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ON THE KRIEGER-ARAKI-WOODS RATIO SET

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Abstract. We show how to calculate the ratio sets of G-measures as limit points of infinite products of the associated g-functions. In particular, we show that every g-measure is of type III_1.

1. Introduction. By Dye's celebrated theorem, every ergodic system of type II or type III is orbit equivalent to one of the form (X, Γ, μ), where X is the infinite product of two-point spaces, Γ the (countable) group of finite coordinate changes in X, and μ some measure on X which is quasi-invariant and ergodic with respect to the action of Γ. The Krieger-Araki-Woods ratio set, discussed in [9], is an invariant for orbit equivalence, allowing classification into systems of types II_1, II_∞, III_1, III_λ $(0<\lambda<1)$, and III_0. We will discuss here only probability measures.

In a recent paper [1], two of the authors introduced the G-measure formalism, showing that all ergodic measures may be regarded as a generalization of the g-measures of M. Keane, that is, there are functions g_k on X such that

$$\frac{d\mu}{d\mu^{(n)}}(x)=g_1(x)g_2(x)\cdots g_n(x)=G_n(x)\,.$$

Here, $\mu^{(n)}$ denotes the measure μ averaged over the first n coordinates, and the function g_i depends on the coordinates $(x_i, x_{i+1}, \cdots)$ and satisfies

$$\frac{1}{2}(g_i(0, x_{i+1}, x_{i+2}, \cdots)+g_i(1, x_{i+1}, x_{i+2}, \cdots))=1 \quad \text{for every} \quad x\in X\,.$$

In this paper, we shall seek to characterise the ratio set of μ in terms of the limit points of infinite products of the form $\prod_{i=n}^{\infty} g_i(u)/g_i(v)$, $u, v\in X$. Our major result, Theorem 4.4, gives a necessary and a different sufficient condition which are nevertheless rather close to each other, for a number r to belong to the ratio set. In Section 5, this theorem is applied to show that provided the image of g contains an interval, every g-measure is of type III_1. Hence by a theorem of Connes-Krieger [3], [5], they are all orbit equivalent. In the last section, we apply our results to infinite product measures.

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2. Preliminaries.

(2.1) NOTATION. For each integer $i \in \mathbf{N}$, suppose we are given a finite space X_i which we may identify with $\mathbf{Z}_{l(i)}$, the integers modulo $l(i)$, for some positive integer $l(i)$. We shall denote by X the infinite product $\prod_{i=1}^{\infty} X_i$, and by Γ the group of finite coordinate changes

$$\{\gamma \in X : \exists N \text{ with } \gamma_n = 0 \text{ for } n > N\}.$$

Here, and in the sequel, we are thinking of elements in X as sequences $x = (x_1, x_2, x_3, \cdots)$.

An element $\gamma \in \Gamma$ acts on $x \in X$ by the rule $(\gamma x)_n = \gamma_n + x_n \pmod{l(n)}$. We will say that two elements $u, v \in X$ are eventually equal if there exists $\gamma \in \Gamma$ with $\gamma u = v$. This is equivalent to demanding that there exists $n \in \mathbf{N}$ with $u_n = v_n$ for all $n \geq N$. For $n \in \mathbf{N}$, let $X^n = \prod_{i=n+1}^{\infty} X_i$; it will be convenient also to identify X^n with $\{x \in X : x_1 = x_2 = \cdots = x_n = 0\}$. Thus we define $\Gamma^n = \Gamma \cap X^n$, and $\Gamma_n = \{\gamma \in \Gamma : \gamma_k = 0 \text{ for } k > n\}$. Notice that one may write X as a disjoint union

$$X = \bigcup_{\gamma \in \Gamma_n} \gamma X^n .$$

As usual, we shall assume that X is equipped with its Borel σ-algebra $\mathscr{C}$ derived from the product topology. Let μ be a measure on X, and suppose that μ is quasi-invariant for the action of Γ, i.e. we define $\mu \circ \gamma(E) = \mu(\gamma^{-1} E)$ for E a Borel subset of X, $\gamma \in \Gamma$, and assume that $\mu \circ \gamma \sim \mu$ for all $\gamma \in \Gamma$.

Then $\mu^{(n)} \sim \mu$, where $\mu^{(n)} = (1/|\Gamma_n|) \sum_{\gamma \in \Gamma_n} \mu \circ \gamma$. We shall always assume that μ is a probability measure, in which case $\mu^{(n)}$ is also a probability measure. Notice that for all $\gamma \in \Gamma_n$, we have $\mu^{(n)}(\gamma X^n) = \mu^{(n)}(X^n) = 1/|\Gamma_n|$. Recall that a quasi-invariant probability measure is said to be *ergodic* if for every Γ-invariant Borel set A, either $\mu(A) = 0$ or $\mu(A) = 1$.

(2.2) DEFINITION. We recall from [9, §2] the definition of the Krieger-Araki-Woods *ratio set*. Let μ be a measure on X, and let $r \in [0, \infty]$. We shall say that $r \in r(X, \Gamma, \mu)$ if for all $\varepsilon > 0$ and for every set A of positive μ-measure, there exists a set $B \subset A$ of positive μ-measure and there exists $\gamma \in [\Gamma]$ such that

$$\gamma B \subseteq A \qquad \text{and} \qquad \left| \frac{d\mu \circ \gamma}{d\mu}(x) - r \right| < \varepsilon$$

for almost every $x \in B$.

In this definition, the full group $[\Gamma]$ of Γ consists of all those automorphisms

$S: X \to X$ such that for all $x \in X$ there is $\gamma = \gamma(x) \in \Gamma$ with $Sx = \gamma x$. A moment's reflection shows that in the case of our group Γ, the action of the full group may be replaced by the action of Γ itself. Thus, we may state:

(2.3) LEMMA. (i) *Let* $r \in [0, \infty[$. *Then* $r \in r(X, \Gamma, \mu)$ *if and only if for every* $\varepsilon > 0$ *and for every set* A *with* $\mu(A) > 0$, *there exists* $\gamma \in \Gamma$ *such that*

$$\mu\left\{x : \gamma x \in A \text{ and } \left|\frac{d\mu \circ \gamma}{d\mu}(x) - r\right| < \varepsilon\right\} > 0 .$$

(ii) $\infty \in r(X, \Gamma, \mu)$ *if and only if for every* $M > 0$ *and for every set* A *with* $\mu(A) > 0$ *there exists* $\gamma \in \Gamma$ *such that*

$$\mu\left\{x : \gamma x \in A \text{ and } \frac{d\mu \circ \gamma}{d\mu}(x) > M\right\} > 0 .$$

(2.4) We recall from [1] that a probability measure on X is a *G-measure*, where $G = (G_n)_{n=1}^{\infty}$ is a family of non-negative Borel functions on X which are

(i) *normalized* in the sense that

$$(1/|\Gamma_n|) \sum_{\gamma \in \Gamma_n} G_n(\gamma x) = 1 \qquad \text{for all } x \in X, \text{ and}$$

(ii) *compatible* in the sense that

$$G_n(\gamma x) G_m(x) = G_m(\gamma x) G_n(x) , \qquad \text{where } n > m,\ \gamma \in \Gamma_m \text{ and } x \in X .$$

The G-measure condition is to require that for all n

$$\frac{d\mu}{d\mu^{(n)}}(x) = G_n(x) \qquad \text{for all } x \in X .$$

It is shown in [1, Proposition 1] that, after passage to an equivalent measure, the G_n may actually be assumed continuous on X.

An equivalent formulation, somewhat preferable from the point of view of the present work, involves functions

$$g_n(x) = \begin{cases} G_n(x)/G_{n-1}(x) & \text{if } G_{n-1}(x) \neq 0 \\ 0 & \text{if } G_{n-1}(x) = 0 . \end{cases}$$

These satisfy two relations:

(i) $g_n(x)$ depends only on the coordinates $(x_n, x_{n+1}, \cdots)$, and

(ii) for each n, $(1/l(n)) \sum_{\gamma \in \boldsymbol{Z}(l(n))} g_n(\gamma x) = 1$ for all $x \in X$.

One has $G_n(x) = g_1(x) g_2(x) \cdots g_n(x)$.

Let $\boldsymbol{T}$ be the unit circle, and consider the map $q_n : X^n \to \boldsymbol{T}$ defined by

$$q_n(x) = \sum_{j=n+1}^{\infty} \frac{x_j}{l(n+1) \cdots l(j)} .$$

The function g_n on X^n is said to be *q_n-continuous* if there is a continuous function g_n' on $\boldsymbol{Z}_{l(n)} \times \boldsymbol{T}$ such that $g_n(x) = g_n'(x_n, q_n(x))$.

The family G is said to be *q-continuous* if g_n is q_n-continuous for all $n \in N$.

A measure μ on X is said to be *circle adapted* if $q_0(x) = q_0(y)$ implies $\mu(\{x\}) = \mu(\{y\})$ for all $x, y \in X$. Notice that every measure on X differs from a circle adapted one by a discrete measure of countable support, hence every continuous measure is circle adapted.

Proposition 1 of [1] asserts that every circle adapted measure is equivalent to a G-measure, the family G being q-continuous.

Henceforth, we shall assume that G is a normalized compatible q-continuous family, and that μ is a G-measure. We shall seek to describe the ratio set of μ in terms of the functions g_k and g_k'. As a preliminary to this, let us note the following obvious fact:

(2.5) LEMMA. *Let μ be a quasi-invariant G-measure, with G as above. Then for each k, we have $g_k(x) > 0$ for μ a.e. x, and for $\gamma \in \Gamma$,*

$$\frac{d\mu \circ \gamma}{d\mu}(x) = \prod_{k=1}^{\infty} \frac{g_k(\gamma x)}{g_k(x)} .$$

In this infinite product, only finitely many terms are different from 1, for if $\gamma \in \Gamma_n$, then by the property (i) of the functions g_k, we have $g_k(\gamma x) = g_k(x)$ for $k > n$.

3. The basic theorems. We give two theorems, a necessary and a (different) sufficient condition for a number to belong to the ratio set.

(3.1) THEOREM. *Suppose that μ is a G-measure on X which is quasi-invariant for Γ.*

(i) *Let $r \in [0, \infty[$. Then if $r \in r(X, \Gamma, \mu)$ then for every $\varepsilon > 0$, for every n and for every $\gamma_0 \in \Gamma_n$ there exists $\gamma \in \Gamma^n$ such that*

$$\mu\left\{u \in \gamma_0 X^n : \left| \prod_{i=1}^{\infty} \frac{g_i(\gamma u)}{g_i(u)} - r \right| < \varepsilon \right\} > 0 .$$

(ii) *If $\infty \in r(X, \Gamma, \mu)$, then for every $M > 0$, for every n and for every $\gamma_0 \in \Gamma_n$, there exists $\gamma \in \Gamma^n$ such that*

$$\mu\left\{u \in \gamma_0 X^n : \prod_{i=1}^{\infty} \frac{g_i(\gamma u)}{g_i(u)} > M \right\} > 0 .$$

PROOF. This is a direct consequence of Lemmas (2.3) and (2.5), applied to $A = \gamma_0 X^n$. □

We have the following sufficient condition:

(3.2) THEOREM. *Suppose that μ is a G-measure on X which is quasi-invariant for Γ.*

(i) *Let $r \in]0, \infty[$. Suppose that for every $\varepsilon > 0$ there exists $\beta > 0$ such that for every n and for every $\gamma_0 \in \Gamma_n$ there exists $\gamma \in \Gamma^n$ such that*

$$\mu\left(\left\{u\in\gamma_0X^n:\left|\prod_{i=1}^{\infty}\frac{g_i(\gamma u)}{g_i(u)}-r\right|<\varepsilon\right\}\right)>\beta\mu(\gamma_0X^n)\,.$$

Then $r\in r(X,\Gamma,\mu)$.

(ii) *Suppose that for every* $m>0$ *there exists* $\beta>0$ *such that for every* n *and for every* $\gamma_0\in\Gamma_n$ *there exists* $\gamma\in\Gamma^n$ *such that*

$$\mu\left(\left\{u\in\gamma_0X^n:\prod_{i=1}^{\infty}\frac{g_i(\gamma u)}{g_i(u)}>m\right\}\right)>\beta\mu(\gamma_0X^n)\,.$$

Then $\infty\in r(X,\Gamma,\mu)$.

PROOF. We shall prove only (i); the proof of (ii) is similar and left to the reader.

Let $\varepsilon>0$, and suppose that $\varepsilon<r$. Choose β according to (i). Let A be an arbitrary set of positive μ-measure. By a theorem of Carathéodory [6, (10.30)] there exists n and $\gamma_0\in\Gamma_n$ so that

$$\mu(A\cap\gamma_0X^n)>\left(1-\frac{\beta}{2}\right)\mu(\gamma_0X^n)$$

and

$$\mu(A\cap\gamma_0X^n)>\left(1-\frac{\beta}{2}(r-\varepsilon)\right)\mu(\gamma_0X^n)\,.$$

We may choose $\gamma\in\Gamma^n$ such that

$$\left|\frac{d\mu\circ\gamma}{d\mu}(u)-r\right|<\varepsilon$$

on a subset of measure greater than $\beta\mu(\gamma_0X^n)$. Letting B be the intersection of this subset with A, we see that $\mu(B)>\beta\mu(\gamma_0X^n)/2$, and $|d\mu\circ\gamma/d\mu(u)-r|<\varepsilon$ for all $u\in B$.

It follows that $\gamma B\subseteq\gamma_0X^n$, and that

$$\mu(\gamma B)>\frac{\beta}{2}(r-\varepsilon)\mu(\gamma_0X^n)\,.$$

Hence $\mu(A\cap\gamma B)>0$. By definition, we have $r\in r(X,\Gamma,\mu)$. □

Notice that the above proof breaks down for the case $r=0$. We shall present an example later to show that the above theorem is false for $r=0$.

4. The condition (E_1). We recall from [1], the condition (E_1).

For every $\varepsilon>0$ there is $k\in N$ such that for all $n\in N$ and for all $\gamma\in\Gamma^{n+k}$,

$$\sup\left\{\left|1-\frac{G_n(\gamma x)}{G_n(x)}\right|:x\in X\right\}<\varepsilon\,.$$

We shall assume henceforth that this condition is satisfied.

It was shown in [1] that this condition implies that there is a unique G-measure. Here, we show that in the presence of this condition the necessary and sufficient condition of § 3 can be ameliorated.

Indeed, for n fixed, the infinite product occuring in (3.1) and (3.2) may be written as

$$\prod_{i=0}^{\infty} \frac{g_i(\gamma u)}{g_i(u)} = \frac{G_n(\gamma u)}{G_n(u)} \cdot \prod_{i=n+1}^{\infty} \frac{g_i(\gamma u)}{g_i(u)} .$$

Thus, letting $\tilde{\mu}_{\gamma_0,n}$ denote the probability measure $(1/\mu(\gamma_0 X^n))\mu|_{\gamma_0 X^n}$, we have:

(4.1) PROPOSITION. *Let $r \in]0, \infty[$. Consider the conditions:*

(a) *For every $\varepsilon > 0$ there exist $\beta > 0$ and $k > 0$ such that for all $n \in N$ and for all $\gamma_0 \in \Gamma_{n+k}$ there exist $\gamma \in \Gamma^{n+k}$ such that*

$$\tilde{\mu}_{\gamma_0,n+k}\left(\left\{u \in \gamma_0 X^{n+k} : \left| \prod_{i=n+1}^{\infty} \frac{g_i(\gamma u)}{g_i(u)} - r \right| < \varepsilon \right\}\right) > \beta .$$

(b) $r \in r(X, \Gamma, \mu)$.

(c) *For every $\varepsilon > 0$ there exists $k > 0$ such that for all $n \in N$ and for all $\gamma_0 \in \Gamma_{n+k}$ there exists $\gamma \in \Gamma^{n+k}$ such that*

$$\tilde{\mu}_{\gamma_0,n+k}\left(\left\{u \in \gamma_0 X^{n+k} : \left| \prod_{i=n+1}^{\infty} \frac{g_i(\gamma u)}{g_i(u)} - r \right| < \varepsilon \right\}\right) > 0 .$$

One has (a)$\Rightarrow$(b)$\Rightarrow$(c). *The implication* (b)$\Rightarrow$(c) *holds also for* $r = 0$.

A similar statement, whose formulation is left to the reader, holds for $r = \infty$.

We may use the condition (E_1) to control the "tail" of the infinite product. The technicalities are contained in the next two lemmas.

(4.2) LEMMA. *Let $\varepsilon, \delta > 0$, and suppose $u \in X$, $\gamma \in \Gamma_N$, and that $|\prod_{i=1}^{\infty} g_i(\gamma u)/g_i(u) - r| < \delta$. Choose k according to the condition (E_1). Then for all v such that $u_1 = v_1, \ldots, u_{N+k} = v_{N+k}$, one has*

$$\left| \prod_{i=1}^{\infty} \frac{g_i(\gamma v)}{g_i(v)} - r \right| < (r+\delta)(2+\varepsilon)\varepsilon + \delta .$$

PROOF. Notice firstly that

$$\left| \prod_{i=1}^{\infty} \frac{g_i(\gamma v)}{g_i(v)} - r \right| \le \left| \prod_{i=1}^{\infty} \frac{g_i(\gamma v)}{g_i(v)} - \prod_{i=1}^{\infty} \frac{g_i(\gamma u)}{g_i(u)} \right| + \delta .$$

Secondly

$$\left|\prod_{i=1}^{\infty}\frac{g_i(\gamma v)}{g_i(v)}-\prod_{i=1}^{\infty}\frac{g_i(\gamma u)}{g_i(u)}\right|=\prod_{i=1}^{\infty}\frac{g_i(\gamma u)}{g_i(u)}\left|\prod_{i=1}^{\infty}\frac{g_i(\gamma v)}{g_i(\gamma u)}\cdot\prod_{i=1}^{\infty}\frac{g_i(u)}{g_i(v)}-1\right|.$$

Since

$$\left|\prod_{i=1}^{\infty}\frac{g_i(\gamma v)}{g_i(\gamma u)}-1\right|<\varepsilon \quad\text{and}\quad \left|\prod_{i=1}^{\infty}\frac{g_i(u)}{g_i(v)}-1\right|<\varepsilon\,,$$

we have $|\prod_{i=1}^{\infty}(g_i(\gamma v)g_i(u))/g_i(\gamma u)g_i(v))-1|<(2+\varepsilon)\varepsilon$. It follows that

$$\left|\prod_{i=1}^{\infty}\frac{g_i(\gamma v)}{g_i(v)}-r\right|\le(r+\delta)(2+\varepsilon)\varepsilon+\delta\,.$$

□

(4.3) LEMMA. *Let $\varepsilon>0$. Suppose that $u, v\in X$ and $L\in N$ are given so that $|\prod_{i=1}^{L}g_i(v)/g_i(u)-r|<\delta$. Choose k according to the condition (E_1). Then there exists $\gamma\in\Gamma_{L+k}$ such that*

$$\left|\prod_{i=1}^{L}\frac{g_i(\gamma u)}{g_i(u)}-r\right|<\delta+(r+\delta)\varepsilon\,.$$

PROOF. Choose $\gamma\in\Gamma_{L+k}$ by $\gamma_j u_j=v_j$, for $i\le L+k$. Then $\gamma u=v+w$, for $w\in X^L$. Thus $\prod_{i=1}^{L}g_i(\gamma u)/g_i(u)=\prod_{i=1}^{L}(g_i(v)/g_i(u))(g_i(v+w)/g_i(v))$.
Now, since $|\prod_{i=1}^{L}g_i(v)/g_i(u)-r|<\delta$, and $|\prod_{i=1}^{L}g_i(v+w)/g_i(v)-1|<\varepsilon$, we have

$$\left|\prod_{i=1}^{L}\frac{g_i(\gamma u)}{g_i(u)}-r\right|<\delta+(r+\delta)\varepsilon\,.$$

□

Notice that if u and v are eventually equal, for L sufficiently large, we have $\prod_{i=1}^{L}g_i(v)/g_i(u)=\prod_{i=1}^{\infty}g_i(v)/g_i(u)$ and $\prod_{i=1}^{L}g_i(\gamma u)/g_i(u)=\prod_{i=1}^{\infty}g_i(\gamma u)/g_i(u)$, where γ is chosen as in the above proof.

Using these two lemmas, we may refine the conditions in Proposition (4.1), obtaining:

(4.4) THEOREM. *Let $r\in\,]0,\infty[$. Consider the following conditions:*

(a) *For every $\varepsilon>0$, there exist $\beta>0$, $k>0$ and $L\in N$ such that for all $n\in N$ and all $\gamma_0\in\Gamma_{n+k}$,*

$$\tilde{\mu}_{\gamma_0,n+k}\left(\left\{u\in\gamma_0X^{n+k}:\ \exists v\in\gamma_0X^{n+k},\text{ eventually equal to }u,\text{ such that }l\ge n+L\text{ implies}\right.\right.$$
$$\left.\left.\left|\prod_{i=n+1}^{l}g_i(v)/g_i(u)-r\right|<\varepsilon\right\}\right)>\beta\,.$$

(b) $r\in r(X,\Gamma,\mu)$.

(c) *For every* $\varepsilon>0$, *there exists* $k>0$ *such that for all* $n\in N$ *and all* $\gamma_0\in\Gamma_{n+k}$ *there exists* $L\geq n$ *and* $\gamma\in\Gamma_{L+k}$ *with*

$$\tilde{\mu}_{\gamma_0,n+k}\left(\left\{u\in\gamma_0X^{n+k}:\ \exists v\in\gamma_0X^{n+k},\ \text{eventually equal to } u, \text{ such that}\right.\right.$$
$$\left.\left. l>L \text{ implies} \left|\prod_{i=n+1}^{l} g_i(v)/g_i(u)-r\right|<\varepsilon\right\}\right)>\tilde{\mu}_{\gamma_0,n+k}(\gamma_0\gamma X^{L+k})\,.$$

We have (a)⇒(b)⇒(c).

PROOF. If $|\prod_{i=n+1}^{l} g_i(v)/g_i(u)-r|<\varepsilon$ and v is eventually equal to u, then by Lemma 4.3 there is $\gamma\in\Gamma_{L+k}$ so that $|\prod_{i=n+1}^{l} g_i(\gamma u)/g_i(u)-r|<\delta+(r+\delta)\varepsilon$.

Thus, the condition (a) implies the condition (a) of Proposition (4.1), and it follows that $r\in r(X,\Gamma,y)$.

The proof of (b)⇒(c) is similar. □

The point of Theorem (4.4) is that it expresses in a clear way our "motherhood" statement that the ratio set consists of the infinite products $\prod_{i=n}^{\infty}(g_i(v)/g_i(u))$. We will see in the next section that in certain circumstances the integer L may be chosen so that for all n and γ_0

$$\tilde{\mu}_{\gamma_0,n+k}(\gamma_0\gamma X^{L+k})=\frac{\mu(\gamma_0\gamma X^{L+k})}{\mu(\gamma_0X^{n+k})}\geq\beta\,,$$

so that the conditions (a) and (c) coincide.

5. Tail conditions. The functions g'_k introduced in §2 allow us to use a kind of shift, identifying each of the tail spaces X^k with the circle $\boldsymbol{T}$. We are able thereby to refine the conditions of Theorem 4.4. The essential technique of this section is based upon considering limits of the form

$$\lim_{N\to\infty}\prod_{k=n}^{N}\frac{g'_k(\gamma_k,\ s/l(k+1)\ \cdots\ l(N))}{g'_k(\gamma_k,\ t/l(k+1)\ \cdots\ l(N))}\,,$$

where $\gamma\in X$ is fixed and $s,t\in\boldsymbol{T}$.

Indeed, we have:

(5.1) LEMMA. *Let* $r\in]0,\infty[$. *Suppose that for all* $\varepsilon>0$, *for all* $n\in\boldsymbol{N}$ *and for all* $\gamma\in X$, *there exists* $N=N(n)$ *so that for all* $m>N$, $\mu(\gamma_m(q_m^{-1}(W_m)))>0$, *where*

$$W_m=\left\{s\in\boldsymbol{T}:\ \exists t\in\boldsymbol{T}\ \text{with}\ \left|\prod_{k=n}^{m}\frac{g'_k(\gamma_k,\ q_k(\gamma_m^k)+s/l(k)\ \cdots\ l(m))}{g'_k(\gamma_k,\ q_k(\gamma_m^k)+t/l(k)\ \cdots\ l(m))}-r\right|<\varepsilon\right\}.$$

Then for all $m>N$, *for all* $u\in\gamma_m^1q_m^{-1}(W_m)$, *there exists* $v\in\gamma_m^1q_m^{-1}(W_m)$, *eventually equal to* u, *such that*

$$\left|\prod_{k=n}^{m} \frac{g_k(u)}{g_k(v)} - r\right| < \varepsilon .$$

NOTATION. The notation in the above statement is as follows:

$$\gamma_m^k \in \Gamma \quad \text{is defined by} \quad (\gamma_m^k)_p = \begin{cases} \gamma_p & \text{if } m \le p \le k \\ 0 & \text{otherwise} . \end{cases}$$

PROOF. It is easy to see that

$$q_k(\gamma_m^k) + s/l(k) \cdots l(m) = q_k(\gamma_m^1 + q_m^{-1}(s)) .$$

Hence, taking $u = \gamma_m^1 q_m^{-1}(s)$ and $v = \gamma_m^1 q_m^{-1}(t)$, we see that the product

$$P(s, t) = \prod_{k=n}^{m} \frac{g_k'(\gamma_n, q_n(\gamma_m^k) + s/l(n) \cdots l(m))}{g_k'(\gamma_n, q_n(\gamma_m^k) + t/l(n) \cdots l(m))} \quad \text{is also equal to} \quad \prod_{k=n}^{m} \frac{g_k(u)}{g_k(v)} .$$

By the continuity of the function $g_n'(\gamma_n, \cdot)$, we see that if $|P(s, t) - r| < \delta$ for some t, then this is also true in a neighbourhood of t. Thus, we may choose t so that $q_m^{-1}(s)$ and $q_m^{-1}(t)$ are eventually equal. □

(5.2) THEOREM. *Let* $r \in]0, \infty[$. *Suppose that* $\sup_i l(i) = l < \infty$. *Suppose that for all* $\varepsilon > 0$, *and for all* $n \in \boldsymbol{N}$, *the integer* $N(n)$ *of* (5.1) *may be chosen satisfying* $\sup_{n \in \boldsymbol{N}}(N(n)) \le \mathscr{K}(\varepsilon)$ *for some* $\mathscr{K} = \mathscr{K}(\varepsilon) \in \boldsymbol{R}^+$. *Then* $r \in r(X, \Gamma, \mu)$.

PROOF. This is a direct result of Theorem 4.4 and Lemma 5.1, modulo the fact that if $m - n = \mathscr{K}$, then for some $\beta > 0$ independent of m and n

$$\tilde{\mu}_{\gamma_{n,m}}(\gamma_m q_m^{-1}(W_m)) > \beta .$$

This is an easy consequence of the boundedness of $l(i)$. □

We apply this result to the g-measures of Keane (cf. [8]); see [1, 2.8] for their relationship to G-measures.

(5.2) PROPOSITION. *Let* $l(i) = l$ *(constant) for all i. For each* $\gamma \in \boldsymbol{Z}_l$, *let* $g'(\gamma, \cdot)$ *be a log Lipschitz function of order* α *on* $\boldsymbol{T}$, *satisfying* $(1/l)\sum_{\gamma \in \boldsymbol{Z}_l} g'(\gamma, \cdot) = 1$. *Let* μ *denote the (unique) g-measure. Then if g is not the constant function* 1, μ *is of type* III_1.

PROOF. In this case, the functions g_k' are all identical, so the product in Lemma 5.1 reduces to

$$\prod_{k=n}^{m} \frac{g'(\gamma_k, s/l^{m-k})}{g'(\gamma_k, t/l^{m-k})} = \prod_{i=0}^{m-n} \frac{g'(\gamma_{m-i}, s/l^i)}{g'(\gamma_{m-i}, t/l^i)} .$$

Since the $g'(\gamma, \cdot)$'s are log Lipschitz, we may find a constant $\mathscr{K}$ so that for $i = 1, \cdots, m-n$,

$$|\log g'(\gamma_{m-i}, s/l^i) - \log g'(\gamma_{m-i}, t/l^i)| < \mathscr{K}(|s-t|/l^i)^{\alpha}$$

and hence, since $\sum_{i=1}^{\infty}(l^{\alpha})^{-i}<\infty$, the infinite product

$$r=\prod_{i=1}^{\infty}\frac{g_i'(\gamma_i, s/l^i)}{g_i'(\gamma_i, t/l^i)}$$

exists, for all $s, t\in \boldsymbol{T}$ and furthermore, for $\varepsilon>0$ there exists k such that

$$\left|\prod_{i=1}^{k}\frac{g_i'(\gamma_i, s/l^i)}{g_i'(\gamma_i, t/l^i)}-r\right|<\varepsilon .$$

Since the g_i' are nonconstant, and Lipschitz, one can choose pairs (s, t), (s_0, t_0) so that

$$\prod_{i=1}^{\infty}\frac{g_i'(\gamma_i, s/l^i)}{g_i'(\gamma_i, t/l^i)} \quad\text{and}\quad \prod_{i=1}^{\infty}\frac{g_i'(\gamma_i, s_0/l^i)}{g_i'(\gamma_i, t_0/l^i)}$$

approach rationally independent limits, r_1, r_2. Thus, the ratio set must consist of all of $[0, \infty]$ and we are in a type III_1 situation. □

The above theorem implies in particular that the Riesz product $\prod_{k=1}^{\infty}(1+a\cos 3^k t)dt$ $(a\neq 0)$ is of the III_1. This result was also obtained by Yoshida; his methods are based on our Lemma (2.3), but are particular to Riesz products.

The next proposition analyses Riesz products of the form $\prod_{k=1}^{\infty}(1+a_k\cos 3^k t)dt$, where $a_k\to 0$ as $k\to\infty$. These are of type II_∞ or III_0.

(5.4) PROPOSITION. *Let ν be the Riesz product which is the weak*-limit of the measures*

$$\prod_{k=1}^{n}(1+a_k 2\pi\cos 3^k t)dt .$$

Suppose that $a_k\searrow 0$ as $k\to\infty$. Then the ratio set of ν is contained in the set $\{0, 1, \infty\}$.

PROOF. We may as well suppose $a_k\leq 1/2$ for all k. According to lemma (5.1), we should consider the infinite product

$$P_{n,m}=\prod_{k=n}^{m}\left(\frac{1+a_k\cos 2\pi(\gamma+3^{k-m}t)}{1+a_k\cos 2\pi(\gamma+3^{k-m}s)}\right),$$

where γ is a triadic rational whose denominator is at most 3^n and $s, t\in\boldsymbol{T}$. A simple manipulation shows that the product is equal to

$$\exp\left\{\sum_{k=n}^{m}\log\left(1+a_k\left(\sin 2\pi 3^{k-m}\left(\frac{t-s}{2}\right)\frac{2\sin 2\pi\left(\gamma+3^{k-m}\left(\frac{t+s}{2}\right)\right)}{1+a_k\cos 2\pi\left(\gamma+3^{k-m}\left(\frac{t+s}{2}\right)\right)}\right)\right)\right\}.$$

We claim that as $n\to\infty$, this approaches 1. In fact, $|2\sin\theta/(1+a_k\cos\theta)|\leq 4$, so

$$\left|\log\left(1+a_k\left(\sin 2\pi 3^{k-m}\left(\frac{t-s}{2}\right)\frac{2\sin 2\pi\left(\gamma+3^{k-m}\left(\frac{t+s}{2}\right)\right)}{\left(1+a_k\cos 2\pi\left(\gamma+3^{k-m}\left(\frac{t+s}{2}\right)\right)\right)}\right)\right)\right|$$

$$\leq\left|4a_k\sin 2\pi 3^{k-m}\left(\frac{t-s}{2}\right)\right|\leq 4\pi a_k 3^{k-m}.$$

(We have used the facts that $\log(1+x)\leq x$, that $\sin x\leq x$ and that $(t-s)/2\leq 1$.) Now $\sum_{k=n}^{m}a_k 3^{k-m}\leq a_n\sum_{k=n}^{m}3^{k-m}=3a_n/2$.

Thus, we see that

$$\exp(-3a_n)\leq P_{n,m}\leq\exp(3a_n)\,,\quad\text{for all choices of } t \text{ and } s\,.$$

Since $a_n\to 0$, we see that $P_{n,m}\to 1$ uniformly in s and t. By Lemma (5.1), no element of $]0,\infty[$ apart from 1 can belong to the ratio set. This is therefore contained in $\{0, 1, \infty\}$. □

REMARKS. Let us suppose that $\sum a_k^2=\infty$ and $\sum(1-a_k)=\infty$. By standard Riesz product arguments, μ is neither equivalent to Haar measure, nor does it have any atoms. Hence μ is neither of type II_1 nor of type I. For these Riesz products measures, μ is either of type II_∞ or type III_0. It would be desirable to have criterion for deciding which, but at the present time none is available.

6. Product measures. It is instructive to apply Theorem 4.4 to product measures on infinite products of two point spaces. Thus, let $l(k)=2$ for each k, and choose the functions $g_k(x)$ to depend only on the coordinate x_k. Choose $a_i\in(0, 1)$ and write

$$g_k(x)=\begin{cases}(1-a_i)/2 & \text{if}\quad x_i=1\\ (1+a_i)/2 & \text{if}\quad x_i=0\,.\end{cases}$$

The G-measure corresponding to this choice is the infinite product $\mu=\bigotimes_{i=1}^{\infty}\mu_i$, where for $\gamma\in\{0, 1\}$,

$$\mu_i(\{\gamma\})=\frac{1+(-1)^{\gamma}a_i}{2}\,.$$

Moore [11] has calculated the ratio sets for product measures. In the present notation, we may re-state his result as follows

(6.1) THEOREM. (1) *μ is type* I *if and only if* $\sum_n(1-a_i)<\infty$.

(2) *μ is type* II_1 *if and only if*

$$\sum a_i^2<\infty\,.$$

(3) *μ is of type* III *if and only if*

$$\sum(\min(2a_i, 1-a_i))^2=\infty .$$

In the remaining cases, μ is of type II_∞.

The present techniques allow us to somewhat refine this theorem. We will need some notation. Let σ_i denote $\log\{(1+a_i)/(1-a_i)\}$. Then $\log(g_i(\gamma)/g_i(\eta))=(\eta-\gamma)\sigma_i$ for $\gamma, \eta\in\{0, 1\}$.

It follows that for $u, v\in X^n$, and for $m>n$

$$\prod_{i=n}^{m}\frac{g_i(u)}{g_i(v)}=\exp\left(\sum_{i=n}^{m}(v_i-u_i)\sigma_i\right). \tag{1}$$

We shall consider two cases, representing two extremes; $a_i\to 0$ and $a_i\to 1$. These correspond to $\sigma_i\to 0$ and $\sigma_i\to\infty$, respectively.

Our first result is:

(6.2) PROPOSITION. *If $a_i\to 0$ and $\sum^\infty a_i^2=\infty$, then the measure μ is of type* III_1.

PROOF. We may as well assume that $a_i\le 1/2$ for all i. The estimates

$$a_i\ge\sigma_i=\log\left(1+\frac{2a_i}{1-a_i}\right)\ge\frac{a_i}{1-a_i}\ge a_i$$

show that $\sum\sigma_i=\infty$ and $\sigma_i\to 0$ as $i\to\infty$.

Let $r\in[1, \infty[$, $\varepsilon>0$ and let n be any integer. Let u be an arbitrary element of $\gamma_0 X^n$. Since $\sum\sigma_i=\infty$, we may choose v differing from u in finitely many coordinates such that $|\sum_{i=n}^{\infty}(v_i-u_i)\sigma_i-\log r|<\varepsilon$. From (4.4) (a), together with formula (1), it follows that $r\in r(X, \Gamma, \mu)$, and hence μ is of type III_1.

(6.3) The above examples have dealt with measures of type III_1. As was indicated in Section (5.4) the present methods do not allow us to readily distinguish between Type II_∞ and Type III_0. A major reason for this is that the statement of Theorem 3.2 fails for $r=0$. The referee was kind enough to provide the following example of this.

Define a product measure by taking $a_i=0$ if i is even and $a_i=(1-2^{-i})/(1+2^{-i})$ if i is odd. By Moore's criterion, the measure is of type II_∞. But if Theorem 3.2 held for $r=0$, we would be able to use the same arguments given in the paper to obtain a version of Theorem 4.4 valid for all $r\ge 0$. Yet one can easily check that for all γ_0 and u

$$\tilde{\mu}_{\gamma_0,n+2}\Big\{u\in\gamma_0 x^{n+2}: \exists v\in\gamma_0 X^{n+2} \text{ eventually equal to } u \text{ such that}$$

$$l>n+2 \text{ implies } \prod_{c=n+1}^{l}\frac{g_i(v)}{g_i(u)}<\varepsilon\Big\}>\frac{1}{2},$$

and this would imply that 0 belonged to the ratio set.

(6.4) It seems to be an open problem to give an explicit construction of a product measure of type III_0 on an infinite product of two point spaces. (The standard examples found for example in [7] can be realized on product spaces where the number of points in each space is unbounded.)

We would like to offer a conjecture which would resolve this problem.

CONJECTURE. If $\{a_i\}$ is a sequence with $0<a_i<1$ and such that $a_i \nearrow 1$ as $i \to \infty$, then the ratio set of the product measure μ formed from $\{a_i\}$ has ratio set contained in $\{0, 1, \infty\}$.

If our conjecture could be proved, it would combine with Moore's criterion to give easy examples of measures of Type III_0.

Added in Proof (December 26, 1994). Dooley and Klemes have recently proved the conjecture false, but are able to give examples of sequences for which the measure is of Type III_0.

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