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ON THE LAGRANGE RESOLVENTS OF A DIHEDRAL QUINTIC POLYNOMIAL

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The cyclic quartic field generated by the fifth powers of the Lagrange resolvents of a dihedral quintic polynomial $f(x)$ is explicitly determined in terms of a generator for the quadratic subfield of the splitting field of $f(x)$.

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Let $f(x) = x^5 + px^3 + qx^2 + rx + s \in \mathbb{Q}[x]$ be an irreducible quintic polynomial with a solvable Galois group. Let $x_1, x_2, x_3, x_4, x_5 \in \mathbb{C}$ be the roots of $f(x)$. The splitting field of f is $K = \mathbb{Q}(x_1, x_2, x_3, x_4, x_5)$. Let ζ be a primitive fifth root of unity. The Lagrange resolvents of the root x_1 are

$$\begin{aligned}
r_1 &= (x_1, \zeta) = x_1 + x_2\zeta + x_3\zeta^2 + x_4\zeta^3 + x_5\zeta^4 \in K(\zeta),\\
r_2 &= (x_1, \zeta^2) = x_1 + x_2\zeta^2 + x_3\zeta^4 + x_4\zeta + x_5\zeta^3 \in K(\zeta),\\
r_3 &= (x_1, \zeta^3) = x_1 + x_2\zeta^3 + x_3\zeta + x_4\zeta^4 + x_5\zeta^2 \in K(\zeta),\\
r_4 &= (x_1, \zeta^4) = x_1 + x_2\zeta^4 + x_3\zeta^3 + x_4\zeta^2 + x_5\zeta \in K(\zeta).
\end{aligned} \tag{1}$$

We set

$$R_i = r_i^5, \quad i = 1, 2, 3, 4. \tag{2}$$

By [1, Theorem 2] we know that the Galois group of f is $\mathbb{Z}_5$ (cyclic group of order 5), D_5 (dihedral group of order 10), or F_{20} (Frobenius group of order 20). When $\mathrm{Gal}(f) \simeq D_5$, the splitting field K of f contains a unique quadratic subfield, say $\mathbb{Q}(\sqrt{m})$ (m squarefree integer $\neq 1$). In this note we show, for quintic polynomials f with $\mathrm{Gal}(f) \simeq D_5$, that the fields $\mathbb{Q}(R_i)$ $(i = 1, 2, 3, 4)$ are the same cyclic quartic field and we give a simple explicit generator for this field. We prove the following theorem.

Theorem 1. *If* $\mathrm{Gal}(f) \simeq D_5$*, then*

$$\mathbb{Q}(R_i) = \mathbb{Q}\left(\sqrt{-m(5 + 2\sqrt{5})}\right), \quad i = 1, 2, 3, 4, \tag{3}$$

where $\mathbb{Q}(\sqrt{m})$ *is the unique quadratic subfield of the splitting field* K *of* f.

Proof. Expanding $(x_1,\zeta)^5 = (x_1+x_2\zeta+x_3\zeta^2+x_4\zeta^3+x_5\zeta^4)^5$ we obtain

$$R_1 = l_0+l_1\zeta+l_2\zeta^2+l_3\zeta^3+l_4\zeta^4, \tag{4}$$

where $l_0,l_1,l_2,l_3,l_4 \in K$ are given in [1, page 391] and satisfy

$$l_0+l_1+l_2+l_3+l_4 = (x_1+x_2+x_3+x_4+x_5)^5 = 0. \tag{5}$$

As $\mathrm{Gal}(f) \simeq D_5$, by [1, Theorem 2, page 397] the discriminant D of f is a square in $\mathbb{Q}$. Thus, by [1, pages 392–397], l_1, l_2, l_3, l_4 are the roots of a quartic polynomial belonging to $\mathbb{Q}[x]$, which factors over $\mathbb{Q}$ into two irreducible conjugate quadratics

$$\left(x^2+(T_1+T_2\sqrt{D})x+(T_3+T_4\sqrt{D})\right)\left(x^2+(T_1-T_2\sqrt{D})x+(T_3-T_4\sqrt{D})\right) \tag{6}$$

with T_1, T_2, T_3, $T_4 \in \mathbb{Q}$. The roots of one of these quadratics (without loss of generality the first) are l_1 and l_4, and the roots of the other are l_2 and l_3. Thus

$$\begin{aligned} l_1+l_4 &= -T_1-T_2\sqrt{D}, \qquad & l_2+l_3 &= -T_1+T_2\sqrt{D}, \\ l_1l_4 &= T_3+T_4\sqrt{D}, \qquad & l_2l_3 &= T_3-T_4\sqrt{D}. \end{aligned} \tag{7}$$

Clearly $[\mathbb{Q}(l_i):\mathbb{Q}] = 2$ $(i=1,2,3,4)$. Also $l_i \in K$ $(i=1,2,3,4)$ so that $\mathbb{Q}(l_i) \subseteq K$ $(i=1,2,3,4)$. However K has a unique quadratic subfield $\mathbb{Q}(\sqrt{m})$. Thus $\mathbb{Q}(l_i) = \mathbb{Q}(\sqrt{m})$, $i=1,2,3,4$. Hence

$$l_1 = a+b\sqrt{m}, \qquad l_4 = a-b\sqrt{m}, \qquad l_2 = c+d\sqrt{m}, \qquad l_3 = c-d\sqrt{m}, \tag{8}$$

where $a,b,c,d \in \mathbb{Q}$, $b \neq 0$ and $d \neq 0$. Thus

$$l_0 = -l_1-l_2-l_3-l_4 = -2a-2c. \tag{9}$$

Next we define

$$g(x) = (x-R_1)(x-R_2)(x-R_3)(x-R_4) \in K(\zeta)[x]. \tag{10}$$

Hence, as $1+\zeta+\zeta^2+\zeta^3+\zeta^4 = 0$, we obtain

$$\begin{aligned} R_1 &= l_0+l_1\zeta+l_2\zeta^2+l_3\zeta^3+l_4\zeta^4 \\ &= (a+b\sqrt{m}+2a+2c)\zeta+(c+d\sqrt{m}+2a+2c)\zeta^2 \\ &\quad +(c-d\sqrt{m}+2a+2c)\zeta^3+(a-b\sqrt{m}+2a+2c)\zeta^4 \in \mathbb{Q}(\sqrt{m},\zeta). \end{aligned} \tag{11}$$

Similarly

$$\begin{aligned} R_2 &= (a+b\sqrt{m}+2a+2c)\zeta^2+(c+d\sqrt{m}+2a+2c)\zeta^4 \\ &\quad +(c-d\sqrt{m}+2a+2c)\zeta+(a-b\sqrt{m}+2a+2c)\zeta^3 \in \mathbb{Q}(\sqrt{m},\zeta), \\ R_3 &= (a+b\sqrt{m}+2a+2c)\zeta^3+(c+d\sqrt{m}+2a+2c)\zeta \\ &\quad +(c-d\sqrt{m}+2a+2c)\zeta^4+(a-b\sqrt{m}+2a+2c)\zeta^2 \in \mathbb{Q}(\sqrt{m},\zeta), \\ R_4 &= (a+b\sqrt{m}+2a+2c)\zeta^4+(c+d\sqrt{m}+2a+2c)\zeta^3 \\ &\quad +(c-d\sqrt{m}+2a+2c)\zeta^2+(a-b\sqrt{m}+2a+2c)\zeta \in \mathbb{Q}(\sqrt{m},\zeta). \end{aligned} \tag{12}$$

Using Maple we find that

$$
\begin{aligned}
g(x) = {} & x^4+(10c+10a)x^3+(5b^2m+5d^2m+80ac+35a^2+35c^2)x^2 \\
& +(30cd^2m+50c^3+200a^2c-20bcdm+30ab^2m+20ad^2m \\
& \quad +20b^2cm+200ac^2+50a^3+20abdm)x-10b^3dm^2+150a^3c \\
& +25a^2d^2m+25b^2c^2m-5b^2d^2m^2+275a^2c^2+25c^4+10bd^3m^2 \\
& +50acd^2m-50bc^2dm+150ac^3+50a^2bdm+50c^2d^2m+5d^4m^2 \\
& +25a^4+5b^4m^2+50a^2b^2m+50ab^2cm.
\end{aligned} \tag{13}
$$

The roots of $g(x)$ are (again using Maple)

$$
\begin{aligned}
& -\frac{5}{2}a-\frac{5}{2}c+\frac{1}{2}(-a+c)\sqrt{5}+\frac{1}{2}\sqrt{-m(10(b^2+d^2)-(2b^2+8bd-2d^2)\sqrt{5})}, \\
& -\frac{5}{2}a-\frac{5}{2}c+\frac{1}{2}(-a+c)\sqrt{5}-\frac{1}{2}\sqrt{-m(10(b^2+d^2)-(2b^2+8bd-2d^2)\sqrt{5})}, \\
& -\frac{5}{2}a-\frac{5}{2}c-\frac{1}{2}(-a+c)\sqrt{5}+\frac{1}{2}\sqrt{-m(10(b^2+d^2)+(2b^2+8bd-2d^2)\sqrt{5})}, \\
& -\frac{5}{2}a-\frac{5}{2}c-\frac{1}{2}(-a+c)\sqrt{5}+\frac{1}{2}\sqrt{-m(10(b^2+d^2)+(2b^2+8bd-2d^2)\sqrt{5})}.
\end{aligned} \tag{14}
$$

The quantities under the radicals are $X+Y\sqrt{5}$ and $X-Y\sqrt{5}$, where

$$X=-10m(b^2+d^2), \qquad Y=m(2b^2+8bd-2d^2). \tag{15}$$

As

$$X^2-5Y^2=5m^2(4b^2-4bd-4d^2)^2, \tag{16}$$

the roots of $g(x)$ belong to the cyclic quartic field $\mathbb{Q}(\sqrt{X\pm Y\sqrt{5}})$ [2, Theorem 1, page 134]. Further

$$X+Y\sqrt{5}=(-10+2\sqrt{5})m\left(\frac{2b-d-d\sqrt{5}}{2}\right)^2 \tag{17}$$

so that (as $b\neq 0$ and $d\neq 0$)

$$\mathbb{Q}\left(\sqrt{X+Y\sqrt{5}}\right)=\mathbb{Q}\left(\sqrt{(-10+2\sqrt{5})m}\right)=\mathbb{Q}\left(\sqrt{-m(5+2\sqrt{5})}\right), \tag{18}$$

as $(-10+2\sqrt{5})(-5-2\sqrt{5})=(5+\sqrt{5})^2$. □

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