On the Kronecker Product of Schur Functions of Two Row Shapes

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Abstract

The Kronecker product of two homogeneous symmetric polynomials P_1 and P_2 is defined by means of the Frobenius map by the formula $P_1 \otimes P_2 = F(F^{-1}P_1)(F^{-1}P_2)$. When P_1 and P_2 are Schur functions s_λ and s_μ respectively, then the resulting product $s_\lambda \otimes s_\mu$ is the Frobenius characteristic of the tensor product of the irreducible representations of the symmetric group corresponding to the diagrams λ and μ. Taking the scalar product of $s_\lambda \otimes s_\mu$ with a third Schur function s_ν gives the so-called Kronecker coefficient $g_{\lambda\mu\nu} = \langle s_\lambda \otimes s_\mu, s_\nu \rangle$ which gives the multiplicity of the representation corresponding to ν in the tensor product. In this paper, we prove a number of results about the coefficients $g_{\lambda\mu\nu}$ when both λ and μ are partitions with only two parts, or partitions whose largest part is of size two. We derive an explicit formula for $g_{\lambda\mu\nu}$ and give its maximum value.

0 Introduction

Let $A(S_n)$ denote the group algebra of S_n, the symmetric group on n letters, i.e. $A(S_n) = \{f : S_n \rightarrow \mathbb{C}\}$ where $\mathbb{C}$ denotes the complex numbers. Let $C(S_n)$ denote the set of class functions of $A(S_n)$, i.e. those $f \in A(S_n)$ which are constant on

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conjugacy classes. Then every homogeneous symmetric polynomial P of degree n can be written in the form

$$P = \frac{1}{n!} \sum_{\sigma \in S_n} \chi(\sigma) p_1^{\nu_1(\sigma)} \dots p_n^{\nu_n(\sigma)}, \tag{0.1}$$

where χ is a class function uniquely determined by P, p_i denotes the i-th power symmetric function, and $\nu_i(\sigma)$ denotes the number of cycles of length i in σ. We refer to P as the *characteristic* of χ. The map $\chi \to P$ defined by (0.1) shall simply be written as

$$P = F\chi.$$

This map was first considered in this connection by Frobenius who proved that

$$s_\lambda = F\chi^\lambda,$$

where s_λ is the Schur function indexed by the Ferrers' diagram λ and χ^λ represents the irreducible character of S_n corresponding to the partition λ. The so called *Kronecker product* of homogeneous symmetric polynomials of degree n is defined in terms of F by setting

$$P_1 \otimes P_2 = F\chi_1\chi_2, \tag{0.2}$$

where $P_1 = F\chi_1$, $P_2 = F\chi_2$, and $\chi_1\chi_2(\sigma) = \chi_1(\sigma)\chi_2(\sigma)$ for all $\sigma \in S_n$. Now if χ_1 and χ_2 are characters of representations of S_n, then $\chi_1\chi_2$ is the character of the tensor product of these representations. Hence the expansion of $P_1 \otimes P_2$ gives the multiplicities of the corresponding irreducible characters in this tensor product. Thus it is of fundamental importance to determine the coefficients $g_{\lambda\mu\nu}$ defined by

$$g_{\lambda\mu\nu} = \langle s_\lambda \otimes s_\mu, s_\nu \rangle \tag{0.3}$$

where $\langle P, Q \rangle$ denotes the usual Hall inner product on symmetric functions.

The main purpose of this paper is to explore these coefficients $g_{\lambda\mu\nu}$ for λ and μ restricted to shapes with only two parts and ν an arbitrary shape. The techniques used are mainly combinatorial, and rely both on the Jacobi-Trudi identity and on a rule for expanding the Kronecker product of two homogeneous symmetric functions due to Garsia and Remmel [1] which gives the expansion in terms of decompositions of the shape μ. In this instance, studying $g_{\lambda\mu\nu}$ reduces to studying a collection of signed diagrams with 4 or less rows. We then give two involutions on this set which allow us to give a formula for these coefficients and to calculate their maximum value, as well as the partition at which this maximum is attained. This maximum depends on the size of the partitions λ and μ, which shows that these coefficients are unbounded. This is in marked contrast to the values of the Kronecker coefficients when λ and μ are both hook shapes (shapes of the form $(1^t, n-t)$) or when λ is a hook shape and μ is a two-row shape. In these instances, Remmel shows in [4] and [5] that the coefficients are always strictly less than four for all n.

Let $\lfloor x \rfloor$ be the largest integer less than or equal to x and $\lceil x \rceil$ be the smallest integer greater than or equal to x. Then with $h + k = l + m = n$ where $l \leq h$ and

$g_{(h,k)(l,m)\nu}$ defined by $s_{(h,k)} \otimes s_{(l,m)} = \sum_\nu g_{(h,k)(l,m)\nu} s_\nu$, we show that $g_{(h,k)(l,m)\nu} = 0$ if ν has more than 4 parts. Otherwise, with $\nu = (a, b, c, d)$,

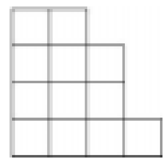
$$\begin{aligned} g_{(h,k)(l,m)\nu} = & \sum_{r=L_1}^{U_1} 1 + \min(b-a, l+h-a-c-2r) + \\ & \sum_{r=L_2}^{U_2} 1 + \min(d-c, l+h-a-b-c-r) - \\ & \sum_{r=L_3}^{U_3} 1 + \min(c-b, l+h-a-b-1-2r) - \\ & \sum_{r=L_4}^{U_4} 1 + \min(d-c, l+h-a-b-c-1-r) - \\ & \sum_{r=L_5}^{U_5} 1 + \min(b-a, l+h+b+c-n-1-2r) - \\ & \sum_{r=L_6}^{U_6} 1 + \min(c-b, l+h+c-n-1-r) + \\ & \sum_{r=L_7}^{U_7} 1 + \min(b-a, l+h-a-c-2-2r) + \\ & \sum_{r=L_8}^{U_8} 1 + \min(b-a, l+h-a-c-2-2r) \end{aligned}$$

where the upper and lower limits of the above sums depend on n, l, h, a, b, c and d and are given in the following section. Admitedly, this formula is rather messy, but it can easily be evaluated by computer. Moreover, in several special cases, for example when ν is a two-part partition, or a four-part partition whose two smallest parts are equal, the above sums simplify to exceedingly simple formulas. Also, our approach allows us to compute the maximum of $g_{(h,k)(l,m)\nu}$ for any fixed value of l over all possible values of h and n. In fact, we show that for fixed l, the maximum of $g_{(h,k)(l,m)\nu}$ grows like $9l^2/44$ and we specify the partition which attains this maximum. Finally, because the coefficients $g_{\lambda\mu\nu}$ are symmetric in λ, μ and ν, and $g_{\lambda'\mu\nu}$, $g_{\lambda\mu'\nu}$ and $g_{\lambda\mu\nu'}$ are easily expressed in terms of $g_{\lambda\mu\nu}$, where λ' is the congugate partition of λ, our formula gives the values of $g_{\lambda\mu\nu}$ for any triple of partitions λ, μ, and ν where two of the three partitions have either at most two parts or largest part of size two.

The paper is organized as follows. In Section 1 we develop our notation and give basic results about the Kronecker product. In Section 2 we give the formula for $g_{(h,k)(l,m)\nu}$ and its proof. In Section 3 we examine the formula for special values of ν. Lastly, in Section 4, we compute the maximum of $g_{(h,k)(l,m)\nu}$ for fixed l as h and ν varies.

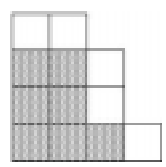
1 Basic Formulas and Algorithms

Given the partition $\lambda = (\lambda_1, \lambda_2, \ldots \lambda_k)$ where $0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_k$ and $\sum \lambda_j = n$, we let F_λ denote the Ferrers' diagram of λ, i.e. F_λ is the set of left-justified squares or boxes with λ_1 squares in the top row, λ_2 squares in the second row, etc. For example,

$F_{(2,3,3,4)} =$ 

For the sake of convenience, we will often refer to the diagram F_λ simply by λ.

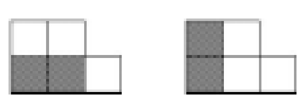
Given two partitions $\lambda = (\lambda_1, \ldots, \lambda_k)$ and $\mu = (\mu_1, \ldots, \mu_l)$, we write $\lambda \leq \mu$ if and only if $k \leq l$ and $\lambda_{k-p} \leq \mu_{l-p}$ for $0 \leq p \leq k-1$. If $\lambda \leq \mu$, we let $F_{\mu/\lambda}$ denote the Ferrers' diagram of the skew shape μ/λ where $F_{\mu/\lambda}$ is the diagram that results by removing the boxes corresponding to F_λ from the diagram F_μ. For example, $F_{(2,3,3,4)/(2,2,3)}$ consists of the unshaded boxes below:



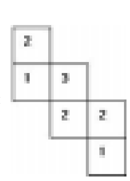
Let $\lambda \vdash n$ and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ be a sequence of positive integers such that $\sum \alpha_i = n$. Define a *decomposition* of λ of type α, denoted by $D_1 + D_2 + \ldots + D_k = \lambda$, as a sequence of shapes

$$\lambda^1 \subset \lambda^2 \subset \ldots \subset \lambda^k = \lambda$$

with λ^i/λ^{i-1} a skew shape, $D_i = \lambda^i/\lambda^{i-1}$ and $|D_i| = \alpha_i$. For example, for $k = 2$, $\lambda = (2,3)$, $\alpha_1 = 2$, $\alpha_2 = 3$, the two decompositions of λ of type α are pictured below where the shaded portion corresponds to D_1 and the unshaded portion corresponds to D_2:



A column strict tableau T of shape μ/λ is a filling of $F_{\mu/\lambda}$ with positive integers so that the numbers weakly increase from left to right in each row and strictly increase from bottom to top in each column. T is said to be *standard* if the entries of T are precisely the numbers $1, 2, \ldots, n$ where n equals $|\mu/\lambda|$. We let $CS(\mu/\lambda)$ and $ST(\mu/\lambda)$ denote the set of all column strict tableaux and standard tableaux of shape μ/λ respectively. Given $T \in CS(\mu/\lambda)$, the *weight* of T, denoted by $\omega(T)$, is the monomial obtained by replacing each i in T by x_i and taking the product over all boxes. For example, if

$T =$ 

then $\omega(T) = x_1^2 x_2^3 x_3$.

This given, the skew Schur function $s_{\mu/\lambda}$ is defined by

$$s_{\mu/\lambda}(x_1, x_2, \ldots) = \sum_{T \in CS(\mu/\lambda)} \omega(T). \tag{1.1}$$

The special case of (1.1) where λ is the empty diagram, i.e. $\lambda = \emptyset$, defines the usual Schur function s_μ. For emphasis, we shall often refer to those shapes which arise directly from partitions μ as straight shapes so as to distinguish them among the general class of skew shapes.

For an integer n, the Schur function indexed by the partition (n) is also called the nth *homogeneous symmetric function* and will be denoted by h_n. Thus,

$$h_n = s_{(n)}.$$

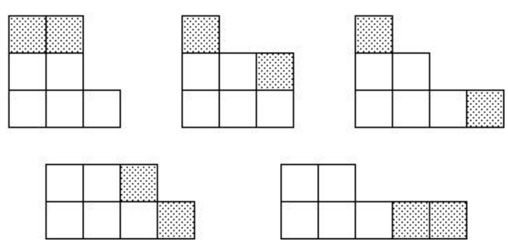
Also for a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$, let

$$h_\lambda = h_{\lambda_1} \ldots h_{\lambda_k}.$$

Then with these definitions, we state Pieri's rule which gives a combinatorial rule for the Schur function expansion of a product of a Schur function and homogeneous symmetric function:

$$h_r \cdot s_\lambda = \sum_\mu s_\mu \tag{1.2}$$

where the sum is over all μ such that μ/λ is a horizontal r-strip, i.e. a skew shape consisting of r boxes, with no two boxes lying in the same column. For example, to multiply $h_2 \cdot s_{(2,3)}$ the sum in (1.2) is over the following shapes, with the shaded portions corresponding to the horizontal 2-strip:



and thus

$$h_2 \cdot s_{(2,3)} = s_{(2,2,3)} + s_{(1,3,3)} + s_{(1,2,4)} + s_{(3,4)} + s_{(2,5)}.$$

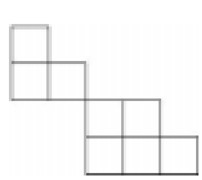
We also have the following identity for Schur functions, called the Jacobi-Trudi identity:

$$s_\lambda = det||h_{\lambda_j - i + j}||_{1 \le i,j \le l(\lambda)} \tag{1.3}$$

where $h_0 = 1$ and for $r < 0$, $h_r = 0$. Proofs of both of these theorems can be found in [3] .

For two shapes λ and μ let $\lambda * \mu$ represent the skew diagram obtained by joining at the corners the rightmost, lowest box of F_λ to the leftmost, highest box of F_μ. For example, if $\lambda = (1,2)$ and $\mu = (2,3)$, we have

$$F_{(1,2)*(2,3)} =$$



Obviously, in light of the combinatorial definition of Schur functions, $s_{\lambda * \mu} = s_\lambda \cdot s_\mu$. Clearly the same idea can be used to express an arbitrary product of Schur functions as a single skew Schur function, i.e. $s_{\lambda_1} \cdots s_{\lambda_k} = s_{\lambda_1 * \lambda_2 * \ldots * \lambda_k}$. Thus we have for $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$,

$$h_\mu = s_{\mu_1} s_{\mu_2} \ldots s_{\mu_k} = s_{\mu_1 * \mu_2 * \ldots * \mu_k}.$$

We now state some properties of the Kronecker product. (1.4) through (1.7) are easily established from its definition. A proof of (1.8) can be found in [2] .

$$h_n \otimes s_\lambda = s_\lambda \tag{1.4}$$

$$s_{(1^n)} \otimes s_\lambda = s_{\lambda'} \text{ where } \lambda' \text{ denotes the conjugate of } \lambda \tag{1.5}$$

$$s_\lambda \otimes s_\mu = s_\mu \otimes s_\lambda = s_{\lambda'} \otimes s_{\mu'} = s_{\mu'} \otimes s_{\lambda'} \tag{1.6}$$

$$(P+Q) \otimes R = P \otimes R + Q \otimes R. \tag{1.7}$$

$$g_{\lambda\mu\nu} = g_{\lambda\nu\mu} = g_{\nu\lambda\mu} = g_{\nu\mu\lambda} = g_{\mu\lambda\nu} = g_{\mu\nu\lambda} \tag{1.8}$$

Littlewood [2] proved the following:

$$s_\alpha\, s_\beta \otimes s_\lambda = \sum_{\gamma \vdash |\alpha|} \sum_{\delta \vdash |\beta|} c_{\gamma\delta\lambda} (s_\alpha \otimes s_\gamma)\,(s_\beta \otimes s_\delta) \tag{1.9}$$

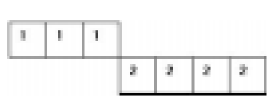
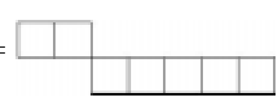
where γ, δ and λ are straight shapes and $c_{\gamma\delta\lambda}$ is the Littlewood-Richardson coefficient, i.e. $c_{\gamma\delta\lambda} = \langle s_\gamma\, s_\delta, s_\lambda \rangle$. Garsia and Remmel [1] then used (1.9) to prove the following:

$$(s_H \cdot s_K) \otimes s_D = \sum_{\substack{D_1 + D_2 = D \\ |D_1| = |H| \\ |D_2| = |K|}} (s_H \otimes s_{D_1}) \cdot (s_K \otimes s_{D_2}) \tag{1.10}$$

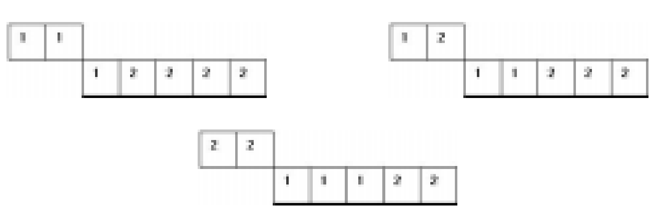
where H,K, and D are skew shapes and the sum runs over all decompositions of the skew shape D. In particular, one can easily establish by induction from (1.10) that

$$(h_{a_1} \ldots h_{a_k}) \otimes s_D = \sum_{\substack{D_1 + \ldots D_k = D \\ |D_i| = a_i}} s_{D_1} \ldots s_{D_k} \tag{1.11}$$

where the sum runs over all decompositions of D of length k such that $|D_i| = a_i$ for all i. For example, if we want to expand $h_3 \cdot h_4 \otimes h_2 \cdot h_5$, then we have that

$H * K =$  and $D =$ 

and we have the following decompositions:



and so

$$h_3 \cdot h_4 \otimes h_2 \cdot h_5 = h_1 \cdot h_2 \cdot h_4 + h_1 \cdot h_1 \cdot h_2 \cdot h_3 + h_2 \cdot h_2 \cdot h_3,$$

each term of which can be expanded by using Pieri's rule (1.2).

We are now in a position to give the results mentioned in the introduction.

2 A Formula for the Coefficients in $S_{(k,n-k)} \otimes S_{(l,n-l)}$

In this section we shall give a formula for the coefficient $g_{(h,k)(l,m)\nu}$. The motivation for studying such a problem can be found in [4] and [5] . There it is shown that the coefficients that occur in the expansion of the Kronecker product of Schur functions indexed by two hook shapes (shapes of the form $(1^t, n-t)$) and by a hook shape and a two-row shape are strictly bounded for all n. Specifically, if

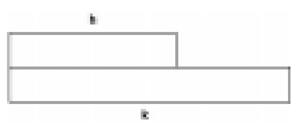
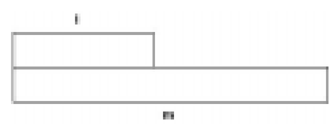
$$s_{(1^t,n-t)} \otimes s_{(1^s,n-s)} = \sum_\gamma g_{(1^t,n-t)\,(1^s,n-s)\,\gamma} s_\gamma,$$

then $g_{(1^t,n-t)\,(1^s,n-s)\,\gamma} \leq 2$ for all γ and n and if

$$s_{(1^t,n-t)} \otimes s_{(k,n-k)} = \sum_\gamma g_{(1^t,n-t)\,(k,n-k)\,\gamma} s_\gamma,$$

then $g_{(1^t,n-t)\,(k,n-k)\,\gamma} \leq 3$ for all γ and n.

Let the two-row shapes be denoted by (h,k) and (l,m) with $h+k = l+m = n$ and $l \leq h$. So our two shapes look like

 and 

We now give a combinatorial proof of the following.

(2.1) Theorem. *Let $h+k = l+m = n$ where $l \leq h$ and define $g_{(h,k)(l,m)\nu}$ by $s_{(h,k)} \otimes s_{(l,m)} = \sum_\nu g_{(h,k)(l,m)\nu} s_\nu$. Then $g_{(h,k)(l,m)\nu} = 0$ if ν has more than 4 parts. Otherwise, with $\nu = (a,b,c,d)$,*

$$\begin{aligned} g_{(h,k)(l,m)\nu} = &\sum_{r=L_1}^{U_1} 1 + \min(b-a, l+h-a-c-2r) + \\ &\sum_{r=L_2}^{U_2} 1 + \min(d-c, l+h-a-b-c-r) - \\ &\sum_{r=L_3}^{U_3} 1 + \min(c-b, l+h-a-b-1-2r) - \\ &\sum_{r=L_4}^{U_4} 1 + \min(d-c, l+h-a-b-c-1-r) - \\ &\sum_{r=L_5}^{U_5} 1 + \min(b-a, l+h+b+c-n-1-2r) - \\ &\sum_{r=L_6}^{U_6} 1 + \min(c-b, l+h+c-n-1-r) + \\ &\sum_{r=L_7}^{U_7} 1 + \min(b-a, l+h-a-c-2-2r) + \end{aligned}$$

$$\sum_{r=L_8}^{U_8} 1 + \min(b-a, l+h-a-c-2-2r)$$

where $L_1 = \max(b, h-c, \lceil \frac{l+h+a+c-n}{2} \rceil)$, $U_1 = \min(l, \lfloor \frac{h}{2} \rfloor, \lfloor \frac{l+h-a-c}{2} \rfloor)$,
$L_2 = \max(a, h-c, \lceil \frac{l+h+a+c-n}{2} \rceil)$, $U_2 = \min(l, \lfloor \frac{h}{2} \rfloor, b-1, h-b, l+h-a-b-c)$,
$L_3 = \max(c, \lceil \frac{h}{2} \rceil, h-b, l+h+b-n-1)$, $U_3 = \min(l-1, h-a, \lfloor \frac{l+h-a-b-1}{2} \rfloor)$,
$L_4 = \max(\lceil \frac{h}{2} \rceil, \lceil \frac{l+h+b+c-n-1}{2} \rceil, h-b, b)$, $U_4 = \min(l-1, c-1, h-a, l+h-a-b-c-1)$,
$L_5 = \max(b, l+h+a-n-1)$, $U_5 = \min(l, \lfloor \frac{h-1}{2} \rfloor, h-c-1, \lfloor \frac{l+h+b+c-n-1}{2} \rfloor)$,
$L_6 = \max(a, \lceil \frac{l+h+a+b-n-1}{2} \rceil)$, $U_6 = \min(l, \lfloor \frac{h-1}{2} \rfloor, b-1, h-c-1, l+h+c-n-1)$,
$L_7 = \max(\lceil \frac{h-1}{2} \rceil, c, h-c-1, l+h+a-n-2)$, $U_7 = \min(l-1, h-b-1, \lfloor \frac{l+h-a-c-2}{2} \rfloor)$,
$L_8 = \max(\lceil \frac{h-1}{2} \rceil, \lceil \frac{h+l+a+c-n-2}{2} \rceil)$, $U_8 = \min(l-1, h-b-1, \lfloor \frac{l+h-a-c-2}{2} \rfloor, c-1)$.

Proof. Using the Jacobi-Trudi identity, (1.3) we have

$$s_{(h,k)} = \det \begin{pmatrix} s_h & s_{k+1} \\ s_{h-1} & s_k \end{pmatrix} = s_h\, s_k - s_{h-1}\, s_{k+1}$$

with a similar expression for $s_{(l,m)}$. These expansions and (1.7) give

$$\begin{aligned} & s_{(h,k)} \otimes s_{(l,m)} = \\ & (s_h\, s_k - s_{h-1}\, s_{k+1}) \otimes (s_l\, s_m - s_{l-1}\, s_{m+1}) = \\ & s_h\, s_k \otimes s_l\, s_m - s_h\, s_k \otimes s_{l-1} s_{m+1} - \\ & s_{h-1}\, s_{k+1} \otimes s_l\, s_m + s_{h-1}\, s_{k+1} \otimes s_{l-1}\, s_{m+1}. \end{aligned}$$

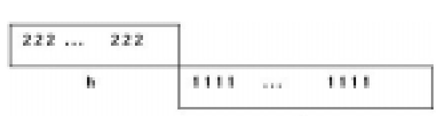
So let

$$A = s_h\, s_k \otimes s_l\, s_m \qquad B = s_h\, s_k \otimes s_{l-1} s_{m+1}$$
$$C = s_{h-1}\, s_{k+1} \otimes s_l\, s_m \qquad D = s_{h-1}\, s_{k+1} \otimes s_{l-1}\, s_{m+1}.$$

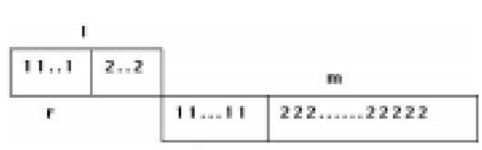
Then

$$s_{(h,k)} \otimes s_{(l,m)} = A - B - C + D. \tag{2.1}$$

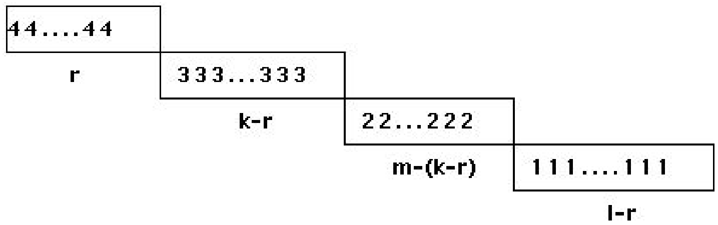
Using (1.10) we have that the expansion of A can be obtained by forming all decompositions $D_1 + D_2 = (l) * (m)$ with $|D_1| = h$ and $|D_2| = k$. The Schur functions indexed by the resulting skew shapes, s_{D_1} and s_{D_2}, each occur in the expansion with coefficient one. It is easy to see that all decompositions of (l, m) using



must be of the form



for $0 \le r \le l$. This decomposition then gives the term $s_r\, s_{l-r}\, s_{k-r}\, s_{m-(k-r)}$ which can be multiplied using Pieri's rule (1.2). The same holds true for the terms B, C, and D. In order to keep track of the four different parts that occur in the above decomposition of (l, m), we will fill each part with a number corresponding to the order in which we will multiply the parts. The diagram below gives the lengths of the parts for a term in A.



Thus we have

$$A = \sum_{r=0}^{l} s_r \; s_{k-r} \; s_{m-k+r} \; s_{l-r} \tag{2.2}$$

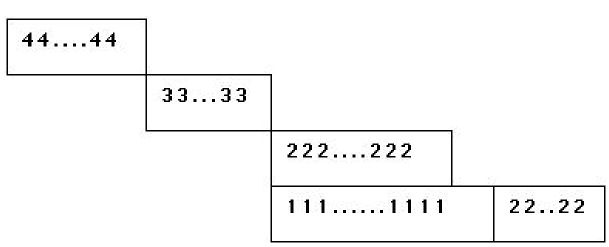
and

$$B = \sum_{r=0}^{l-1} s_r \; s_{k-r} \; s_{m+1-k+r} \; s_{l-1-r} \tag{2.3}$$

Under each Schur function we have placed the numbered part to which it corresponds. The expansion for C (resp. D) is obtainable from the expansion of A (resp. B) by replacing k by $k+1$.

From now on, we will examine the shapes (and their associated fillings with the numbers $1, 2, 3, 4$) that index the Schur functions which result from performing the multiplications in each term of (2.2) and (2.3). Thus we will let **A**,-**B**,-**C** and **D** represent the sets of these configurations in A, B, C, and D respectively. The minus signs in front of **B** and **C** are to remind us that those shapes have an associated sign of -1.

Pieri's rule tells us that if we multiply the part filled with 2's with the part filled with 1's then the 2's cannot form a column of height greater than 1. Also, they must fall on top of or to the right of the row filled with 1's. So after this first multiplication, we have shapes like



We now want to define an involution

$$I : \begin{cases} \mathbf{A} \longleftrightarrow -\mathbf{B} \\ -\mathbf{C} \longleftrightarrow \mathbf{D} \end{cases}$$

which will pair two identical shapes with opposite signs. For a fixed r, shapes in **A** have $l-r$ 1's and $m-k+r$ 2's while shapes in -**B** have $l-r-1$ 1's and $m-k+r+1$

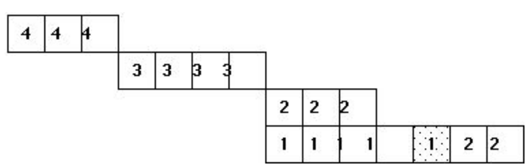
2's. A similar relationship holds for **-C** and **D**. Thus our involution will have to change a 1 to a 2.

With this in mind, for a configuration S in $\mathbf{A}$ or $-\mathbf{C}$ let S_1 be the configuration obtained from S by changing the last 1 in the bottom row of S to a 2. For a configuration T in $-\mathbf{B}$ or $\mathbf{D}$ let T_2 be the configuration obtained from T by changing the first 2 in the bottom row of T to a 1. Note that S_1 may violate Pieri's rule, meaning that it may have a column of height 2 filled with 2's. Also note that if T has no 2's in its bottom row, then T_2 cannot be made. However, forming T_2 will never cause a violation of Pieri's rule because there are no 1's in the second row of T. So define $\mathbf{A}_1$ and $-\mathbf{C}_1$ to be those configurations S in $\mathbf{A}$ and $-\mathbf{C}$ for which S_1 can be formed without violating Pieri's rule and let $\mathbf{A}_2$ and $-\mathbf{C}_2$ be those configurations S in $\mathbf{A}$ and $-\mathbf{C}$ for which S_1 violates Pieri's rule. Break up $-\mathbf{B}$ and $\mathbf{D}$ in a similar manner: $-\mathbf{B} = -\mathbf{B}_1 \cup -\mathbf{B}_2$ where $-\mathbf{B}_1$ consists of those configurations T in $-\mathbf{B}$ for which T_2 can be formed and $-\mathbf{B}_2$ consists of those configurations T in $-\mathbf{B}$ for which T_2 cannot be formed. Define $\mathbf{D}_1$ and $\mathbf{D}_2$ analogously.

Then the involution is

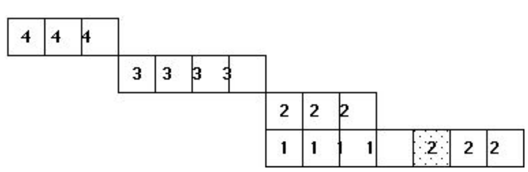
$$I(S) = \begin{cases} S_1 \text{ if } S \in \ \mathbf{A}_1 \text{ or } -\mathbf{C}_1 \\ S_2 \text{ if } S \in \ -\mathbf{B}_1 \text{ or } \mathbf{D}_1 \\ S \text{ otherwise} \end{cases} \tag{2.4}$$

I^2 is clearly the identity and I pairs off a configuration in $\mathbf{A}_1$ (resp. $-\mathbf{C}_1$) with a configuration of the same shape in $-\mathbf{B}_1$ (resp. $\mathbf{D}_1$) thus canceling their associated Schur functions in (2.1). For example, I pairs off the following two configurations, with the shaded box indicating the number changed by the involution:



configuration in $\mathbf{A}_1$ or $-\mathbf{C}_1$

is paired with

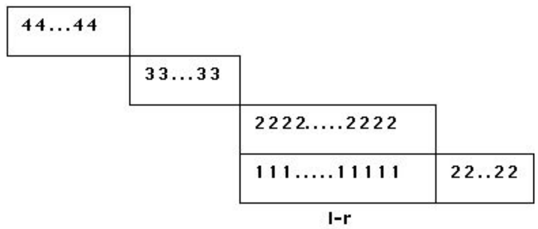


configuration in $-\mathbf{B}_1$ or $\mathbf{D}_1$

Thus we have

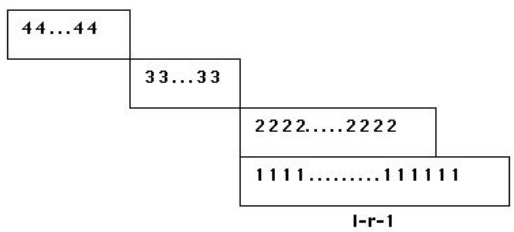
$$s_{(h,k)} \otimes s_{(l,m)} = \sum_{T_a \in \mathbf{A}_2} s_{sh(T_a)} - \sum_{T_b \in -\mathbf{B}_2} s_{sh(T_b)} - \sum_{T_c \in -\mathbf{C}_2} s_{sh(T_c)} + \sum_{T_d \in \mathbf{D}_2} s_{sh(T_d)}$$

where for any skew tableaux T, $sh(T)$ denotes the shape of T. We need to study these configurations fixed by I. We have already stated that a configuration in $-\mathbf{B}_2$ or $\mathbf{D}_2$ has no 2's in its bottom row. For a configuration in $\mathbf{A}_2$ or $-\mathbf{C}_2$, changing the last 1 in the bottom row to a 2 causes a column of height 2 filled with 2's to be formed. This means that if the first 2 in the bottom row is in position i then the last 2 in the next row up must be in position $i-1$. When this happens we will say the 2's "meet at the corner". Note that this condition includes the case when the number of 1's in the first row equals the number of 2's in the second row and there are no 2's in the first row. Thus the configurations fixed by I must look like



configuration in $\mathbf{A}_2$ or $-\mathbf{C}_2$

and



configuration in $-\mathbf{B}_2$ or $\mathbf{D}_2$

Remember that $\mathbf{A}_2$ configurations have $l-r$ 1's and $m-k+r$ 2's while $-\mathbf{B}_2$ configurations have $l-1-r$ 1's and $m-k+r+1$ 2's. By looking at the above configurations one sees that a necessary condition for a configuration to be in $\mathbf{A}_2$ is that $l-r \le m-k+r$. Similarly for a configuration to be in $-\mathbf{B}_2$, it must be that $m-k+r+1 \le l-1-r$. So if in A we remove all terms involving Schur functions cancelled by I, and if we let the index of summation be r' instead of r, we have

$$A = \sum_{r'=0}^{l} \chi(l-r' \le m-k+r') s_{(l-r',m-k+r')} \overset{s_{r'}}{\boxed{44..44}} \overset{s_{k-r'}}{\boxed{33..33}} ,$$

where for a statement S, $\chi(S)$ equals 1 if S is true and 0 if S is false. Let $r = l - r'$. Then

$$A = \sum_{r=0}^{l} \chi(r \le m+l-k-r) s_{(r,m+l-k-r)} \overset{s_{l-r}}{\boxed{44..44}} \overset{s_{k+r-l}}{\boxed{33..33}} .$$

But since $h+k = l+m = n$, $m+l-k = h$. So A simplifies to

$$A = \sum_{r=0}^{l} \chi(r \le h-r) s_{(r,h-r)} \overset{s_{l-r}}{\boxed{44..44}} \overset{s_{k+r-l}}{\boxed{33..33}} . \tag{2.5}$$

Similarly for B, by removing all Schur functions cancelled by I, and with an index of summation of r', we have

$$B = \sum_{r'=0}^{l-1} \chi(m-k+r'+1 \le l-1-r') s_{(m-k+r'+1,l-1-r')} \overset{s_{r'}}{\boxed{44..44}} \overset{s_{k-r'}}{\boxed{33..33}} .$$

By letting $r = l - 1 - r'$ and again noting that $l + m - k = h$,we have

$$B = \sum_{r=0}^{l-1} \chi(h - r \le r) s_{(h-r,r)} \underset{\boxed{44..44}}{s_{l-r-1}} \underset{\boxed{33..33}}{s_{k+r+1-l}}. \tag{2.6}$$

By replacing h by $h - 1$ and k by $k + 1$ in A and B, we get

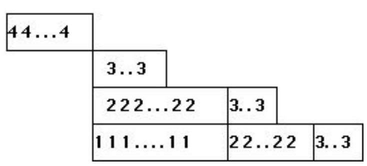
$$C = \sum_{r=0}^{l} \chi(r \le h - 1 - r) s_{(r,h-1-r)} \underset{\boxed{44..44}}{s_{l-r}} \underset{\boxed{33..33}}{s_{k+r-l+1}}, \tag{2.7}$$

$$D = \sum_{r=0}^{l-1} \chi(h - 1 - r \le r) s_{(h-1-r,r)} \underset{\boxed{44..44}}{s_{l-r-1}} \underset{\boxed{33..33}}{s_{k+r+2-l}}. \tag{2.8}$$

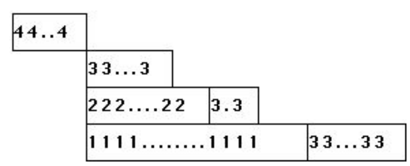
With these expansions, $s_{(h,k)} \otimes s_{(l,m)} = A - B - C + D$.

Now we multiply the Schur function indexed by the row filled with 3's and the Schur function indexed by the two-row shape that occur in A, B, C and D above. Again Pieri's rule tells us that the 3's cannot form a column of height greater than one. Also, the 3's must fall on top of the 1's and 2's that are already making the configuration so we cannot have a 3 falling beneath a 1 or a 2.

Thus our configurations in $\mathbf{A}_2$, $-\mathbf{B}_2$, $-\mathbf{C}_2$ and $\mathbf{D}_2$ look like



configuration in $\mathbf{A}_2$ or $-\mathbf{C}_2$



configuration in $-\mathbf{B}_2$ or $\mathbf{D}_2$

We now want to perform another involution on these configurations. Configurations in $\mathbf{A}_2$ have $h-r$ 2's and $k+r-l$ 3's while $-\mathbf{C}_2$ configurations have $h-r-1$ 2's and $k+r-l+1$ 3's. A similar relation exists between the number of 2's and 3's in $-\mathbf{B}_2$ and $\mathbf{D}_2$ configurations. Thus our involution will involve changing a 2 to a 3 and vice versa. This will have the effect of pairing off identical shapes in $\mathbf{A}_2$ and $-\mathbf{C}_2$ as well as in $-\mathbf{B}_2$ and $\mathbf{D}_2$. As in our first involution, we will encounter configurations in which this type of switch is impossible because it results in a configuration which violates Pieri's rule or because there is no 3 to change to a 2. We now define the involutions.

For a configuration S in $\mathbf{A}_2$ define S' to be the configuration obtained from S by changing the last 2 in the bottom row of S to a 3. As before, we break $\mathbf{A}_2$ into two subsets: $\mathbf{A}_{2,1}$, which consists of those configurations for which S' does not violate Pieri's rule, and $\mathbf{A}_{2,2}$, those configurations for which S' does violate Pieri's rule.

For a configuration S in $-\mathbf{C}_2$ define S'' to be the configuration obtained from S by changing the first 3 in the bottom row of S to a 2. Let $-\mathbf{C}_{2,1}$ be those configurations for which it is possible to form S'', and let $-\mathbf{C}_{2,2}$ be those configurations for which it is impossible.

For a configuration T in $-\mathbf{B}_2$ let T^* be the configuration obtained from T by changing the last 2 in the second row (starting at the bottom) of T to a 3. Let

$-\mathbf{B}_{2,1}$ be those configurations for which T^* does not violate Pieri's rule and $-\mathbf{B}_{2,2}$ be those configurations for which it does.

Lastly, for a configuration T in $\mathbf{D}_2$ let T^{**} be the configuration obtained from T by changing the first 3 in the second row of T to a 2. Let $\mathbf{D}_{2,1}$ consist of those configurations for which T^{**} can be formed and $\mathbf{D}_{2,2}$ consist of those configurations for which T^{**} cannot be formed.

For a configuration T, we define two involutions:

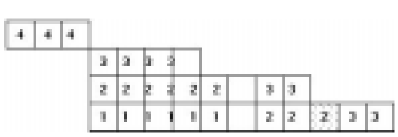
$$I_1 : \mathbf{A}_2 \longleftrightarrow -\mathbf{C}_2 \text{ given by } I_1(T) = \begin{cases} T' \text{ if } T \in \mathbf{A}_{2,1} \\ T'' \text{ if } T \in -\mathbf{C}_{2,1} \\ T \text{ otherwise} \end{cases}$$

and

$$I_2 : -\mathbf{B}_2 \longleftrightarrow \mathbf{D}_2 \text{ given by } I_2(T) = \begin{cases} T^* \text{ if } T \in -\mathbf{B}_{2,1} \\ T^{**} \text{ if } T \in \mathbf{D}_{2,1} \\ T \text{ otherwise} \end{cases}$$

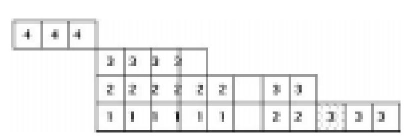
It is clear that both I_1^2 and I_2^2 are the identity. For T in $\mathbf{A}_2$, unless there is a violation of Pieri's rule, T' will be in $-\mathbf{C}_2$ because we increased the number of 3's by one and decreased the number of 2's by one. For T in $-\mathbf{C}_2$, changing a 3 to a 2 cannot violate Pieri's rule or cause a larger number to have a smaller number on top of it. Thus T'' is an element of $\mathbf{A}_2$. Note that for I_2 we are working in the second row because the first row contains no 2's. For T in $-\mathbf{B}_2$, changing a 2 to a 3 may violate Pieri's rule, but that is the only possible problem. Since the configurations for which this happens are fixed by I_2, T^* is in $\mathbf{D}_2$. For T in $\mathbf{D}_2$ the only possible problem in changing a 3 to a 2 is if there are no 3's to change. In making this change, Pieri's rule will not be violated nor will a bigger number have a smaller number on top of it. Hence T^{**} is an element of $-\mathbf{B}_2$ and both I_1 and I_2 are involutions on the sets given in the definition.

As an example, the following shapes are paired by these involutions.

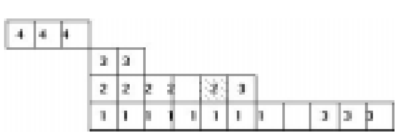


configuration in $\mathbf{A}_{2,1}$

$\overset{I_1}{\leftrightarrow}$

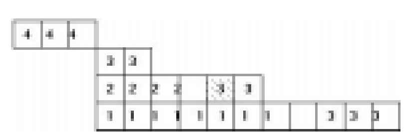


configuration in $-\mathbf{C}_{2,1}$



configuration in $-\mathbf{B}_{2,1}$

$\overset{I_2}{\leftrightarrow}$

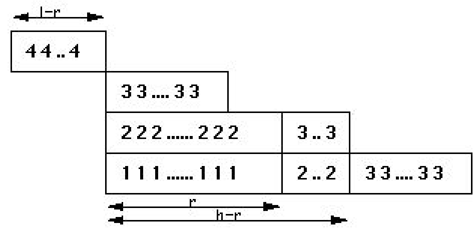


configuration in $\mathbf{D}_{2,1}$

As in the case of the first involution, I_1 pairs off identical shapes in $\mathbf{A}_{2,1}$ and $-\mathbf{C}_{2,1}$ which corresponds to canceling their Schur functions in $A - C$. Likewise, I_2 corresponds to canceling Schur functions indexed by the same shape in $-B + D$. Thus we have

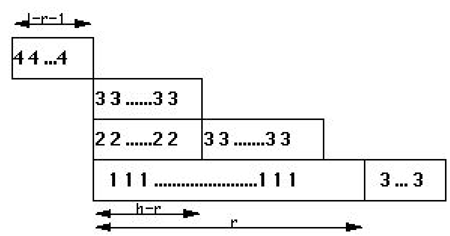
$$s_{(h,k)} \otimes s_{(l,m)} = \sum_{T_1 \in \mathbf{A}_{2,2}} s_{sh(T_1)} - \sum_{T_2 \in -\mathbf{B}_{2,2}} s_{sh(T_2)} - \sum_{T_3 \in -\mathbf{C}_{2,2}} s_{sh(T_3)} + \sum_{T_4 \in \mathbf{D}_{2,2}} s_{sh(T_4)}.$$

So we need to determine what configurations in these sets look like. For T in $\mathbf{A}_{2,2}$: Since T was fixed by involution I, the 2's in the bottom row and second row of T must meet at the corner. Also since we cannot change a 2 in the bottom row to a 3, the 3's in the bottom row and second row must meet at the corner. Thus T looks like:



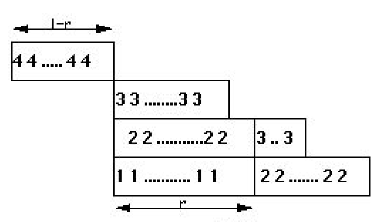
configuration in $\mathbf{A}_{2,2}$ (2.9)

For T in $-\mathbf{B}_{2,2}$: Since T was fixed by involution I, it cannot have 2's in the bottom row. Since it was fixed by I_2, a 2 in the second row could not be changed to a 3 and so the 3's in rows two and three must meet at the corner. So we have T like



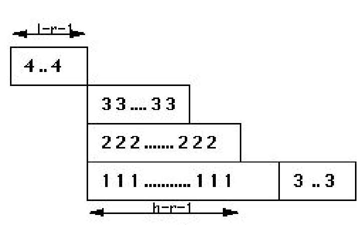
configuration in $-\mathbf{B}_{2,2}$ (2.10)

For T in $-\mathbf{C}_{2,2}$: As in the case for $\mathbf{A}_{2,2}$, since T is fixed by I, the 2's in the bottom and second row must meet at the corner. Also because of I_1, there must be no 3's in the bottom row. Hence T is of the form



configuration in $-\mathbf{C}_{2,2}$ (2.11)

For T in $\mathbf{D}_{2,2}$: As in the case for $-\mathbf{B}_{2,2}$, T has no 2's in the bottom row. And since T is fixed by I_2 there must be no 3's in row two. So T looks like



configuration in $\mathbf{D}_{2,2}$ (2.12)

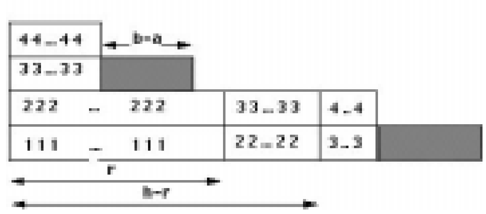
Let $\hat{\nu}$ be the shape of a configuration in $\mathbf{A}_{2,2}$. According to Pieri's rule (1.2) when the 4's are added to $\hat{\nu}$, they must form a horizontal strip of length $l-r$. Only those configurations whose final shape is ν and whose skew shape $\nu/\hat{\nu}$ is a horizontal strip of length $l-r$ will contribute to the coefficient $g_{(h,k)(l,m)\nu}$. A similar conclusion holds true for $-\mathbf{B}_{2,2}$, $-\mathbf{C}_{2,2}$, and $\mathbf{D}_{2,2}$. This observation and the above involutions lead us to the following

(2.2) Theorem. *With $l \leq h$ and $h+k = l+m = n$,*

$$\begin{aligned} g_{(h,k)(l,m)\nu} = &\textit{ the number of } \mathbf{A}_{2,2} \textit{ configurations } C \textit{ such that } \nu/sh(C) \\ &\qquad\qquad \textit{is an horizontal } l-r \textit{ strip} \\ &- \textit{the number of } -\mathbf{B}_{2,2} \textit{ configurations } C \textit{ such that } \nu/sh(C) \\ &\qquad\qquad \textit{is a horizontal } l-r-1 \textit{ strip} \\ &- \textit{the number of } -\mathbf{C}_{2,2} \textit{ configurations } C \textit{ such that } \nu/sh(C) \\ &\qquad\qquad \textit{is a horizontal } l-r \textit{ strip} \\ &+ \textit{the number of } \mathbf{D}_{2,2} \textit{ configurations } C \textit{ such that } \nu/sh(C) \\ &\qquad\qquad \textit{is a horizontal } l-r-1 \textit{ strip} \end{aligned}$$

We now are in a position to calculate the coefficient $g_{(h,k)(l,m)\nu}$. This will be accomplished by counting the number of $\mathbf{A}_{2,2}$, $-\mathbf{B}_{2,2}$, $-\mathbf{C}_{2,2}$, and $\mathbf{D}_{2,2}$ configurations which result in a final shape of ν. To this end, let $\nu = (a,b,c,d)$. We will count the number of diagrams of shape ν in a given set by first filling ν with all its 1's and 2's, and then filling in all 3's and 4's whose placements are forced by the two involutions and the shape ν. We then count the number of ways of filling in the rest of the shape with the remaining 3's and 4's so as to obtain a legal diagram.

Consider a configuration T in $\mathbf{A}_{2,2}$. Recall that the number of 1's in T is r. We have two possibilities: $r < b$ or $r \geq b$. We will examine each case separately.
Case 1: $T \in \mathbf{A}_{2,2}$, $r \geq b$



Because of (2.5) we need that $r \leq l$. In order to insure the filling as pictured above, we also require that $r \leq h-r$, which gives $r \leq \lfloor \frac{h}{2} \rfloor$. Because our configuration was fixed by the second involution, the 3's meet at the corner in the first and second rows. For this to happen, we must have $h-r \leq c$, or $r \geq h-c$. We also need enough 3's to fill in the forced parts of the diagram. In other words, the number of placed 3's must be less than or equal to the total number of 3's. We have placed $a+(h-(h-r))+(c-(h-r))$ 3's so far. We have $n-$ (the total number of 1's, 2's and 4's) $= n-h-(l-r)$ 3's altogether. Thus we require $a+c-r \leq n-h-l+r$ which gives $\lceil \frac{l+h+a+c-n}{2} \rceil \leq r$. We also must insure that we have enough 4's to fill the diagram as pictured above. This gives $a+(c-(h-r)) \leq l-r$, or $r \leq \lfloor \frac{l+h-a-c}{2} \rfloor$. We are now ready to count the number of ways of filling the shaded rows above so as to produce a valid configuration. Note that once we decide how many 4's to place in the shaded area of length $b-a$, the placement of the remaining 3's and 4's is entirely determined: we must fill the rest of the shaded area with 3's (placed to the left of the 4's) and all remaining numbers fill the shaded area in the first row, again with the 3's to the left of the 4's. We can place from 0 up to the minimum of the number of unused 4's and $b-a$, the length to be filled. Thus we have $1+\min(b-a, l-r-a-(c-(h-r)))$ ways of filling the configuration. Thus defining $g_\nu(A,1)$ to be the number of configurations in $\mathbf{A}_{2,2}$ of shape $\nu=(a,b,c,d)$ with $r \geq b$ we have

$$g_\nu(A,1) = \sum_{r=\max(b,h-c,\lceil \frac{l+h+a+c-n}{2} \rceil)}^{\min(l,\lfloor \frac{h}{2} \rfloor, \lfloor \frac{l+h-a-c}{2} \rfloor)} 1+\min(b-a, l+h-a-c-2r). \tag{2.13}$$

We note that all the other cases that we shall analyze to count the number of $\mathbf{A}_{2,2}$, $-\mathbf{B}_{2,2}$, $-\mathbf{C}_{2,2}$, and $\mathbf{D}_{2,2}$ configurations which result in a final shape of ν will follow the same pattern. That is, there are a number of inequalites on r in the resulting sum which we shall classify as follows.

(I) The basic range of r plus any extra assumptions.
In case 1, our basic range of r is $0 \leq r \leq l$ and our extra assumption is that $b \leq r$.

(II) Conditions on the relative lengths in the diagram.
In case 1, this resulted in two inequalites, namely,
$h-r \leq c \Rightarrow h-c \leq r$, and
$r \leq h-r \Rightarrow r \leq \lfloor \frac{h}{2} \rfloor$.

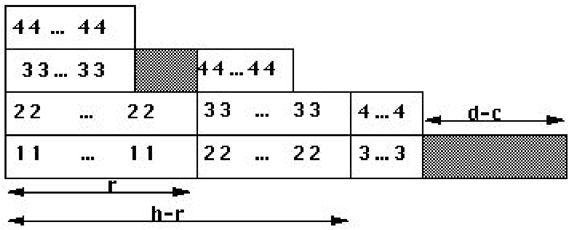
(III) Conditions that there are enough 3's to fill the forced 3's in the diagram.
In case 1, this resulted in the inequality
$a+((h-r)-r)+(c-(h-r)) \leq n-h-(l-r) \Rightarrow \lceil \frac{l+h+a+c-n}{2} \rceil \leq r$.

(IV) Conditions that there are enough 4's to fill the forced 4's in the diagram.
In case 1, this resulted in the inequality
$a+(c-(h-r)) \leq l-r \Rightarrow r \leq \lfloor \frac{l+h-a-c}{2} \rfloor$.

Finally for fixed r, we are left to place the free 4's in the diagram. In case 1, we keep track of the amount of free 4's in the shaded region whose length is marked. The marked region is of length $b-a$ in this case and the number of free 4's is $l+h-a-c-2r$ which leads to the summand $1+\min(b-a, l+h-a-c-2r)$.

Rather than give a detailed agument for the rest of the cases, we shall simply draw the picture as in case 1, give the inequalities of type (I)–(IV), list under (V) the number of free 4's for a given r, and then give the final formula for the number of configurations of shape $\nu=(a,b,c,d)$.

Case 2: $T \in \mathbf{A}_{2,2}$, $r < b$.

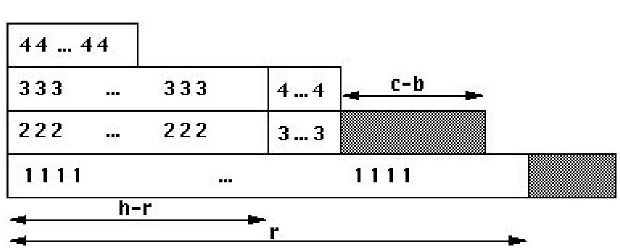


(I)
$0 \leq r \leq l$,
$r \leq b-1$.
(II)
$a \leq r$,
$h-r \leq c \Rightarrow h-c \leq r$,
$r \leq h-r \Rightarrow r \leq \lfloor \frac{h}{2} \rfloor$,
$b \leq h-r \Rightarrow r \leq h-b$.
(III)
$a+((h-r)-r)+(c-(h-r)) \leq n-h-(l-r) \Rightarrow \lceil \frac{l+h+a+c-n}{2} \rceil \leq r$.
(IV)
$a+(c-(h-r))+(b-r) \leq l-r \Rightarrow r \leq l+h-a-b-c$.
(V) $l+h-a-b-c-r$.

$$g_\nu(A,2) = \sum_{r=\max(a,h-c,\lceil \frac{l+h+a+c-n}{2} \rceil)}^{\min(l,\lfloor \frac{h}{2} \rfloor,b-1,h-b,l+h-a-b-c)} 1+\min(d-c,l+h-a-b-c-r). \tag{2.14}$$

We now give the contribution to $g_{(h,k)(l,m)\nu}$ of the remaining sets of configurations. The quantities $g_\nu(B,i), g_\nu(C,i), g_\nu(D,i)$ with $i=1,2$ are defined in a manner analogous to $g_\nu(A,1)$ and $g_\nu(A,2)$.
Diagrams in $-\mathbf{B}_{2,2}$.

Case 1: $T \in -\mathbf{B}_{2,2}$, $r \geq c$

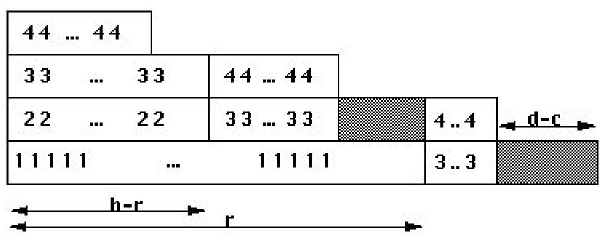


(I)
$0 \leq r \leq l-1$,
$c \leq r$.
(II)
$a \leq h-r \Rightarrow r \leq h-a$.

$h-r \le b \Rightarrow h-b \le r$,
$h-r \le r \Rightarrow \lfloor \frac{h}{2} \rfloor \le r$.
(III)
$b \le n-h-(l-1-r) \Rightarrow l+h+b-n-1 \le r$.
(IV)
$a+(b-(h-r)) \le l-1-r \Rightarrow r \le \lfloor \frac{l+h-a-b-1}{2} \rfloor$.
(V) $l+h-a-b-2r-1$.

$$g_\nu(B,1) = \sum_{r=\max(c,\lceil \frac{h}{2} \rceil, h-b, l+h+b-n-1)}^{\min(l-1,h-a,\lfloor \frac{l+h-a-b-1}{2} \rfloor)} 1+\min(c-b, l+h-a-b-2r-1). \quad (2.15)$$

Case 2: $T \in -\mathbf{B}_{2,2}$, $r < c$

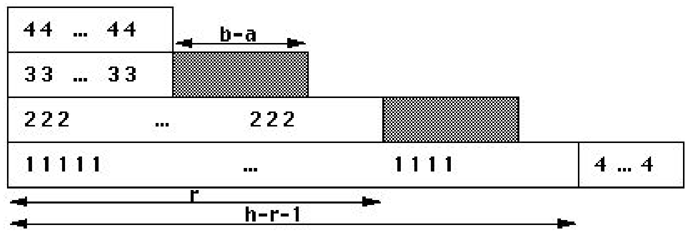


(I)
$0 \le r \le l-1$,
$r \le c-1$.
(II)
$b \le r$,
$a \le h-r \Rightarrow r \le h-a$,
$h-r \le b \Rightarrow h-b \le r$,
$h-r \le r \Rightarrow \lfloor \frac{h}{2} \rfloor \le r$.
(III)
$b+(c-r) \le n-h-(l-1-r) \Rightarrow \lceil \frac{l+h+b+c-n-1}{2} \rceil \le r$.
(IV)
$a+(b-(h-r))+(c-r) \le l-1-r \Rightarrow r \le l+h-a-b-c-1$.
(V) $l+h-a-b-c-1-r$.

$$g_\nu(B,2) = \sum_{r=\max(\lceil \frac{h}{2} \rceil, h-b, b, \lceil \frac{l+h+b+c-n-1}{2} \rceil)}^{\min(l-1,c-1,h-a,l+h-a-b-c-1)} 1+\min(d-c, l+h-a-b-c-1-r). \quad (2.16)$$

Diagrams in $-\mathbf{C}_{2,2}$.

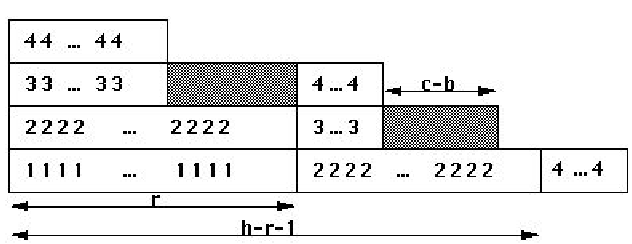
Case 1: $T \in -\mathbf{C}_{2,2}$, $r \geq b$



(I)
$0 \leq r \leq l$,
$b \leq r$.
(II)
$r \leq h-r-1 \Rightarrow r \leq \lfloor \frac{h-1}{2} \rfloor$,
$c \leq h-r-1 \Rightarrow r \leq h-c-1$.
(III)
$a \leq n-(h-1)-(l-r) \Rightarrow l+h+a-n-1 \leq r$.
(IV)
$a+(d-(h-r-1)) \leq l-r \Rightarrow r \leq \lfloor \frac{l+h+b+c-n-1}{2} \rfloor$.
(V) $l+h+b+c-n-1-2r$.

$$g_\nu(C,1) = \sum_{r=\max(b,l+h+a-n-1)}^{\min(l,\lfloor \frac{h-1}{2} \rfloor, h-c-1, \lfloor \frac{l+h+b+c-n-1}{2} \rfloor)} 1+\min(b-a, l+h+b+c-n-1-2r). \tag{2.17}$$

Case 2: $T \in -\mathbf{C}_{2,2}$, $r < b$



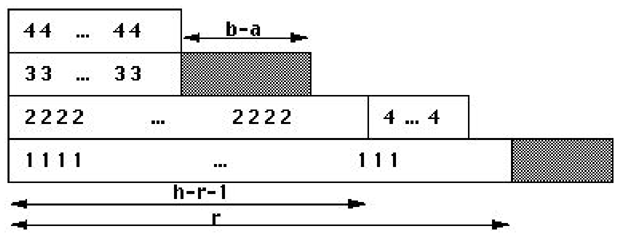
(I)
$0 \leq r \leq l$,
$r \leq b-1$.
(II)
$a \leq r$,
$r \leq h-r-1 \Rightarrow r \leq \lfloor \frac{h-1}{2} \rfloor$,
$c \leq h-r-1 \Rightarrow r \leq h-c-1$.
(III)
$a+(b-r) \leq n-(h-1)-(l-r) \Rightarrow \lceil \frac{l+h+a+b-n-1}{2} \rceil \leq r$.

(IV)
$a+(b-r)+(d-(h-r-1)) \leq l-r \Rightarrow r \leq l+h+c-n-1.$
(V) $l+h+c-n-1-r.$

$$g_\nu(C,2) = \sum_{r=\max(a,\lceil\frac{l+h+a+b-n-1}{2}\rceil)}^{\min(l,\lfloor\frac{h-1}{2}\rfloor,b-1,h-c-1,l+h+c-n-1)} 1+\min(c-b,l+h+c-n-1-r). \tag{2.18}$$

Diagrams in $\mathbf{D}_{2,2}$.

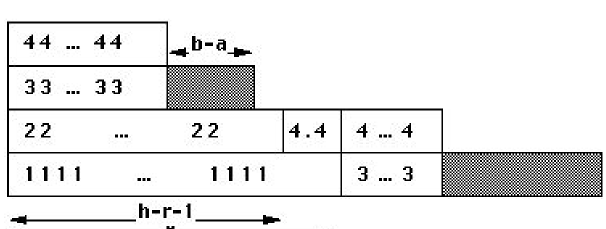
Case 1: $T \in\mathbf{D}_{2,2}$, $r \geq c$



(I)
$0 \leq r \leq l-1,$
$c \leq r.$
(II)
$h-r-1 \leq c \Rightarrow h-c-1 \leq r,$
$h-r-1 \leq r \Rightarrow \lfloor\frac{h-1}{2}\rfloor \leq r,$
$b \leq h-r-1 \Rightarrow r \leq h-b-1.$
(III)
$a \leq n-(h-1)-(l-1-r) \Rightarrow l+h+a-n-2 \leq r.$
(IV)
$a+(c-(h-r-1)) \leq l-1-r \Rightarrow r \leq \lfloor\frac{l+h-a-c-2}{2}\rfloor.$
(V) $l+h-a-c-2-2r.$

$$g_\nu(D,1) = \sum_{r=\max(c,\lceil\frac{h-1}{2}\rceil,h-c-1,l+h+a-n-2)}^{\min(l-1,h-b-1,\lfloor\frac{l+h-a-c-2}{2}\rfloor)} 1+\min(b-a,l+h-a-c-2r-2). \tag{2.19}$$

Case 2: $T \in\mathbf{D}_{2,2}$, $r < c$



(I)
$0 \le r \le l-1,$
$r \le c-1.$
(II)
$h-r-1 \le r \Rightarrow \lfloor \frac{h-1}{2} \rfloor \le r,$
$b \le h-r-1 \Rightarrow r \le h-b-1.$
(III)
$a+(c-r) \le n-h-(l-1-r) \Rightarrow \lceil \frac{l+h+a-c-n-2}{2} \rceil \le r.$
(IV)
$a+(c-(h-r-1)) \le l-1-r \Rightarrow r \le \lfloor \frac{l+h-a-c-2}{2} \rfloor.$
(V) $l+h-a-c-2-2r.$

$$g_\nu(D,2) = \sum_{r=\max(\lceil \frac{h-1}{2} \rceil, \lceil \frac{l+h+a+c-n-2}{2} \rceil)}^{\min(l-1,c-1,h-b-1,\lfloor \frac{l+h-a-c-2}{2} \rfloor)} 1+\min(b-a, l+h-a-c-2-2r). \tag{2.20}$$

Thus, recalling that configurations in $-\mathbf{B}_{2,2}$ and $-\mathbf{C}_{2,2}$ come with an associated minus sign, we have
$g_{(h,k)(l,m)(a,b,c,d)} =$

$$g_\nu(A,1)+g_\nu(A,2)-g_\nu(B,1)-g_\nu(B,2)-g_\nu(C,1)-g_\nu(C,2)+g_\nu(D,1)+g_\nu(D,2)$$

as stated. □

Notice that in light of (1.6) and (1.8) this also gives us a formula for $g_{\lambda\mu\nu}$ when two of the three partitions have either at most two parts or largest part of size two.

3 Special Cases

We now consider special values of $\nu = (a,b,c,d)$ and examine the coefficient $g_{(h,k)(l,m)\nu}$. First, let $a=b$ so that ν is a 4-part partition whose first two parts are equal. We consider the two sums for each of our four sets separately.
$\mathbf{A}_{2,2}$: Here with $a=b$, the sum (2.14) is empty, because the lower limit is of the form $\max(a,\ldots)$ and the upper limit is $\min(b-1,\ldots)$. Hence the contribution of configurations in $\mathbf{A}_{2,2}$ to the coefficient $g_{(h,k)(l,m)(a,a,c,d)}$ is

$$\begin{aligned} &\sum_{r=\max(a,h-c,\lceil \frac{l+h+a+c-n}{2} \rceil)}^{\min(l,\lfloor \frac{h}{2} \rfloor,\lfloor \frac{l+h-a-c}{2} \rfloor)} 1+\min(a-a, l+h-a-c-2r) \\ &= \sum_{r=\max(a,h-c,\lceil \frac{l+h+a+c-n}{2} \rceil)}^{\min(l,\lfloor \frac{h}{2} \rfloor,\lfloor \frac{l+h-a-c}{2} \rfloor)} 1. \end{aligned} \tag{3.1}$$

$-\mathbf{B}_{2,2}$: In this case, (2.16) has its lower limit of the form $\max(h-a,\ldots)$ and its upper limit of the form $\min(h-a,\ldots)$ and thus contains at most one term. Also, since $l \leq h$, $\lfloor (l+h-a-a-1)/2 \rfloor < \lfloor 2(h-a)/2 \rfloor = h-a$ so that (2.15) is empty. Hence, the contribution to $g_{(h,k)(l,m)(a,a,c,d)}$ is

$$(1+\min(c-a, l-a-c-1))\chi(\max(\lceil \frac{h}{2} \rceil, \lceil \frac{l+h+a+c-n-1}{2} \rceil, a)$$
$$\leq h-a \leq \min(c-1, l-1, l+h-2a-c-1))$$

$-\mathbf{C}_{2,2}$: Here, the sum given in (2.18) is empty because the lower limit is $r = \max(a, ...)$ and the upper limit is $\min(a-1, ...)$. Hence, the contribution of $-\mathbf{C}_{2,2}$ configurations to the Kronecker coefficient is

$$\sum_{r=\max(a, l+h+a-n-1)}^{\min(l, \lfloor \frac{h-1}{2} \rfloor, h-c-1, \lfloor \frac{l+h+a+c-n-1}{2} \rfloor)} 1. \tag{3.2}$$

$\mathbf{D}_{2,2}$: With $a = b$, (2.19) and (2.20) simplify to

$$\sum_{r=\max(\lceil \frac{h-1}{2} \rceil, c, h-c-1, l+h+a-n-2)}^{\min(l-1, h-a-1, \lfloor \frac{l+h-a-c-2}{2} \rfloor)} 1 + \sum_{r=\max(\lceil \frac{h-1}{2} \rceil, \lceil \frac{l+h+a+c-n-2}{2} \rceil)}^{\min(l-1, h-a-1, \lfloor \frac{l+h-a-c-2}{2} \rfloor, c-1)} 1.$$

However, we have that

$$\lfloor \frac{l+h-a-c-2}{2} \rfloor = \lfloor \frac{h-a-1+l-c-1}{2} \rfloor \leq \lfloor \frac{2(h-a-1)}{2} \rfloor = h-a-1,$$

$$l+h+a-n-2 = a+(l+h-n-2) \leq a \leq c \text{ since } l+h \leq n,$$

and

$$\lceil \frac{l+h+a+c-n-2}{2} \rceil = \lceil \frac{h-1+l+a+c-n-1}{2} \rceil \leq \lceil \frac{h-1}{2} \rceil$$
$$\text{since } l \leq \lfloor \frac{n}{2} \rfloor, a+c \leq \lfloor \frac{n}{2} \rfloor.$$

Thus, (3.2) becomes

$$\sum_{r=\max(\lceil \frac{h-1}{2} \rceil, c, h-c-1)}^{\min(l-1, \lfloor \frac{l+h-a-c-2}{2} \rfloor)} 1 + \sum_{r=\lceil \frac{h-1}{2} \rceil}^{\min(l-1, \lfloor \frac{l+h-a-c-2}{2} \rfloor, c-1)} 1 \tag{3.3}$$

Now, if $\lfloor (h-1)/2 \rfloor \geq c$, the second sum in (3.3) is empty, and (3.3) reduces to

$$\sum_{r=\max(\lceil \frac{h-1}{2} \rceil, h-c-1)}^{\min(l-1, \lfloor \frac{l+h-a-c-2}{2} \rfloor)} 1. \tag{3.4}$$

If $\lfloor (h-1)/2 \rfloor < c$, we have that $h-c-1 \leq \lfloor (h-1)/2 \rfloor$ and so (3.3) becomes

$$\sum_{r=c}^{\min(l-1, \lfloor \frac{l+h-a-c-2}{2} \rfloor)} 1 + \sum_{r=\lceil \frac{h-1}{2} \rceil}^{\min(l-1, \lfloor \frac{l+h-a-c-2}{2} \rfloor, c-1)} 1 = \sum_{r=\lceil \frac{h-1}{2} \rceil}^{\min(l-1, \lfloor \frac{l+h-a-c-2}{2} \rfloor)} 1 \tag{3.5}$$

Noting that, with $\lfloor (h-1)/2 \rfloor < c$, $\max(\lfloor (h-1)/2 \rfloor, h-c-1) = \lfloor (h-1)/2 \rfloor$, we see that (3.4) and (3.5) are identical, thus giving in both cases, the contribution of $\mathbf{D}_{2,2}$ configurations to $g_{(h,k)(l,m)(a,a,c,d)}$ to be

$$\sum_{r=\max(\lceil \frac{h-1}{2} \rceil, h-c-1)}^{\min(l-1, \lfloor \frac{l+h-a-c-2}{2} \rfloor)} 1 = \sum_{r=\max(\lceil \frac{h+1}{2} \rceil, h-c)}^{\min(l, \lfloor \frac{l+h-a-c}{2} \rfloor)} 1 \tag{3.6}$$

with the last equality obtained by adding one to each of the quantities in the upper and lower limits.

Hence we have

$$\begin{aligned} g_{(h,k)(l,m)(a,a,c,d)} = & \sum_{r=\max(a,h-c,\lceil \frac{l+h+a+c-n}{2} \rceil)}^{\min(l, \lfloor \frac{h}{2} \rfloor, \lfloor \frac{l+h-a-c}{2} \rfloor)} 1 - \sum_{r=\max(a, l+h+a-n-1)}^{\min(l, \lfloor \frac{h-1}{2} \rfloor, h-c-1, \lfloor \frac{l+h+a+c-n-1}{2} \rfloor)} 1 \\ & + \sum_{r=\max(\lceil \frac{h+1}{2} \rceil, h-c)}^{\min(l, \lfloor \frac{l+h-a-c}{2} \rfloor)} 1 \\ & - (1 + \min(c-a, l-a-c-1))\chi(V_1 \le h-a \le W_1) \end{aligned} \tag{3.7}$$

where $V_1 = \max(\lceil \frac{h}{2} \rceil, \lceil \frac{l+h+a+c-n-1}{2} \rceil, a)$ and $W_1 = \min(c-1, l-1, l+h-2a-c-1))$.

We now consider (3.7) under the two separate assumptions that $\lceil \frac{h+1}{2} \rceil \le h-c$ and $\lceil \frac{h+1}{2} \rceil > h-c$.

Case 1:$\lceil \frac{h+1}{2} \rceil \le h-c$.

This case is equivalent to $c \le \lfloor \frac{h}{2} \rfloor$ which shows that the last expression in (3.7) is zero. Because $\lfloor h/2 \rfloor + 1 = \lceil (h+1)/2 \rceil$, the first sum in (3.7) above is empty. Notice also that $h-c-1 \ge \lfloor \frac{h-1}{2} \rfloor$. Hence, we have

(3.1) Theorem. *When $a = b > 0$ and $\lceil \frac{h+1}{2} \rceil \le h-c$,*

$$g_{(h,k)(l,m)(a,a,c,d)} = \sum_{r=h-c}^{\min(l, \lfloor \frac{l-a+h-c}{2} \rfloor)} 1 - \sum_{r=\max(a, l+h+a-n-1)}^{\min(l, \lfloor \frac{h-1}{2} \rfloor, \lfloor \frac{l+h+a+c-n-1}{2} \rfloor)} 1.$$

□

Case 2:$\lceil \frac{h+1}{2} \rceil > h-c$.

This is equivalent to $c > \lfloor \frac{h}{2} \rfloor$. The first and last sums in (3.7) become

$$\begin{aligned} & \sum_{r=\max(a,h-c,\lceil \frac{l+h+a+c-n}{2} \rceil)}^{\min(l, \lfloor \frac{h}{2} \rfloor, \lfloor \frac{l+h-a-c}{2} \rfloor)} 1 + \sum_{r=\lceil \frac{h+1}{2} \rceil}^{\min(l, \lfloor \frac{l+h-a-c}{2} \rfloor)} 1 \\ = & \sum_{r=\max(a,h-c,\lceil \frac{l+h+a+c-n}{2} \rceil)}^{\min(l, \lfloor \frac{l-a+h-c}{2} \rfloor)} 1 \end{aligned}$$

which give the following

(3.2) Theorem. *With $a = b > 0$ and $\lceil \frac{h+1}{2} \rceil > h - c$,*

$$g_{(h,k)(l,m)(a,a,c,d)} = \sum_{r=\max(a,h-c,\lceil \frac{l+h+a+c-n}{2} \rceil)}^{\min(l,\lfloor \frac{l-a+h-c}{2} \rfloor)} 1 - \sum_{r=\max(a,l+h+a-n-1)}^{\min(l,h-c-1,\lfloor \frac{l+h+a+c-n-1}{2} \rfloor)} 1$$
$$- (1 + \min(c-a, l-a-c-1))\chi(X_1 \leq h-a \leq Y_1)$$

where $X_1 = \max(\lceil \frac{h}{2} \rceil, \lceil \frac{l+h+a+c-n-1}{2} \rceil)$ *and* $Y_1 = \min(c-1, l-1, l+h-a-c-1)$. □

We now consider the case where $a = b = 0$ so that $\nu = (c,d)$, a partition with only two parts. Thus we are computing $g_{(l,k)(h,k)(c,d)}$. Because of the symmetry of the Kronecker coefficients (1.9), there is no loss in generality in assuming that $l \leq h \leq c$. Thus we assume that $l \leq h \leq c$.

Under our assumptions, the index of summation in (2.14) and (2.18) ranges from a to $b-1$ and so (2.14) and (2.18) are empty. Both (2.15) and (2.16) will have $h \leq r \leq l-1$ and since by assumption, $l \leq h$, these sums are also empty. In (2.17), the upper limit is less than or equal to $h-c-1 \leq -1$ while the lower limit is at least 0. Thus (2.17) is empty. So, using the first expression in (3.6) for the contribution of $\mathbf{D}_{2,2}$, Theorem 2.1 simplifies to

$$g_{(h,k)(l,m)(c,d)} = \sum_{r=\max(0,h-c,\lceil \frac{l+h+c-n}{2} \rceil)}^{\min(l,\lfloor \frac{h}{2} \rfloor,\lfloor \frac{l+h-c}{2} \rfloor)} 1 + \sum_{r=\max(\lceil \frac{h-1}{2} \rceil,h-c-1)}^{\min(l-1,\lfloor \frac{l+h-c-2}{2} \rfloor)} 1 \tag{3.8}$$

Again because $l \leq h \leq c$, it is easy to see that (3.8) simplifies to

$$\sum_{r=\max(0,\lceil \frac{l+h+c-n}{2} \rceil)}^{\lfloor \frac{l+h-c}{2} \rfloor} 1 + \sum_{r=\lceil \frac{h-1}{2} \rceil}^{\lfloor \frac{l+h-c-2}{2} \rfloor} 1. \tag{3.9}$$

We now show that the second sum above is empty. For if $h = 2j+1$, the lower limit of this sum is j and the upper limit is $\lfloor (2j-1+l-c)/2 \rfloor \leq j-1$ since $l-c \leq 0$. If $h = 2j$, then this sum runs from j to $\lfloor (2j+2+l-c)/2 \rfloor = j-1+\lfloor (l-c)/2 \rfloor < j$. Thus in both cases, the second sum in (3.9) equals zero, giving the following

(3.3) Theorem. *With $l \leq h \leq c$,*

$$g_{(h,k)(l,m)(c,d)} = (1 + W_1 - V_1)\chi(W_1 \geq V_1)$$

where $V_1 = \max(0, \lceil \frac{l+h+c-n}{2} \rceil)$ *and* $W_1 = \lfloor \frac{l+h-c}{2} \rfloor$. □

We finish now with a formula for the maximum of $g_{(h,k)(l,m)\nu}$.

4 The Maximum of $g_{(h,k)(l,m)\nu}$.

We now want to find a maximum value for the coefficient $g_{(h,k)(l,m)\nu}$ and produce the partition which attains this maximum. In doing so, we will use the notation developed in Section 2.

We begin with a preliminary result.

(4.1) Proposition. *When $h > 2l-1$ and $m > 2h-2$,*

$$g_{(h,k)(l,m)\nu} = \sum_{r=\max(b,h-c,\lceil\frac{l+h+a+c-n}{2}\rceil)}^{\min(l,\lfloor\frac{h}{2}\rfloor,\lfloor\frac{l+h-a-c}{2}\rfloor)} 1+\min(b-a,l+h-a-c-2r)$$
$$+\sum_{r=\max(a,h-c,\lceil\frac{l+h+a+c-n}{2}\rceil)}^{\min(l,\lfloor\frac{h}{2}\rfloor,b-1,h-b,l+h-a-b-c)} 1+\min(d-c,l+h-a-b-c-r).$$

Proof. Let $h > 2l-1$. Looking at (2.6) and (2.8) one sees that $r \le l-1$. Putting these together gives that

$$h-r > 2l-1-r > 2l-1-(l-1) = l > r$$
$$\Rightarrow h-r > r.$$

But the condition for (2.6) and (2.8) to be nonempty is $h-r \le r$. Thus for $h > 2l-1$ both $-\mathbf{B}_{2,2}$ and $\mathbf{D}_{2,2}$ are empty.

Now consider a configuration in $-\mathbf{C}_{2,2}$. It contains $k+1+r-l$ 3's. From looking at such a configuration as illustrated in (2.11), one sees that the number of 3's that the configuration can contain is no greater than the length of the bottom row since the 3's must fall on top of the first and second rows. As seen in (2.11), the length of the bottom row is $h-r-1$. Since $k = l+m-h$, we have that in order for the configuration to exist,

$$\begin{aligned}\text{the number of 3's} &= k+1+r-l\\ &= l+m-h+1+r-l\\ &= m+1+r-h\\ &\le \text{ the length of the bottom row}\\ &= h-r-1\\ &\Longrightarrow m \le 2h-2r-2 \le 2h-2.\end{aligned}$$

Thus if $m > 2h-2$, $-\mathbf{C}_{2,2}$ will be empty. And if we specify that $h > 2l-1$ and $m > 2h-2$ only configurations in $\mathbf{A}_{2,2}$ will exist. Thus, under these two assumptions, we have

$$g_{(h,k)(l,m)\nu} = g_\nu(A,1) + g_\nu(A,2)$$

which gives the proposition. □

We now prove the following:

(4.2) Theorem. *With $h+k = l+m = n$ where $l \le h$,*

$$g_{(h,k)(l,m)(a,b,c,d)} \le (1+\lceil\frac{3l-3}{11}\rceil)(\left\lceil\frac{l-\lceil\frac{3l-3}{11}\rceil}{2}\right\rceil - \lceil\frac{3l-3}{11}\rceil)$$
$$+(1+l)(\left\lfloor\frac{l}{2}\right\rfloor - \left\lceil\frac{l-\lceil\frac{3l-3}{11}\rceil}{2}\right\rceil + 1) - 2\binom{\lfloor\frac{l}{2}\rfloor+1}{2}$$
$$+2\binom{\left\lceil\frac{l-\lceil\frac{3l-3}{11}\rceil}{2}\right\rceil}{2} + (1+l-\lceil\frac{3l-3}{11}\rceil)(\lceil\frac{3l-3}{11}\rceil) - \binom{\lceil\frac{3l-3}{11}\rceil}{2}.$$

This grows like $9l^2/44$ and can be expressed as a polynomial in s, where $l = 22s + t$, $0 \leq t \leq 21$. This maximum is attained when $\nu = (0, \lceil \frac{3l-3}{11} \rceil, h, d)$ with $m > 2h - 2$, $n >> 0$,$c >> l$, $d >> c, h$, and $h = c$.

Proof. In order to determine the maximum value of $g_{(h,k)(l,m)\nu}$, we examine the sums in Theorem (2.1). Of course because configurations in $-\mathbf{B}_{2,2}$ and $-\mathbf{C}_{2,2}$ come with an associated negative sign, we need only consider those sums arising from $\mathbf{A}_{2,2}$ and $\mathbf{D}_{2,2}$ to get an upper bound for $g_{(h,k)(l,m)\nu}$. For convenience, we give them below. Defining $g_\nu(A)$ and $g_\nu(D)$ to be the sums from $\mathbf{A}_{2,2}$ and $\mathbf{D}_{2,2}$ respectively, we have

$$g_\nu(A) = \sum_{r=\max(b,h-c,\lceil \frac{l+h+a+c-n}{2} \rceil)}^{\min(l,\lfloor \frac{h}{2} \rfloor,\lfloor \frac{l+h-a-c}{2} \rfloor)} 1 + \min(b-a, l+h-a-c-2r)$$

$$+ \sum_{r=\max(a,h-c,\lceil \frac{l+h+a+c-n}{2} \rceil)}^{\min(l,\lfloor \frac{h}{2} \rfloor,b-1,h-b,l+h-a-b-c)} 1 + \min(d-c, l+h-a-b-c-r) \quad (4.1)$$

and

$$g_\nu(D) = \sum_{r=\max(c,\lceil \frac{h-1}{2} \rceil,h-c-1,l+h+a-n-2)}^{\min(l-1,h-b-1,\lfloor \frac{l+h-a-c-2}{2} \rfloor)} 1 + \min(b-a, l+h-a-c-2-2r)$$

$$+ \sum_{r=\max(\lceil \frac{h-1}{2} \rceil,\lceil \frac{l+h+a+c-n-2}{2} \rceil)}^{\min(l-1,c-1,h-b-1,\lfloor \frac{l+h-a-c-2}{2} \rfloor)} 1 + \min(b-a, l+h-a-c-2-2r). \quad (4.2)$$

Looking at the upper and lower limits of all four sums above, one sees that they will be maximized when $a = 0$ Also, since a small value of n will only make the lower limit larger, there is no harm in assuming that n is large. We shall write $x >> y$ to indicate the x is much larger than y. Hence (4.1) and (4.2) simplify to

$$g_\nu(A) = \sum_{r=\max(b,h-c,)}^{\min(l,\lfloor \frac{h}{2} \rfloor,\lfloor \frac{l+h-c}{2} \rfloor)} 1 + \min(b, l+h-c-2r)$$

$$+ \sum_{r=\max(0,h-c)}^{\min(l,\lfloor \frac{h}{2} \rfloor,b-1,h-b,l+h-c-b)} 1 + \min(d-c, l+h-b-c-r) \quad (4.3)$$

and

$$g_\nu(D) = \sum_{r=\max(c,\lceil \frac{h-1}{2} \rceil,h-c-1)}^{\min(l-1,h-b-1,\lfloor \frac{l+h-c-2}{2} \rfloor)} 1 + \min(b, l+h-c-2-2r)$$

$$+ \sum_{r=\lceil \frac{h-1}{2} \rceil}^{\min(l-1,c-1,h-b-1,\lfloor \frac{l+h-c-2}{2} \rfloor)} 1 + \min(b, l+h-c-2-2r). \quad (4.4)$$

We now examine (4.4). If $c \leq \lceil (h-1)/2 \rceil$, then the second sum is empty and, letting the index of summation be r', $g_\nu(D)$ equals

$$\sum_{r'=\lceil \frac{h-1}{2} \rceil}^{\min(l-1,h-b-1,\lfloor \frac{l+h-c-2}{2} \rfloor)} 1+\min(b, l+h-c-2-2r'). \tag{4.6}$$

If $\lceil (h-1)/2 \rceil < c \leq \min(l-1, h-b-1, \lfloor (l+h-2-c)/2 \rfloor)$ then the two sums in (4.4) combine, again giving that $g_\nu(D)$ equals (4.5). Finally, if $\min(l-1, h-b-1, \lfloor (l+h-2-c)/2 \rfloor) < c$, the first sum in (4.4) is empty and the c in the upper limit of the second sum is unnecessary, and we have once more that $g_\nu(D)$ equals (4.5). Now set $r' = r-1$. (4.5) then becomes

$$\sum_{r=\lceil \frac{h+1}{2} \rceil}^{\min(l,h-b,\lfloor \frac{l+h-c}{2} \rfloor)} 1+\min(b, l+h-c-2r). \tag{4.7}$$

Notice that the summand in (4.6) is exactly the summand in (4.3) Consider the following two cases. Case 1: $\lfloor h/2 \rfloor < l, \lfloor (l+h-c)/2 \rfloor$ and $\lceil (h+1)/2 \rceil \leq h-b$. Here (4.3) and (4.6) give that

$$\begin{aligned} g_{(h,k)(l,m)\nu} \leq & \sum_{r=\max(b,h-c)}^{\lfloor \frac{h}{2} \rfloor} 1+\min(b, l+h-c-2r) \\ & + \sum_{r=\max(0,h-c)}^{\min(\lfloor \frac{h}{2} \rfloor, b-1, h-b, l+h-c-b)} 1+\min(d-c, l+h-b-c-r) \\ & + \sum_{r=\lceil \frac{h+1}{2} \rceil}^{\min(l,h-b,\lfloor \frac{l+h-c}{2} \rfloor)} 1+\min(b, l+h-c-2r). \end{aligned} \tag{4.7}$$

But the first and third sums in (4.7) combine since $\lfloor h/2 \rfloor + 1 = \lceil (h+1)/2 \rceil$, giving

$$\begin{aligned} g_{(h,k)(l,m)\nu} \leq & \sum_{r=\max(b,h-c)}^{\min(l,h-b,\lfloor \frac{l+h-c}{2} \rfloor)} 1+\min(b, l+h-c-2r) \\ & + \sum_{r=\max(0,h-c)}^{\min(\lfloor \frac{h}{2} \rfloor, b-1, h-b, l+h-c-b)} 1+\min(d-c, l+h-b-c-r) \\ \leq & \sum_{r=\max(b,h-c)}^{\min(l,\lfloor \frac{l+h-c}{2} \rfloor)} 1+\min(b, l+h-c-2r) \\ & + \sum_{r=\max(0,h-c)}^{\min(b-1,l+h-c-b)} 1+\min(d-c, l+h-b-c-r). \end{aligned} \tag{4.8}$$

Here the last inequality results from dropping the term $h-b$ from the maximum in the upper limit in the first sum and dropping the terms $\lfloor \frac{h}{2} \rfloor$ and $h-b$ from the maximum in the upper limit in the second sum. We note that dropping these terms can only increase the respective sums because the term $\lfloor \frac{l+h-c}{2} \rfloor$ in the upper limit of the first sum insures that the summand $1+\min(b, l+h-c-2r)$ in the first sum is always ≥ 0 and the term $l+h-c-b$ in the upper limit of the second sum insures that the summand $1+\min(d-c, l+h-b-c-r)$ in the second sum is alway ≥ 0.
Case 2: $\lfloor h/2 \rfloor \geq l, \lfloor (l+h-c)/2 \rfloor$ or $\lceil (h+1)/2 \rceil > h-b$.
In this case, (4.6) is empty and we have

$$\begin{aligned} g_{(h,k)(l,m)\nu} \leq & \sum_{r=\max(b,h-c)}^{\min(l,\lfloor \frac{h}{2} \rfloor, \lfloor \frac{l+h-c}{2} \rfloor)} 1+\min(b, l+h-c-2r) \\ & + \sum_{r=\max(0,h-c)}^{\min(l,\lfloor \frac{h}{2} \rfloor, b-1, h-b, l+h-c-b)} 1+\min(d-c, l+h-b-c-r) \\ & \leq \sum_{r=\max(b,h-c)}^{\min(l,\lfloor \frac{l+h-c}{2} \rfloor)} 1+\min(b, l+h-c-2r) \\ & + \sum_{r=\max(0,h-c)}^{\min(b-1,l+h-c-b)} 1+\min(d-c, l+h-b-c-r). \end{aligned} \tag{4.9}$$

We now want to maximize the last sums of (4.8) and (4.9). First, we may assume that $h-c \leq l$ because otherwise both sums are empty. Also, because of the lower limit on the sums, there is no advantage to having $h-c<0$. Thus we may assume that $0 \leq h-c \leq l$. With this restriction on $h-c$, there is no loss in assuming that $h, c >> b$. This given, and with $j = h-c$, (4.8) and (4.9) both simplify to

$$\sum_{r=\max(b,j)}^{\lfloor \frac{l+j}{2} \rfloor} 1+\min(b, l+j-2r) + \sum_{r=j}^{\min(b-1,l+j-b)} 1+\min(d-c, l+j-b-r). \tag{4.10}$$

Notice that both sums are zero if $b > l$ so we may assume that $b \leq l$. Also, since n is large, there is no harm in assuming that $d >> c$. Hence, (4.10) becomes

$$g_{(h,k)(l,m)\nu} \leq \sum_{r=\max(b,j)}^{\lfloor \frac{l+j}{2} \rfloor} 1+\min(b, l+j-2r) + \sum_{r=j}^{\min(b-1,l+j-b)} 1+l+j-b-r. \tag{4.11}$$

Now fix b and set $f(j)$ equal to the first sum on the righthand side of (4.11) and set $g(j)$ equal to the second sum on the righthand side of (4.11). Let $h(j) = f(j)+g(j)$. Note that by assumption $0 \leq j = h-c \leq l$. We claim that $h(j)$ reaches its maximum when $j=0$. We shall prove this by showing that $h(j+1)-h(j) \leq 0$ for all $0 \leq j \leq l-1$. We shall consider 3 cases.
Case 1. $j \geq b$
In this case, both $g(j+1)$ and $g(j)$ equal 0 since their sums are empty. Moreover,

we have

$$f(j+1) = \sum_{j+1}^{\lfloor \frac{l+j+1}{2} \rfloor} 1 + \min(b, l+j+1-2r)$$

$$f(j) = \sum_{j}^{\lfloor \frac{l+j}{2} \rfloor} 1 + \min(b, l+j-2r).$$

We now consider two subcases.

Subcase 1A. $l+j$ is even.
In this case

$$h(j+1) - h(j) = -(1+\min(b, l-j)) + \sum_{j+1}^{\lfloor \frac{l+j}{2} \rfloor} \min(b, l+j+1-2r) - \min(b, l+j-2r). \tag{4.12}$$

Now if $b \geq l-j$, then clearly the righthand side of (4.12) is less than or equal to

$$(\lfloor \frac{l+j}{2} \rfloor - j) - (1+l-j) = \lfloor \frac{l+j}{2} \rfloor - l - 1 \leq -1$$

since $j \leq l$. Next suppose that $l-j < b$. Then note that the summand in (4.12) is equal to 1 if and only if

$$l+j+1-2r \leq b \Rightarrow \left\lceil \frac{l+j+1-b}{2} \right\rceil \leq r. \tag{4.13}$$

Note that $l+j+1-b \geq l+j+1-(l-j+1) = 2j$ since we are assuming that $l-j < b$. Moreover $l+j+1-b = 2j$ if and only if $l-j = b-1$. Thus we have two more subcases to consider.
Subcase 1A.1. $\lceil \frac{l+j+1-b}{2} \rceil \geq j+1$.
In this case (4.12) becomes

$$\begin{aligned} h(j+1) - h(j) &= -(1+b) + \sum_{\lfloor \frac{l+j+1-b}{2} \rfloor}^{\lfloor \frac{l+j}{2} \rfloor} 1 \\ &= -(1+b) + \lfloor \frac{l+j}{2} \rfloor - (\lfloor \frac{l+j+1-b}{2} \rfloor - 1) \\ &= -b + \lfloor \frac{l+j}{2} \rfloor - (\lfloor \frac{l+j+1-b}{2} \rfloor). \end{aligned} \tag{4.14}$$

It is then easy to check that the final expression in (4.14) is ≤ -1 unless $l+j$ is even and b is odd in which case it is ≤ 0.

Subcase 1A.2. $\lceil\frac{l+j+1-b}{2}\rceil = j$.
In this case (4.12) becomes

$$\begin{aligned} h(j+1)-h(j) &= -(1+b)+\sum_{\lfloor\frac{l+j+1-b}{2}\rfloor}^{1+\lfloor\frac{l+j}{2}\rfloor} 1 \\ &= -(1+b)+\lfloor\frac{l+j}{2}\rfloor-(\lfloor\frac{l+j+1-b}{2}\rfloor) \qquad (4.15)\\ &= -b-1+\lfloor\frac{l+j}{2}\rfloor-(\lfloor\frac{l+j+1-b}{2}\rfloor). \end{aligned}$$

Note that the final expression in (4.15) is 1 less than the final expression in (4.14) so that the final expression in (4.15) is always ≤ -1.
Case 1B. $l+j$ is odd.
In this case,

$$\begin{aligned} h(j+1)-h(j) &= (1+\min(b,l+j+1-2(\lfloor\frac{l+j+1}{2}\rfloor)-(1+\min(b,l-j)) \\ &+\sum_{j+1}^{\lfloor\frac{l+j}{2}\rfloor}\min(b,l+j+1-2r)-\min(b,l+j-2r). \qquad (4.16) \end{aligned}$$

Note that since $l+j+1$ is even, the first term of (4.16) is just 1. Thus the $h(j+1)-h(j)$ is at most one more than what we found in Case 1A. However, because $l+j$ is odd, our analysis showed that we always found the difference of $h(j+1)$ and $h(j)$ to be ≤ -1 in Case 1A. Thus $h(j+1)-h(j)\leq 0$ in this case.

This completes Case 1 so that for the rest of the cases we may assume that $j\leq b-1$.
Case 2. $b\leq\lfloor(l+j)/2\rfloor$.
Note that this gives $l+j-b\geq l+j-\lfloor(l+j)/2\rfloor=\lceil(l+j)/2\rceil > b-1$. Thus

$$h(j)=\sum_{r=b}^{\lceil\frac{l+j}{2}\rceil}1+\min(b,l+j-2r)+\sum_{r=j}^{b-1}1+l+j-b-r.$$

Similarly

$$h(j+1)=\sum_{r=b}^{\lfloor\frac{l+j+1}{2}\rfloor}1+\min(b,l+j+1-2r)+\sum_{r=j+1}^{b-1}1+l+j+1-b-r.$$

Then

$$\begin{aligned} g(j+1)-g(j) &= -(1+l+j-b-1)+\sum_{r=j+1}^{b-1}1 \\ &= -(l+j+2-2b). \qquad (4.17) \end{aligned}$$

For $f(j+1)-f(j)$, we consider two cases.
Case (i). $l+j$ is even.
In this case

$$\begin{aligned} f(j+1)-f(j) &= \sum_{r=b}^{\lfloor \frac{l+j}{2} \rfloor} \min(b, l+j+1-2r) - \min(b, l+j-2r) \\ &\leq \sum_{r=b}^{\lfloor \frac{l+j}{2} \rfloor} 1 \\ &= \lfloor \frac{l+j}{2} \rfloor - b + 1. \end{aligned}$$

Case (ii). $l+j$ is odd.
In this case

$$\begin{aligned} f(j+1)-f(j) = 1 &+ \min(b, l+j+1-2\lfloor \frac{l+j+1}{2} \rfloor) + \sum_{r=b}^{\lfloor \frac{l+j}{2} \rfloor} \min(b, l+j+1-2r) \\ &- \min(b, l+j-2r) \\ \leq{}& 1 + \sum_{r=b}^{\lfloor \frac{l+j}{2} \rfloor} 1 \\ ={}& \lfloor \frac{l+j}{2} \rfloor - b + 2. \end{aligned}$$

Thus in either case,

$$\begin{aligned} (f(j+1)+g(j+1)) - (f(j)+g(j)) &\leq \lfloor \frac{l+j}{2} \rfloor - b + 2 - (l+j+2-2b) \\ &= b - \lceil \frac{l+j}{2} \rceil \leq 0. \end{aligned}$$

Case 3. $\lfloor \frac{l+j}{2} \rfloor < b$.
In this case the sums corresponding to $f(j+1)$ and $f(j)$ are empty. We then consider two subcases.
Subcase 3A. $\lfloor \frac{l+j+1}{2} \rfloor < b$.
In this case, $l+j+1-b \leq l+j+1-(1+\lfloor \frac{l+j}{2} \rfloor) = \lceil \frac{l+j}{2} \rceil - 1 \leq b-1$. Similarly $l+j-b \leq b-1$. Thus

$$h(j+1) = \sum_{j+1}^{l+j+1-b} 1 + l + j + 1 - b - r = \sum_{s=0}^{l-b} 1 + s$$

and

$$h(j) = \sum_{j}^{l+j-b} 1 + l + j - b - r = \sum_{s=0}^{l-b} 1 + s.$$

Thus $h(j+1)-h(j)=0$ in this case.
Subcase 3B. $\lfloor \frac{l+j}{2}\rfloor < \lfloor \frac{l+j+1}{2}\rfloor = b$.
In this case, $l+j+1$ must be even. Hence

$$\begin{aligned} h(j+1) &= \sum_{r=b}^{\lfloor \frac{l+j+1}{2}\rfloor} 1+\min(b,l+j+1-2r) + \sum_{r=j+1}^{\min(b-1,l+j+1-b)} 1+l+j+1-b-r \\ &= 1+\sum_{r=j+1}^{b-1} 1+l+j+1-b-r \end{aligned}$$

and

$$\begin{aligned} h(j) &= \sum_{r=b}^{\lfloor \frac{l+j}{2}\rfloor} 1+\min(b,l+j-2r) + \sum_{r=j}^{\min(b-1,l+j-b)} 1+l+j+1-b-r \\ &= \sum_{r=j}^{b-1} 1+l+j+1-b-r. \end{aligned}$$

Thus

$$\begin{aligned} h(j+1)-h(j) &= 1+(\sum_{r=j+1}^{b-1} 1) - (1+l+j-b-j) \\ &= 1+(b-1-j)-(1+l-b) \\ &= 2b-(l+j+1)=0 \end{aligned}$$

Thus in all cases we have shown that $h(j+1)-h(j)\le 0$. Hence the righthand side of (4.11) is maximized when $j=0$ so that we have proved that

$$g_{(h,k)(l,m)(a,b,c,d)} \le \sum_{r=b}^{\lfloor \frac{l}{2}\rfloor} 1+\min(b,l-2r) + \sum_{r=0}^{\min(b-1,l-b)} 1+l-b-r \qquad (4.18)$$

We now want to determine what value of b maximizes the above sum. To this end, let $H(b)$ equal the above two sums. Then again we consider two cases.
Case I. $b \le \lfloor l/2 \rfloor$.
In this case, it is easy to check that

$$\begin{aligned} H(b) &= \sum_{r=b}^{\lceil \frac{l-b}{2}\rceil -1} 1+b + \sum_{r=\lceil \frac{l-b}{2}\rceil} 1+l-2r + \sum_{r=0}^{b-1} 1+l-b-r \\ &= (1+b)(\left\lceil \frac{l-b}{2}\right\rceil - b) + (1+l)(\left\lfloor \frac{l}{2}\right\rfloor - \left\lceil \frac{l-b}{2}\right\rceil + 1) \qquad (4.19) \\ &- 2\left(\binom{\lfloor \frac{l}{2}\rfloor+1}{2}. \quad - \binom{\lceil \frac{l-b}{2}\rceil}{2}\right) + (1+l-b)(b) - \binom{b}{2} \end{aligned}$$

We claim that the maximum value of $H(b)$ for $b \leq \lfloor \frac{l}{2} \rfloor$ occurs when $b = \lceil (3l-3)/11 \rceil$. In fact, we shall prove that for all b, $H(b)$ is maximized when $b = \lceil (3l-3)/11 \rceil$. Thus replacing b by this value in (4.19) gives the result stated in Theorem (4.2). Our claim is easy to check directly when $l = 0, 1$, or 2. Thus assume that $l \geq 3$. Using (4.18) it is easy to see that

$$H(b+1) - H(b) = l - 3b - 1 - \min(b, l-2b) + \sum_{r=b+1}^{\lfloor \frac{l}{2} \rfloor} \min(b+1, l-2r) - \min(b, l-2r). \tag{4.20}$$

Note that the summand in the sum of (4.20) is 0 unless $b+1 \leq l-2r$ or equivalently unless $r \leq \lfloor \frac{l-b-1}{2} \rfloor$. Thus

$$H(b+1) - H(b) = l - 3b - 1 - \min(b, l-2b) + \sum_{r=b+1}^{\lfloor \frac{l-b-1}{2} \rfloor} 1. \tag{4.21}$$

We now consider three cases.
Case I1. $b \leq \lfloor \frac{l-3}{3} \rfloor$
In this case, $l - 2b \geq l - 2\lfloor \frac{l-3}{3} \rfloor \geq \lfloor \frac{l-3}{3} \rfloor \geq b$. Morover

$$\begin{aligned} b+1 \leq \lfloor \frac{l-b-1}{2} \rfloor &\Leftrightarrow b+1 \leq \frac{l-b-1}{2} \\ &\Leftrightarrow 3b \leq l-3 \\ &\Leftrightarrow b \leq \lfloor \frac{l-3}{3} \rfloor. \end{aligned} \tag{4.22}$$

Thus in this case, (4.21) becomes

$$H(b+1) - H(b) = l - 3b - b + (\lfloor \frac{l-b-1}{2} \rfloor - b) = \lfloor \frac{3l-11b-3}{2} \rfloor.$$

Now

$$\begin{aligned} \lfloor \frac{3l-11b-3}{2} \rfloor \leq 0 &\Leftrightarrow 3l - 11b - 3 < 1 \\ &\Leftrightarrow 3l - 3 \leq 11b \\ &\Leftrightarrow \lceil \frac{3l-3}{11} \rceil \leq b. \end{aligned}$$

Thus for $b \leq \lfloor \frac{l-3}{3} \rfloor$, $H(b)$ reaches its maximum when $b = \lceil \frac{3l-3}{11} \rceil$.
Case I2. $b \geq \lceil \frac{l}{3} \rceil$.
In this case, $b \geq \lceil \frac{l}{3} \rceil$ implies that $b \geq l - 2b$ and by (4.22) $b+1 > \lfloor \frac{l-b-1}{2} \rfloor$ so that (4.21) becomes

$$H(b+1) - H(b) = l - 3b - 1 - (l - 2b) = -b - 1 \leq 0.$$

Thus $H(b)$ is decreasing for $b \geq \lceil \frac{l}{3} \rceil$.

Case I3. $\lfloor\frac{l-3}{3}\rfloor < b < \lceil\frac{l}{3}\rceil$.
It is easy to check that this happens only if either (i) $l = 3k+1$ for some k and $b = k$ or (ii) $l = 3k+2$ for some k and $b = k$. In case (i), it is easy to check that $H(b+1) - H(b) = -k \leq 0$. Similarly in case (ii), $H(b+1) - H(b) = -k+1 \leq 0$ since we are assuming that $l \geq 3$ and hence $k \geq 1$. Case I3 combined with Case I2 shows that $H(b)$ is decreasing for $b > \lfloor\frac{l-3}{3}\rfloor$.

Thus the maximum of $H(b)$ occurs at $b = \lceil\frac{3l-3}{11}\rceil$ as claimed if $b \leq \lfloor\frac{l}{2}\rfloor$. Now with $l = 22s + t$, where $0 \leq t \leq 21$, $\lfloor\frac{l}{2}\rfloor$, $\lceil(3l-3)/11\rceil$, $l - \lceil(3l-3)/11\rceil$, and $\lceil(l - \lceil(3l-3)/11\rceil)/2\rceil$ simplify to polynomials in s. By plugging these into (4.19) and simplifying, we obtain the following polynomials which give the maximum value of $g_{(h,k)(l,m)\nu}$:

$t = 0,$	$g_{(h,k)(l,m)\nu} \leq 99s^2 + 17s + 1$	$t = 1,$	$g_{(h,k)(l,m)\nu} \leq 99s^2 + 26s + 1$
$t = 2,$	$g_{(h,k)(l,m)\nu} \leq 99s^2 + 35s + 3$	$t = 3,$	$g_{(h,k)(l,m)\nu} \leq 99s^2 + 44s + 5$
$t = 4,$	$g_{(h,k)(l,m)\nu} \leq 99s^2 + 53s + 7$	$t = 5,$	$g_{(h,k)(l,m)\nu} \leq 99s^2 + 62s + 9$
$t = 6,$	$g_{(h,k)(l,m)\nu} \leq 99s^2 + 71s + 13$	$t = 7,$	$g_{(h,k)(l,m)\nu} \leq 99s^2 + 80s + 16$
$t = 8,$	$g_{(h,k)(l,m)\nu} \leq 99s^2 + 89s + 20$	$t = 9,$	$g_{(h,k)(l,m)\nu} \leq 99s^2 + 98s + 24$
$t = 10,$	$g_{(h,k)(l,m)\nu} \leq 99s^2 + 107s + 29$	$t = 11,$	$g_{(h,k)(l,m)\nu} \leq 99s^2 + 116s + 34$
$t = 12,$	$g_{(h,k)(l,m)\nu} \leq 99s^2 + 125s + 39$	$t = 13,$	$g_{(h,k)(l,m)\nu} \leq 99s^2 + 134s + 45$
$t = 14,$	$g_{(h,k)(l,m)\nu} \leq 99s^2 + 143s + 52$	$t = 15,$	$g_{(h,k)(l,m)\nu} \leq 99s^2 + 152s + 58$
$t = 16,$	$g_{(h,k)(l,m)\nu} \leq 99s^2 + 161s + 65$	$t = 17,$	$g_{(h,k)(l,m)\nu} \leq 99s^2 + 170s + 73$
$t = 18,$	$g_{(h,k)(l,m)\nu} \leq 99s^2 + 179s + 81$	$t = 19,$	$g_{(h,k)(l,m)\nu} \leq 99s^2 + 188s + 89$
$t = 20,$	$g_{(h,k)(l,m)\nu} \leq 99s^2 + 197s + 98$	$t = 21,$	$g_{(h,k)(l,m)\nu} \leq 99s^2 + 206s + 107.$

Moreover by Proposition (4.1) and the above analysis, this maximum is attained when
$\nu = (0, \lceil\frac{3l-3}{11}\rceil, h, d)$ with $m > 2h-2$, $n >> 0$, $c >> l$, $d >> c, h$, and $h = c$. By replacing s by $l/22$, we have that $g_{(h,k)(l,m)\nu}$ grows at least like $9l^2/44$.

Case II. $b > \lfloor\frac{l}{2}\rfloor$.
In this case, the first summand of (4.18) is empty so that

$$\begin{aligned}
g_{(h,k)(l,m)(a,b,c,d)} \leq H(b) &= \sum_{r=0}^{l-b} 1 + l - b - r \\
&= \sum_{s=0}^{l-b} 1 + s \\
&= \binom{l-b+2}{2} \\
&\leq \binom{l - \lfloor\frac{l}{2}\rfloor + 1}{2} \\
&= \binom{\lceil\frac{l}{2}\rceil + 1}{2} \\
&= \frac{(\lceil\frac{l}{2}\rceil)^2}{2} + \frac{\lceil\frac{l}{2}\rceil}{2}
\end{aligned}$$

With $l = 22s + t$ $0 \leq t \leq 21$, the above expression becomes

$$60.5s^2 + (11\lceil\frac{t}{2}\rceil + 5.5)s + \binom{\lceil\frac{t}{2}\rceil + 1}{2}. \tag{4.20}$$

For each value of l, by comparing (4.23) to the appropriate polynomial above, one sees that (4.23) is no larger than the bound in the previous case, which completes the proof. □

Thus we have shown that the maximum value of $g_{(h,k)(l,m)\nu}$ grow like $9l^2/44$ when $l \leq h$ and produced the partition that attains that maximum.

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