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ON THE LACUNARY FOURIER SERIES

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1. Introduction. Lacunary trigonometric series have many interesting properties. One of them is as follows (cf. [1] p. 203);

THEOREM OF ZYGMUND. *Let* $\{n_k\}$ *be a sequence of positive integers with a Hadamard's gap, that is,*

(1.1) $$n_{k+1} > n_k(1+c) \quad (c > 0),$$

and $\sum_{k=1}^{\infty} a_k^2$ *a divergent series where* a_k*'s are non-negative real numbers. Then for any sequence of real numbers* $\{\alpha_k\}$ *the trigonometric series*

$$\sum_{k=1}^{\infty} a_k \cos(n_k x + \alpha_k)$$

diverges almost everywhere and also is not a Fourier series.

The purpose of the present note is to weaken the *lacunarity* condition (1.1). In fact we shall prove the following

THEOREM. *Let* $\{n_k\}$ *be a sequence of positive integers and* $\{a_k\}$ *a sequence of non-negative real numbers satisfying*

(1.2) $$n_{k+1} > n_k(1+ck^{-\alpha}) \quad (c > 0 \text{ and } 0 \leqq \alpha \leqq 1/2),$$

(1.3) $$A_N = \left(2^{-1} \sum_{k=1}^{N} a_k^2\right)^{1/2} \to +\infty \text{ and } a_N = O(A_N N^{-\alpha}), \text{ as } N \to +\infty.$$

Then for any sequence of real numbers $\{\alpha_k\}$ *the trigorometric series*

(1.4) $$\sum_{k=1}^{\infty} a_k \cos(n_k x + \alpha_k)$$

diverges almost everywhere and also is not a Fourier series.

If α is zero, then our theorem is that of Zygmund.

2. Some Lemmas. I. The next lemma is easily seen.

LEMMA 1. *Let the functions* $g_n(x)$, $n \geqq 1$, *be in* $L^p(0, 2\pi)$, $p > 1$, *and bounded in* L^p*-norm. If for each* $t \in (0, 2\pi)$

$$\lim_{n\to\infty} \int_0^t g_n(x)\,dx = t\,,$$

then

$$\lim_{n\to\infty} \int_E g_n(x)\,dx = |E|,\ ^{1)}\ \textit{for any set}\ E \subset (0, 2\pi)\,.$$

LEMMA 2. *For any trigonometric series* $\sum_{k=1}^{\infty} c_k \cos(kx+\gamma_k)$ *put*

$$D_0(x) = \sum_{k=1}^{2} c_k \cos(kx+\gamma_k), \qquad D_m(x) = \sum_{k=2^m+1}^{2^{m+1}} c_k \cos(kx+\gamma_k) \quad (m \geqq 1)\,.$$

Then there exists a positive constant K *such that*

$$\int_0^{2\pi} \left\{\sum_{m=0}^{N} D_m(x)\right\}^4 dx \leqq K \int^{2\pi} \left\{\sum_{m=0}^{N} D_m^2(x)\right\}^2 dx \quad (N \geqq 0)\,,$$

and also the constant K *does not depend on the series.*

This lemma is a special case of Theorem (2.1) on p. 224 in [2], but in this case we can prove it more easily by direct computations.

II. From now on let us assume that the sequence $\{n_k\}$ satisfies the gap condition (1.2). First let us put

(2.1) $\qquad p(0) = 0 \quad \text{and} \quad p(k) = \max_m \{m\,;\, n_m \leqq 2^k\} \quad (k \geqq 1)\,.$ [2)]

By (1.2) and (2.1) we haveif $p(k)+1 < p(k+1)$, then

1) $|E|$ denotes the Lebesgue measure of the set E.

2) For some k, $p(k)$ may be equal to $p(k+1)$.

$$2 > n_{p(k+1)}/n_{p(k)+1} > \prod_{m=p(k)+1}^{p(k+1)-1} (1+cm^{-\alpha})$$

$$> 1 + \{p(k+1) - p(k)-1\}\, p^{-\alpha}(k+1),$$

and this implies that

(2. 2) $$p(k+1) - p(k) = O(p^{\alpha}(k)),$$

(2. 3) $$p(k+1)/p(k) \to 1, \quad \text{as} \quad k \to +\infty .$$

LEMMA 3. *For any given integers k, j, q and h satisfying*

(2. 4) $$\begin{cases} k \geqq j+3, \ p(j)+1 < h \leqq p(j+1), \\ p(k)+1 < q \leqq p(k+1), \end{cases}$$

the total number of solutions (n_r, n_i) of the following equations

(2. 5) $$n_q - n_r = (n_h \pm n_i), \ \text{where } p(j) < i < h \text{ and } p(k) < r < q,$$

is at most $K 2^{j-k} p^{\alpha}(k)$, where K does not depend on k, j, q and h.

PROOF. From (2. 5) and (2. 4) it is seen that

$$n_r = n_q - (n_h \pm n_i) > n_q - 2^{j+2} > n_q(1-2^{j-k+2}) \geqq n_q(1+2^{j-k+3})^{-1}.$$

Thus if m_1 (or m_2) is the smallest (or largest) index of n_r's satisfying either of the equations (2. 5), then we have

$$1+2^{j-k+3} > n_{m_2+1}/n_{m_1} > \prod_{k=m_1}^{m_2} (1+ck^{-\alpha}) > 1+(m_2-m_1+1)\, p^{-\alpha}(k+1).$$

Hence, by (2. 3) we can prove the lemma.

3. Proof of the Theorem. From now on we shall assume for simplicity of writing the formulas (1.4) is a cosine series:

$$\sum_{k=1}^{\infty} a_k \cos n_k x .$$

The proof of the general case follows the same lines.

I. First let us put as follows:

$$S_N(x)=\sum_{k=1}^{N} a_k \cos n_k x\,,\quad A_N=\left(2^{-1}\sum_{k=1}^{N} a_k^2\right)^{1/2}$$

$$\Delta_k(x)=\sum_{m=p(k)+1}^{p(k+1)} a_m \cos n_m x,\quad B_k=\left(2^{-1}\sum_{m=p(k)+1}^{p(k+1)} a_m^2\right)^{1/2}\ \text{[1]}\quad \text{and } C_N=\left(\sum_{k=0}^{N} B_k^2\right)^{1/2}.$$

Then, from (1. 3) and (2. 2) it is seen that

$$\text{(3. 1)}\qquad \sup_x|\Delta_k(x)|\leqq\sum_{m=p(k)+1}^{p(k+1)}|a_m|=O(C_k),\ \text{as } k\to+\infty\,.$$

II. By (3. 1) we have

$$\text{(3. 2)}\quad \sum_{k=0}^{N}\sum_{j=k-2}^{k}\int_0^{2\pi}\Delta_k^2(x)\,\Delta_j^2(x)\,dx=O\left(C_N^2\sum_{k=0}^{N}B_k^2\right)=O(C_N^4),\ \text{as } N\to+\infty\,.$$

Further from the definition of $\Delta_k(x)$ we obtain

$$\text{(3. 3)}\qquad \int_0^{2\pi}\Delta_k^2(x)\,\Delta_j^2(x)\,dx\leqq 8\pi\,B_k^2B_j^2+\int_0^{2\pi}V_k(x)\,V_j(x)\,dx\,,$$

where

$$V_k(x)=\sum_{q=p(k)+2}^{p(k+1)}\sum_{r=p(k)+1}^{q-1}a_q\,a_r\{\cos(n_q+n_r)\,x+\cos(n_q-n_r)\,x\}\,.$$

Appling Lemma 3 to $V_k(x)\,V_j(x)$, $k-3\geqq j$, we have

$$\left|\int_0^{2\pi}V_k(x)\,V_j(x)\,dx\right|$$
$$\leqq K\,2^{j-k}p^{\alpha}(k)\sum_{q=p(k)+2}^{p(k+1)}|a_q|\sum_{h=p(j)+2}^{p(j+1)}|a_h|(\max_{p(k)<r<q}|a_r|)(\max_{p(j)<i<h}|a_i|)\,.$$

Since (1. 3), (2. 2) and Schwarz' inequality imply that

$$\sum_{q=p(k)+2}^{p(k+1)}|a_q|\,(\max_{p(k)<r<q}|a_r|)=O(B_k\,C_k\,p^{-\alpha/2}(k)),\ \text{as } k\to+\infty\,,$$

1) If $p(k)=p(k+1)$, then $\Delta_k(x)$ and B_k denote zeroes.

we have

$$\sum_{j=0}^{k-3}\left|\int_0^{2\pi} V_k(x)V_j(x)\,dx\right| = O\left((C_k^2 B_k p^{\alpha/2}(k)\sum_{j=0}^{k-3} 2^{j-k} B_j p^{-\alpha/2}(j)\right)$$

$$= O\left\{C_k^2 B_k p^{\alpha/2}(k)\left(\sum_{j=0}^{k-3} 2^{j-k} B_j^2\right)^{1/2}\left(\sum_{j=0}^{k-3} 2^{j-k} p^{-\alpha}(j)\right)^{1/2}\right\}, \text{ as } k\to+\infty .$$

On the other hand by (2. 3) we have

$$\left(\sum_{j=0}^{k-3} 2^{j-k} p^{-\alpha}(j)\right) = O(p^{-\alpha}(k)), \text{ as } k\to+\infty .$$

Thus we have

$$\sum_{k=3}^{N}\sum_{j=0}^{k-3}\left|\int_0^{2\pi} V_k(x)V_j(x)\,dx\right| = O\left(C_N^2\sum_{k=0}^{N} B_k\left(\sum_{j=0}^{k-3} 2^{j-k} B_j^2\right)^{1/2}\right)$$

$$= O(C_N^2)\left\{\sum_{k=0}^{N} B_k^2\right\}^{1/2}\left\{\sum_{k=0}^{N}\sum_{j=0}^{k-3} 2^{j-k} B_j^2\right\}^{1/2} = O(C_N^4), \text{ as } N\to+\infty .$$

Therefore, by (3. 3) and (3. 2) we have

$$\int_0^{2\pi}\left\{\sum_{k=0}^{N}\Delta_k^2(x)\right\}^2 dx = O(C_N^4), \text{ as } N\to+\infty . \tag{3. 4}$$

By (3. 4) if we apply Lemma 2 to $S_N(x)$, we obtain

$$\int_0^{2\pi}\left\{\sum_{k=0}^{N}\Delta_k(x)\right\}^4 dx = O(C_N^4), \text{ as } N\to+\infty .$$

Further for any q, $p(k) < q \leqq p(k+1)$, (1. 3) and (2. 2) imply that

$$\sum_{m=p(k)+1}^{q} |a_m| = O\left(A_q p^{-\alpha}(k)\{p(k+1)-p(k)\}\right) = O(A_q), \text{ as } q\to+\infty .$$

Thus, we have

$$\int_0^{2\pi}\{A_N^{-1} S_N(x)\}^4\,dx = O(1), \text{ as } N\to+\infty . \tag{3. 5}$$

III. Since $\left|\int_0^t \cos nx\,dx\right| \leqq 2n^{-1}$, we have, by (1. 3),

$$\left|\int_0^t S_N^2(x)\,dx - tA_N^2\right| \leqq \sum_{k=1}^{N} a_k^2/n_k + 4\sum_{k=2}^{N}\sum_{j=1}^{k-1} |a_k a_j|(n_k-n_j)^{-1}$$

$$= o(A_N^2)\left(1+\sum_{k=2}^{N} k(n_k-n_{k-1})^{-1}\right), \quad \text{as} \quad N\to+\infty .$$

Then from (1. 2) it is seen that for some positive constant K

$$n_k-n_{k-1} > ck^{-\alpha}n_{k-1} > ck^{-\alpha}n_1\prod_{m=1}^{k-2}(1+cm^{-\alpha}) > k^{-\alpha}K\exp(Kk^{1-\alpha}),$$

and this implies that for each $t\in(0, 2\pi)$

$$\left|\int_0^t S_N^2(x)\,dx - tA_N^2\right| = o(A_N^2), \quad \text{as} \quad N\to+\infty . \tag{3. 6}$$

By (3. 5), (3. 6) and Lemma 1 we have

$$\lim_{N\to\infty}\int_E \{A_N^{-1}S_N(x)\}^2\,dx = |E|, \text{ for any set } E\subset(0, 2\pi). \tag{3. 7}$$

IV. Suppose, on the contrary, that there exists a subsequence $\{S_{m_k}(x)\}$, $k = 1, 2, \cdots$, which converges on a set E, $E\subset(0, 2\pi)$, of positive measure. Then by the well known theorem of Egoroff we can find a subset E_0 of E, $|E_0| > 0$, and a number M such that $|S_{m_k}(x)| \leqq M$ for $k = 1, 2, \cdots$, $x\in E_0$. Therefore, for this set E_0 we have

$$\lim_{k\to\infty}\int_{E_0}\{A_{m_k}^{-1}S_{m_k}(x)\}^2\,dx = 0. \tag{3. 8}$$

While, (3. 8) contradicts (3. 7). Thus any subsequence of $\{S_n(x)\}$ diverges almost everywhere. This proves the first part of the theorem.

V. Suppose that the series $\sum_{k=1}^{\infty} a_k\cos n_k x$ is a Fourier series. Then it is well known that its partial sums converge in L^p-norm, $0<p<1$. Hence there exists a subsequence $\{S_{m_k}(x)\}$ which converges almost everywhere. Thus by the conclusion of the preceding section we arrive at a contradiction and the series can not be a Fourier series.

REFERENCES

[1] A. ZYGMUND, Trigonometric Series, Vol. I, Cambridge University Press, 1959.
[2] A. ZYGMUND, ibid., Vol. II.

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