ON THE KRULL-SCHMIDT THEOREM FOR INTEGRAL GROUP REPRESENTATIONS OF RANK 1

Andreas Dress

0. INTRODUCTION

Let R be a Dedekind domain with quotient field K (char K = 0). Let G be a finite group. An RG-module M is a finitely generated, torsionfree left R-module on which G acts from the left by R-homomorphisms

$$g: M \to M, \quad m \mapsto gm \qquad (g \in G,\ m \in M).$$

We assume M to be imbedded in $KM = K \otimes_R M$, and we define $\operatorname{rk} M = \operatorname{Dim}_K KM$.

In [5], I. Reiner constructs counterexamples to the Krull-Schmidt Theorem for R' G-modules for the case where R' is a ring of algebraic numbers in some algebraic number field K (integral at all places $\mathfrak{p}$ dividing $|G|$), where the order of G is not a power of a prime, and where G contains a normal subgroup whose index is a prime. He points out that this method does not work for simple groups.

In this note, we provide easy counterexamples for groups G and rings R for which the ideal $|G|\,R$ is not primary in R; in other words, $|G|\,R$ is contained in at least two different prime ideals. Thus we furnish counterexamples in the setting of [5], if G is not a p-group or if G is a p-group, but p decomposes in K.

We consider RG-modules of rank 1 and find conditions for an arbitrary RG-module M to be of the form M' + M" with rk M' = 1. Our starting point is the following observation:

For each $U \leq G$, let $\mathbb{Z}[G/U]$ denote the $\mathbb{Z}G$-module that is spanned as $\mathbb{Z}$-module by the left cosets $gU \in G/U$ with the obvious G-action. Then the trivial $\mathbb{Z}G$-module $\mathbb{Z} = \mathbb{Z}[G/G]$ is a direct summand in $\mathbb{Z}[G/U]$ if and only if $U = G$ (in fact, every module $\mathbb{Z}[G/U]$ is indecomposable). But $\mathbb{Z}$ is a direct summand in

$$\bigoplus_{p \mid |G|} \mathbb{Z}[G/G_p]$$

(where G_p is a p-Sylow subgroup in G), because the map $T_U: \mathbb{Z}[G/U] \twoheadrightarrow \mathbb{Z}$ $(gU \mapsto 1)$ has no right inverse if $U \neq G$, whereas the map

$$\bigoplus_{p \mid |G|} T_{G_p}: \bigoplus_{p \mid |G|} \mathbb{Z}[G/G_p] \to \mathbb{Z}$$

has a right inverse.

Our results obviously do not apply in the situation where R is a discrete valuation ring, and they even seem to support the conjecture that the Krull-Schmidt

Received February 3, 1969.

Michigan Math. J. 17 (1970).

Theorem holds in general for RG-modules in this case. But this conjecture is false, as was shown in [1] and [4].

I conjecture that the Krull-Schmidt Theorem holds for RG-modules if and only if each irreducible KG-module V for which $K^* \otimes V$ is not irreducible contains only projective RG-submodules, where K^* is the $\mathfrak{p}$-adic completion of K ($\mathfrak{p}$ is the maximal ideal in R). One can easily see the sufficiency of this condition by extending the methods in [2]; but, up to now, I was unable to generalize the methods used in [1] and [4] to prove the necessity.

1. NOTATION

For each homomorphism $\phi: G \to R^x$ from G into the group of units in R and each RG-module M, we define

$$M_\phi = \{m \in M \mid gm = \phi(g)m \text{ for each } g \in G\}.$$

Then M_ϕ is an RG-submodule of M, the inclusion $N \subseteq M$ implies that $N_\phi \subseteq M_\phi$, and for $M = M' \oplus M''$, we have that $M_\phi = M'_\phi \oplus M''_\phi$.

Each R-submodule of M_ϕ is also an RG-submodule. Therefore a well-known result [2, Theorem (22.5), p. 147] implies that M_ϕ is a direct sum of RG-modules of rank 1. Moreover, for each RG-submodule M' of M (rk $M' = 1$) there exists exactly one homomorphism $\phi: G \to R^x$ such that $M' \subseteq M_\phi$.

There exists a natural RG-homomorphism

$$T_\phi: M \to KM_\phi, \qquad m \mapsto \frac{1}{|G|} \sum_{g \in G} \phi(g)^{-1} gm$$

that is the identity on M_ϕ and maps M into $\frac{1}{|G|} M_\phi \subseteq KM_\phi$. Thus

$$M_\phi \subseteq T_\phi(M) \subseteq \frac{1}{|G|} M_\phi.$$

We define $s_\phi(M) = \max\{\text{rk } M' \mid M = M' \oplus M'',\ M' \subseteq M_\phi\}$; obviously, $s_\phi(M) \leq \text{rk } M_\phi$.

Finally, for each finitely-generated torsion R-module N, let e(N) denote the minimal number of generators of N (as an R-module). The invariant-factor theorem [2, Theorem (22.12), p. 150] implies that

$$e(N) = \max\{\text{Dim}_{R/\mathfrak{p}}\, R/\mathfrak{p} \otimes_R N \mid \mathfrak{p} \text{ is a maximal ideal in } R\};$$

in particular, $e(N' \oplus N'') = e(N') + e(N'')$ if and only if there exists a maximal ideal $\mathfrak{p}$ with $e(N') = \text{Dim}_{R/\mathfrak{p}}\, R/\mathfrak{p} \otimes_R N'$ and $e(N'') = \text{Dim}_{R/\mathfrak{p}}\, R/\mathfrak{p} \otimes_R N''$.

2. RESULTS

THEOREM 1. *For each* RG-*module* M, *we have the relation*

$$s_\phi(M) + e(T_\phi(M)/M_\phi) = \operatorname{rk} M_\phi .$$

Proof. By the invariant-factor theorem, we can find a basis $m_1, \cdots m_n$ ($n = \operatorname{rk} M_\phi$) of KM_ϕ, fractional ideals $\mathfrak{a}_1, \cdots, \mathfrak{a}_n$, and ideals

$$\mathfrak{e}_1 \supseteq \mathfrak{e}_2 \supseteq \cdots \supseteq \mathfrak{e}_n \supseteq |G| \cdot R$$

with $T_\phi(M) = \bigoplus \mathfrak{a}_i m_i$ and $M_\phi = \bigoplus \mathfrak{a}_i \mathfrak{e}_i m_i$ (these conditions determine the ideals $\mathfrak{e}_i$ uniquely). Assume that $\mathfrak{e}_j = R$ and $\mathfrak{e}_{j+1} \neq R$ for some $j \geq 0$, and choose a maximal $\mathfrak{p}$ with $\mathfrak{e}_{j+1} \subseteq \mathfrak{p}$. Because

$$\bigoplus_1^j \mathfrak{a}_i m_i \subseteq M_\phi \subseteq M,$$

the composition T_1 of T_ϕ with the projection $T_\phi(M) \to \bigoplus_1^j \mathfrak{a}_i m_i$ is an idempotent endomorphism of M; thus

$$M \cong \bigoplus_1^j \mathfrak{a}_i m_i \oplus \operatorname{Ke} T_1 \quad \text{and} \quad s_\phi(M) \geq j .$$

Moreover, with $M(\phi) = T_\phi(M)/M_\phi$, we have the relation $M(\phi) \cong \bigoplus_{j+1}^n R/\mathfrak{e}_i$; thus $n - j \geq e(M(\phi)) \geq \operatorname{Dim}_{R/\mathfrak{p}} (\bigoplus_{j+1}^n R/\mathfrak{p}) = n - j$, that is, $e(M\phi)) = n - j$ and $s_\phi(M) + e(M(\phi)) \geq n = \operatorname{rk} M_\phi \geq e(M(\phi))$.

On the other hand, if $M = M' \oplus M''$ and $M' \subseteq M_\phi$ ($\operatorname{rk} M' = s_\phi(M)$), then $\operatorname{rk} M_\phi = s_\phi(M) + \operatorname{rk} M''_\phi \geq s_\phi(M) + e(M''(\phi))$. But $M(\phi) = M'(\phi) \oplus M''(\phi)$ and $M'(\phi) = 0$; hence $e(M''(\phi)) = e(M(\phi))$ and $\operatorname{rk} M_\phi = s_\phi(M) + e(M(\phi))$.

We remark that each maximal ideal $\mathfrak{p}$ for which the equation

$$e(M(\phi)) = \operatorname{Dim}_{R/\mathfrak{p}} (R/\mathfrak{p} \otimes M(\phi))$$

holds must contain $|G| \cdot R$, because $|G|$ annihilates $M(\phi)$ (unless we have the trivial situation $M(\phi) = 0$, in which case M_ϕ is a direct summand of M).

We mention a few applications.

COROLLARY 1. *For two* RG-*modules* M' *and* M", *the following statements are equivalent:*

(i) $s_\phi(M' \oplus M'') = s_\phi(M') + s_\phi(M'')$,

(ii) $e(M'(\phi) \oplus M''(\phi)) = e(M'(\phi)) + e(M''(\phi))$,

(iii) *there exists a maximal ideal* $\mathfrak{p}$ (*containing* $|G|$ R *in the case where* $|G|$ R $\neq$ R) *for which the relations*

$$e(M'(\phi)) = \operatorname{Dim}_{R/\mathfrak{p}} (R/\mathfrak{p} \otimes M'(\phi)) \quad \textit{and} \quad e(M''(\phi)) = \operatorname{Dim}_{R/\mathfrak{p}} (R/\mathfrak{p} \otimes M''(\phi))$$

hold.

COROLLARY 2. *If* $k[M]$ *denotes the direct sum of* k *copies of* M, *then* $s_\phi(k[M]) = k\, s_\phi(M)$.

COROLLARY 3. *If* $M = M' \oplus M''$, *then* M_ϕ *is a direct summand of* M *if and only if* M'_ϕ *and* M''_ϕ *are direct summands of* M' *and* M'', *respectively.*

For the next two applications, we need the following lemma.

LEMMA. *For each (maximal) ideal* $\mathfrak{p}$ *satisfying the conditions* $|G|\, R \subseteq \mathfrak{p} \subseteq R$, *there exists an* RG-*module* $M^{\mathfrak{p}}$ *for which* $M^{\mathfrak{p}}(\phi) \cong R/\mathfrak{p}$.

Proof. Consider the ring RG ($RG \subseteq KG$) as a left RG-module. We have the relation $T_\phi(RG) = R \circ m$, where

$$m = \frac{1}{|G|} \sum \phi(g)^{-1} g \in KG,$$

and for $M^{\mathfrak{p}} = \mathfrak{p} m + RG \subseteq KG$, we have that $M^{\mathfrak{p}}(\phi) \cong R/\mathfrak{p}$.

Obviously, $e(M^{\mathfrak{p}}(\phi)) = \mathrm{Dim}_{R/\mathfrak{q}}\,(R/\mathfrak{q} \otimes M^{\mathfrak{q}}(\phi))$ if and only if $\mathfrak{p} = \mathfrak{q}$.

COROLLARY 4. *Suppose* M *is an* RG-*module. The relation*

$$s_\phi(M \oplus M') = s_\phi(M) + s_\phi(M')$$

holds for all RG-*modules* M' *if and only if* $e(M(\phi)) = \mathrm{Dim}_{R/\mathfrak{p}}\,(R/\mathfrak{p} \otimes M(\phi))$, *for all maximal ideals containing* $|G|\, R$.

THEOREM 2. *The equation* $s_\phi(M' \oplus M'') = s_\phi(M') + s_\phi(M'')$ *holds for all* RG-*modules* M' *and* M'' *if and only if* $|G|\, R$ *is primary in* R, *that is, if* $|G|\, R$ *is contained in at most one maximal ideal.*

3. REMARKS

a) If M is an indecomposable RG-module and M' is an arbitrary RG-module, one can define $s_M(M') = \max\{k \mid M' \cong k[M] \oplus M''\}$, and one can say that the Krull-Schmidt Theorem holds with respect to M if the relation

$$s_M(M' \oplus M'') = s_M(M') + s_M(M'')$$

holds for all RG-modules M' and M''. Then Theorem 2 shows that the Krull-Schmidt Theorem is valid for RG-modules of rank 1 if R is a valuation ring, or if $R = \mathbb{Z}$ and G is a p-group, or, more generally, if R is some principal-ideal domain in $\mathbb{Q}(\zeta)$ (where ζ is a primitive p^nth root of unity) and G is a p-group. But, as pointed out in the introduction, it seems more interesting that the arguments leading to Theorem 2 provide easy counterexamples to the Krull-Schmidt Theorem, whenever there exist at least two different prime ideals $\mathfrak{p}$ and $\mathfrak{q}$ containing $|G|\, R$: for each homomorphism $\phi: G \to R^\times$, we have the relation $M^{\mathfrak{p}} \oplus M^{\mathfrak{q}} \cong R \oplus M'$, where G acts on R via ϕ ($g \cdot r = \phi(g) r$); but neither $M^{\mathfrak{p}}$ nor $M^{\mathfrak{q}}$ contains a direct summand of rank 1.

b) For each irreducible KG-module V and each RG-module M, one can consider the uniquely determined maximal submodule M_V for which KM_V is a direct sum of submodules isomorphic to V. Define

$$s_V(M) = \max\{\mathrm{rk}\, M' \mid M = M' \oplus M'',\ M' \subseteq M_V\}.$$

There exists a natural map T_V: $M \to KM_V$, $m \mapsto e_V m$, where e_V is the central idempotent in KG associated with V. Consider $M(V) = T_V(M)/M_V$. (In the case where $V = K$, we find that $g \cdot v = \phi(g)v$ ($v \in V$, $g \in G$); we considered these conditions in Section 2.)

The following statements still hold under these more general hypotheses.

(1) $M(V) = M'(V) \oplus M''(V)$ if $M = M' \oplus M''$;

(2) $M(V) = 0$ if and only if M_V is a direct summand of M (thus Corollary 3 still holds);

(3) $s_V(M) + e(M(V)) \leq \mathrm{rk}\ M_V$.

Moreover, one can prove the following result.

THEOREM 3. *Consider the following three statements.*

(i) *If* M, M' *are two* RG-*submodules of* V, *then* $M = \mathfrak{a}M'$ *for some ideal* $\mathfrak{a} \subseteq R$.

(ii) *If* M ($M \neq 0$) *is an* RG-*submodule of* V *and* $\mathfrak{p}$ *is a maximal ideal in* R, *then* $R/\mathfrak{p} \otimes M$ *is an irreducible* $(R/\mathfrak{p})G$-*module.*

(iii) *For each* RG-*module* M, *we have the relation* $s_V(M) + e(M(V)) = \mathrm{rk}\ M_V$.

The implications (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii) *hold in general, and the implication* (ii) $\Leftarrow$ (iii) *holds if* K *is a splitting field for* G.

REFERENCES

1. S. D. Berman and P. M. Gudivok, *Integral representations of finite groups.* Dokl. Akad. Nauk SSSR 145 (1962), 1199-1201.

2. C. W. Curtis and I. Reiner, *Representation theory of finite groups and associative algebras.* Interscience, New York, 1962.

3. H. Jakobinski, *Genera and decomposition of lattices over orders.* Acta Math. 121 (1968), 1-29.

4. A. Jones, *On representations of finite groups over valuation rings.* Illinois J. Math. 9 (1965), 297-303.

5. I. Reiner, *Failure of the Krull-Schmidt Theorem for integral representations.* Michigan Math. J. 9 (1962), 225-231.

Freie Universität Berlin
Berlin, Germany
and
Institute for Advanced Study
Princeton, New Jersey 08540
and
Universität in Bielefeld
48 Bielefeld, Germany