2s}{8}}(\gamma t), \tag{6.7}$$

where E_α is defined in (5.4) and $\gamma = [C(T,\beta)\|\phi\|_1\Gamma(\frac{7-2s}{8})]^{\frac{8}{7-2s}}$. This shows that

$$\sup_{0\le t\le T}\|v(\cdot,t)\|_s \to 0 \tag{6.8}$$

as $\delta_1,\delta_2 \to 0$. Thus there exists $v_0 = \lim_{\delta\to 0} v_\delta$ in $C(0,T;H^s)$, $s < \frac{5}{2}$. For $s \ge \frac{5}{2}$, we estimate (6.4) and (6.5) using (2.3), and Gronwall's lemma then also implies (6.8).

Next it is shown that v_0 satisfies (1.1) with $\delta = 0$. In fact, if v_δ is the solution of (1.1), then

$$v_\delta(\cdot,t)-v_\delta(\cdot,\tau) = -\int_\tau^t [\tfrac{1}{2}(\partial_x v_\delta)^2+\delta\partial_x^3 v_\delta+\beta(\partial_x^2 v_\delta+\partial_x^4 v_\delta)]\,dt'.$$

This implies that at least in H^{s-4}, v_0 satisfies

$$v_0(\cdot,t)-v_0(\cdot,\tau) = -\int_\tau^t [\tfrac{1}{2}(\partial_x v_0)^2+\beta(\partial_x^2 v_0+\partial_x^4 v_0)]\,dt',$$

and so $v_0 \in AC(0,T;H^{s-4}) \cap L_\infty(0,T;H^s)$ from which it follows that v_0 satisfies equation (1.1) with $\delta = 0$ for almost every t. But the local existence result for $\delta = 0$ implies that (1.1) has a unique solution in $C(0,T;H^s)$ corresponding to initial data in H^s. Therefore v_0 coincides with the strong solution of (1.1) with $\delta = 0$ (the KS-equation) and the result is proved.

7. Convergence of solutions of the KdV-KS equation to solutions of KdV-initial-value problem. The next task is to prove that solutions of the initial-value problem for the integrated version (equation (1.1) with $\beta = 0$) of the KdV-equation are obtained as the limit as β tends to zero of the solutions constructed in Theorem 5.1. Here is the result in view.

Theorem 7.1. *Let $\delta > 0$ and $\phi \in H^2$ be given, and let v_β be the solution of* (1.1) *satisfying $v(\cdot, 0) = \phi$. Then the limit $v_0 = \lim_{\beta\to 0} v_\beta$ exists in $C(0, T; H^2)$ and is the unique solution of* (1.1) *with $\delta = 0$. Moreover, the map $\phi \in H^2 \mapsto v_0 \in C(0, T; H^2)$ is continuous with respect to the topologies under consideration.*

Proof. Let $v^{(j)} = v_{\beta j}$, $j = 1, 2$, be solutions of (1.1) with the same initial condition $\phi \in H^3$ but corresponding to different values of the parameter β. Then $v = v^{(1)} - v^{(2)}$ satisfies

$$v_t + \tfrac{1}{2} v_x(v_x^{(1)} + v_x^{(2)}) + \delta v_{xxx} + \beta_1(v_{xx} + v_{xxxx}) + (\beta_1 - \beta_2)(v_{xx}^{(2)} + v_{xxxx}^{(2)}) = 0.$$

Multiplication by v followed by integration over $\mathbb{R}$ gives

$$\begin{aligned}\tfrac{1}{2}\frac{d}{dt}\|v\|^2 &= \tfrac{1}{4}(v^2, v_{xx}^{(1)} + v_{xx}^{(2)}) - \beta_1(v, v_{xx} + v_{xxxx}) - (\beta_1 - \beta_2)(v, v_{xx}^{(2)} + v_{xxxx}^{(2)}) \\ &\le \tfrac{1}{4}\|v_{xx}^{(1)} + v_{xx}^{(2)}\|_{L_\infty}\|v\|^2 + \frac{\beta_1}{4}\|v\|^2 + |\beta_1 - \beta_2|(\|v\|\|v_{xx}^{(2)}\| + \|v_{xx}\|\|v_{xx}^{(2)}\|) \\ &\le c[\|v^{(1)}\|_3 + \|v^{(2)}\|_3 + \beta_1]\|v\|^2 + c|\beta_1 - \beta_2|(\|v^{(1)}\|_2 + \|v^{(2)}\|_2)\|v^{(2)}\|_2.\end{aligned}$$

Integrating over $[0, t]$ and applying Gronwall's lemma once again, we obtain

$$\begin{aligned}\|v(\cdot, t)\|^2 \le cT|\beta_1 - \beta_2| &\sup_t[(\|v^{(1)}\|_2 + \|v^{(2)}\|_2)\|v^{(2)}\|_2] \\ &\times \exp(c\,T \sup_t(\|v^{(1)}\|_3 + \|v^{(2)}\|_3 + \beta_1))\end{aligned} \tag{7.1}$$

for $0 \le t \le T$. From (4.1), (4.2), (4.3) and (4.4), the H^3-norms of $v^{(1)}$ and $v^{(2)}$ are bounded by a function of T and $\|\phi\|_3$, independently of $\beta < 1$, say. If $\beta_1, \beta_2 \to 0$, the right-hand side of (7.1) converges to 0 so that the limit $v_0 = \lim_{\beta\to 0} v_\beta$ exists at least in L_2, uniformly with respect to $t \in [0, T]$.

Arguing as in Proposition 6.1, it is concluded that v_0 satisfies (1.1) with $\beta = 0$ for almost every t. Standard arguments then show that there is at most one solution of (1.1) with $\beta = 0$. Since H^3 is continuously and densely embedded in L_2, it follows that the map $t \in [0, T] \mapsto v_0(\cdot, t) \in H^3$ is weakly continuous. It is not difficult to verify that $\|\phi\|_3 = \liminf_{t\to 0+}\|v_0(\cdot, t)\|_3$, so that the map under consideration is continuous at $t = 0$ with respect to the H^3-topology. Right continuity at $t \in (0, T)$ is a consequence of the continuity at $t = 0$ and the uniqueness of solutions of the initial-value problem. Left continuity follows from the change of variables $(t, x) \mapsto (\tau - t, -x)$ and the fact that all of our previous results remain valid if we change the sign of the nonlinearity in (1.1).

We now turn to the case where the initial data is only in H^2. In fact, consider a sequence $\{\phi_n\}_{n=1}^\infty$ in H^3 converging to ϕ in H^2. For $n = 1, 2, \ldots$, let r_n be the solution of (1.1) with $\beta = 0$ and initial data ϕ_n, obtained as above. We know that $\{r_n\}_{n=1}^\infty$ is a Cauchy sequence in L_2. Indeed,

$$\frac{d}{dt}\|r_n - r_m\| \le \tfrac{1}{2}\|(r_n - r_m)_x\|(\|r_n\|_2 + \|r_m\|_2). \tag{7.2}$$

Now, $q_n = \partial_x r_n$ satisfies (1.2) with $\beta = 0$ and initial data $\phi_n' \in H^1$. In view of the well-posedness result of Kenig, Ponce and Vega ([11]), $q_n \to q$ in $C(0,T;H^1)$, where q is the solution of (1.2) with $\beta = 0$ and $q(\cdot,0) = \phi'$. Since each r_n is a limit of solutions of (1.1) as $\beta \to 0$, its H^2-norm is estimated by a function of $\|\phi_n\|_2$ and T (see (4.2) and (4.3)), which is bounded independently of n, since $\phi_n \to \phi$ in H^2. Thus there exists a constant $C = C(T, \|\phi\|_2) \geq 0$ such that for $0 \leq t \leq T$,

$$\frac{d}{dt}\|r_n - r_m\| \leq C(T, \|\phi\|_2)\|q_n - q_m\|. \tag{7.3}$$

Hence given $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that

$$\frac{d}{dt}\|r_n - r_m\| \leq \varepsilon$$

for $m, n \geq N$, and therefore for $0 \leq t \leq T$, $\|r_n - r_m\| \leq \varepsilon T + \|\phi_n - \phi_m\|$, which proves that there exists $r(\cdot,t) = \lim_{n\to\infty} r_n(\cdot,t)$ in L_2. This, together with the existence of $\lim_{n\to\infty}\partial_x r_n$ in H^1, implies that $r \in H^2$ and $r_n \to r$ in H^2. It is then easy to verify that r is indeed a solution of (1.1) with $\beta = 0$. The uniqueness of the solution follows from the corresponding property for the KdV-equation.

Finally, it is established that the solution depends continuously on the initial data. Take $\phi, \psi \in H^2$ and let $r, s \in C(0,T;H^2)$ be the solutions of (1.1) with $\beta = 0$ satisfying $r(\cdot,0) = \phi$ and $s(\cdot,0) = \psi$, respectively. Using (7.2) with r_m and r_n denoted simply r and s, we have

$$\frac{d}{dt}\|r - s\| \leq \tfrac{1}{2}\|r_x - s_x\|(\|r\|_2 + \|s\|_2) \leq \tfrac{1}{2}\|r_x - s_x\|(f(\|\phi\|_2, T) + g(\|\psi\|_2, T)).$$

Integration of this differential inequality over $[0,T]$ leads to

$$\|r - s\| \leq \|\phi - \psi\| + \tfrac{1}{2}[f(\|\phi\|_2, T) + g(\|\psi\|_2, T)] \int_0^t \|r_x - s_x\|\, dt'$$

Using the continuous dependence proven in [11] we therefore obtain

$$\|r - s\| \leq \|\phi - \psi\|_1 F(\|\phi\|_2, \|\psi\|_2, T).$$

The continuity of the first and second derivatives is of course contained in the results of [9]. This completes the proof of Theorem 7.1.

Remark. It is not difficult to verify that Theorem 7.1 holds, in fact, for every $s \geq 2$.

8. Appendix. Several technicalities that arose in the body of the paper were deferred so as not to interrupt the general development. These points are established in the present section.

We begin by recalling some standard inequalities that were used earlier in the text, and which will figure again in the Appendix. Here, $D = d/dx$, say.

- Gagliardo-Nirenberg inequalities:

$$\|u\|_{L_\infty} \le \|u\|^{\frac{1}{2}} \|Du\|^{\frac{1}{2}} \quad \text{for } u \in H^s(\mathbb{R}), \quad s \ge 1; \tag{8.1}$$
$$\|D^j u\| \le M \|u\|^{1-\frac{j}{m}} \|D^m u\|^{\frac{j}{m}}, \quad 0 \le j \le m, \quad u \in H^m(\mathbb{R}); \tag{8.2}$$

- Young's inequality: $\forall a, b \ge 0$ $\forall \varepsilon > 0$, $1 < p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$,

$$ab \le \varepsilon a^p + C_\varepsilon b^{p'}, \quad \text{where } C_\varepsilon = \varepsilon^{-\frac{1}{p-1}}; \tag{8.3}$$

- Kato's inequality ([8, Lemma (A.5)]): if $k > 2$ and $u \in H^k(\mathbb{R})$,

$$|(u, uDu)_k| \le C\|u\|_2 \|u\|_k^2, \tag{8.4}$$

  where $(\cdot,\cdot)_k$ denotes the scalar product in $H^k(\mathbb{R})$.

The next four inequalities are technical estimates used to prove (4.15):

$$|\int_{-\infty}^{\infty} w^3\,dx| \le \|w\|_{L_\infty}\|w\|^2 \le \|w\|^{\frac{5}{2}}\|w_x\|^{\frac{1}{2}} \le \varepsilon_1\|w_x\|^2 + \varepsilon_1^{-\frac{1}{3}}\|w\|^{\frac{10}{3}}, \tag{8.5}$$
$$|(w_x, w_{xxx})| \le \|w_x\|\|w_{xxx}\| \le \varepsilon_2\|w_x\|^2 + \varepsilon_2^{-1}\|w_{xxx}\|^2, \tag{8.6}$$
$$\begin{aligned}|(w^2, w_{xx})| &\le \|w\|_{L_\infty}\|w\|\|w_{xx}\| \le \|w\|^{\frac{3}{2}}\|w_x\|^{\frac{1}{2}}\|w_{xx}\| \le c\|w\|^{\frac{13}{6}}\|w_{xxx}\|^{\frac{5}{6}}\\ &\le c(\varepsilon_3\|w_{xxx}\| + \varepsilon_3^{-5}\|w\|^{13}),\end{aligned} \tag{8.7}$$
$$\begin{aligned}|(w^2, w_{xxxx})| &= 2|(ww_x, w_{xxx})| \le 2\|w\|_{L_\infty}\|w_x\|\|w_{xxx}\| \le 2\|w\|^{\frac{1}{2}}\|w_x\|^{\frac{3}{2}}\|w_{xxx}\|\\ &\le c\|w\|^{\frac{3}{2}}\|w_{xxx}\|^{\frac{3}{2}} \le c(\varepsilon_4\|w_{xxx}\|^2 + \varepsilon_4^{-3}\|w\|^6),\end{aligned} \tag{8.8}$$

where the ε_i, $i = 1, \dots, 4$, are arbitrary positive constants, chosen suitably in Section 4.

The inequalities (8.9)–(8.15) are estimates of the right-hand side of (4.21) and are used to obtain (4.22):

$$\begin{aligned}|(w^3, w_{xx})| &\le \|w\|_{L_\infty}^2\|w\|\|w_{xx}\| \le \|w\|^2\|w_x\|\|w_{xx}\|\\ &\le \|w\|^{5/2}\|w_{xx}\|^{3/2} \le c(\|w\|^{10} + \|w_{xx}\|^2),\end{aligned} \tag{8.9}$$
$$\begin{aligned}|(w^3, w_{xxxx})| &\le \|w\|_{L_\infty}^2\|w\|\|w_{xxxx}\| \le \|w\|^2\|w_x\|\|w_{xxxx}\|\\ &\le c\|w\|^{\frac{11}{4}}\|w_{xxxx}\|^{\frac{5}{4}} \le c(\varepsilon_1\|w_{xxxx}\|^2 + \varepsilon_1^{-\frac{5}{3}}\|w\|^{\frac{22}{3}}),\end{aligned} \tag{8.10}$$
$$(w_x^2, w_{xx}) = 0, \tag{8.11}$$
$$\begin{aligned}|(w_x^2, w_{xxxx})| &\le \|w_x\|_{L_\infty}\|w_x\|\|w_{xxxx}\| \le \|w_x\|^{\frac{3}{2}}\|w_{xx}\|^{\frac{1}{2}}\|w_{xxxx}\|\\ &\le c\|w\|^{\frac{11}{8}}\|w_{xxxx}\|^{\frac{13}{8}} \le c(\varepsilon_2\|w_{xxxx}\|^2 + \varepsilon_2^{-\frac{13}{3}}\|w\|^{\frac{22}{3}}),\end{aligned} \tag{8.12}$$
$$\begin{aligned}|(ww_{xx}, w_{xx})| &\le \|w_{xx}\|_{L_\infty}\|w\|\|w_{xx}\| \le \|w\|\|w_{xx}\|^{\frac{3}{2}}\|w_{xxx}\|^{\frac{1}{2}}\\ &\le c\|w\|^{\frac{15}{8}}\|w_{xxxx}\|^{\frac{9}{8}} \le c(\varepsilon_3\|w_{xxxx}\|^2 + \varepsilon_3^{-\frac{9}{7}}\|w\|^{\frac{30}{7}}),\end{aligned} \tag{8.13}$$
$$\begin{aligned}|(ww_{xx}, w_{xxxx})| &\le \|w\|_{L_\infty}\|w_{xx}\|\|w_{xxxx}\| \le \|w\|^{\frac{1}{2}}\|w_x\|^{\frac{1}{2}}\|w_{xx}\|\|w_{xxxx}\|\\ &\le c\|w\|^{\frac{11}{8}}\|w_{xxxx}\|^{\frac{13}{8}} \le c(\varepsilon_4\|w_{xxxx}\|^2 + \varepsilon_4^{-\frac{13}{3}}\|w\|^{\frac{22}{3}}),\end{aligned} \tag{8.14}$$
$$|(w_{xxxx}, w_{xx})| \le \|w_{xx}\|\|w_{xxxx}\| \le \varepsilon_5\|w_{xxxx}\|^2 + \varepsilon_5^{-1}\|w_{xx}\|^2. \tag{8.15}$$

Finally the first and second terms on the left-hand side of (4.24) can be bounded as follows:

$$\int_{-\infty}^{\infty} w^4\,dx \le \|w\|_{L_\infty}^2\|w\|^2 \le \|w\|^3\|w_x\| \le c\|w\|^{\frac{7}{2}}\|w_{xx}\|^{\frac{1}{2}}$$

$$\le c(\varepsilon_1\|w_{xx}\|^2 + \varepsilon_1^{-\frac{1}{3}}\|w\|^{\frac{14}{3}}), \tag{8.16}$$

$$|\int_{-\infty}^{\infty} ww_x^2\,dx| \le \|w_x\|_{L_\infty}\|w\|\|w_x\| \le \|w\|\|w_x\|^{\frac{3}{2}}\|w_{xx}\|^{\frac{1}{2}}$$

$$\le c\|w\|^{\frac{7}{4}}\|w_{xx}\|^{\frac{5}{4}} \le c(\varepsilon_2\|w_{xx}\|^2 + \varepsilon_2^{-\frac{5}{3}}\|w\|^{\frac{14}{3}}). \tag{8.17}$$

Acknowledgments. This research was partially supported by the Keck Foundation and the National Science Foundation, USA, and CNPq and FAPESP, Brasil. RJI thanks the Mathematics Departments at Penn State and Unicamp for hospitality and financial support during various stages of the collaboration. HAB, JLB and MS are all grateful for the hospitality accorded them at IMPA during the initial stages of the work. MS also thanks Penn State for hospitality and support.

REFERENCES

[1] J.L. Bona and R. Scott, *The initial-value problem for the Korteweg-de Vries equation in fractional-order Sobolev spaces,* Duke Math. J., 43 (1976), 87–99.

[2] J.L. Bona and R. Smith, *The initial-value problem for the Korteweg-de Vries equation,* Philos. Trans. Royal Soc. London, A 278 (1975), 555–604.

[3] J.L. Bona, M.E. Schonbek and S. Rajopadhye, *Models for propagation of bores I. Two-dimensional theory,* Differential and Integral Equations, 7 (1994), 699–734.

[4] B.I. Cohen, J.A. Krommes, W.M. Tang, and M.N. Rosenbluth, *Non-linear saturation of the dissipative trapped-ion mode by mode coupling,* Nuclear Fusion, 16 (1976), 971–992.

[5] N.M. Ercolani, D.W. McLaughlin, and H. Roitner, *Attractors and transients for a perturbed periodic KdV equation: a nonlinear spectral analysis,* J. of Nonlinear Science, to appear.

[6] D. Henry, "Geometric Theory of Semilinear Parabolic Equations," Lecture Notes in Mathematics 840, Springer, Berlin, 1981.

[7] T. Kato, *On the Korteweg–de Vries equation,* Manuscripta Math., 28 (1979), 89–99.

[8] T. Kato, *On the Cauchy problem for the* (*generalized*), *Korteweg-de Vries equations,* Studies in Appl. Math., Advances in Math. Suppl. Studies, 8 (1983), 93–128.

[9] T. Kato and H. Fujita, *On the non-stationary Navier-Stokes system,* Rend. Sem. Mat. Univ. Padova, 32 (1962), 243–260.

[10] C. Kenig, G. Ponce, and L. Vega, *On the* (*generalized*), *Korteweg-de Vries equation,* Duke Math. J., 59 (1989), 585–610.

[11] C. Kenig, G. Ponce, and L. Vega, *Well-posedness of the initial value problem for the Korteweg-de Vries equation,* J. Amer. Math. Soc., 4 (1991), 323–347.

[12] P.D. Lax and C.D. Levermore, *The small dispersion limit of the Korteweg-de Vries equation, I, II, III,* Comm. Pure Appl. Math., 36 (1983), 253–290 (I), 571–593 (II), 809–829 (III).

[13] B. Nicolaenko and B. Scheurer, *Remarks on the Kuramoto-Sivashinsky equation,* Physica D, 12D (1984), 391–395.

[14] H. Roitner, *Applications of the inverse spectral transform to a Korteweg–de Vries equation with a Kuramoto-Sivashinsky-type perturbation,* Ph.D. Thesis, University of Arizona (1991).

[15] E. Tadmor, *The well-posedness of the Kuramoto-Sivashinsky equation,* SIAM J. Math. Anal., 17 (1986), 884–893.

[16] L. Tartar, *Interpolation non linéaire et régularité,* J. Functional Analysis, 9 (1972), 469–489.

[17] J. Topper and T. Kawahara, *Approximate equations for long nonlinear waves on a viscous fluid,* J. Phys. Soc. Japan, 44 (1978), 663–666.