Prog. Theor. Phys. Vol. 76, No. 1, July 1986, Progress Letters

On the Lagrangian for Many-Body System in General Relativity

Tadayuki OHTA and Toshiei KIMURA*
Department of Physics, Miyagi University of Education, Sendai 980
**Research Institute for Theoretical Physics*
Hiroshima University, Takehara, Hiroshima 725
(Received April 4, 1986)

The gravitational Lagrangian for many-body systems which we first presented to the post-post-Newtonian order of approximation, has recently been reconfirmed. However, the recent criticism of our work has been due to a misunderstanding regarding our approach to the problem. We explain our point of view.

Downloaded from https://academic.oup.com/ptp/article/76/1/329/1909026 by guest on 21 August 2022

More than ten years ago, we published in this journal a series of four papers[1)~4)] on higher order gravitational potential for many-body systems. (In the following these are referred to as [I], [II], [III] and [IV].) Also we derived a post-post-Newtonian many-body Lagrangian (and Hamiltonian),[3)] which depends only on the positions and velocities of the bodies.

Recently, Damour and Schäfer[5)] reconfirmed our result. In their paper, they quoted this in the same notations and symbols we had used, but they called it L^{ADM}. In their discussion, however, they criticized our derivation, saying that the coincidence between the Lagrangians obtained in papers [II] and [III] is an indirect proof of our incorrectness, because of the unjustified substitution of the acceleration in the Lagrangian by means of the Newtonian equations of motion.

This criticism seems to be caused by their misunderstanding about our approach to the problem. Our main work is the calculation in paper [III] where in the canonical formalism we derived, without any substitutions, the acceleration-independent Lagrangian "L^{ADM}" (and Hamiltonian). We had never justified the substitution. It is to be emphasized that they have not given proper attribution to the originality of our calculation of "L^{ADM}".

In the following we explain our point of view and clarify some aspects of our derivations.

Our initial motivation in this problem was concerned with a quantum theory of the gravitational field, namely, we wanted to know to what extent a conventional formalism of quantization of the gravitational field was justified. Here the conventional formalism means that the graviton field $h_{\alpha\beta}$ is defined by means of*)

$$g_{\alpha\beta}=\eta_{\alpha\beta}+\varkappa h_{\alpha\beta}\,, \quad (\varkappa^2=32\pi G) \tag{1}$$

where $g_{\alpha\beta}$ is the metric tensor and G is Newton's gravitational constant, and all higher order terms with $h_{\alpha\beta}$ in the total Lagrangian density of the system are treated as interactions.

In order to investigate this, we adopted the criterion that the result in a quantized theory should coincide, in the limit of the Planck constant h going to zero, with the result in the corresponding classical theory.

However, unfortunately, at that time there had been no results in the literature in which the higher order gravitational potential to PPN order was given. Thus we decided to calculate the classical gravitational potential for a many-body system to PPN order, for which we adopted two methods.

The first method was based on the Fokker action principle, in which the equations of motion of the bodies are given by a variation

$$\delta\int_{x^0}^{x^{0'}} L^*dx^0=0\,, \tag{2}$$

where the Lagrangian L^* is

$$L^*=\int d^3x\Big\{-\sum_a m_a c^2\delta(\boldsymbol{x}-\boldsymbol{z}_a)\times\sqrt{-g_{\mu\nu}\frac{dz_a^{\mu}}{dx^0}\frac{dz_a^{\nu}}{dx^0}}$$

*) In this paper we use the following conventions. Greek indices run from 0 to 3, while Latin indices i and j run from 1 to 3. Repetition of these indices implies summation. A comma in a subscript denotes ordinary differentiation.

$$-\frac{c^4}{16\pi G}\sqrt{-g}\,g^{\mu\nu}(\Gamma^{\rho}_{\mu\nu}\Gamma^{\lambda}_{\rho\lambda}-\Gamma^{\lambda}_{\mu\rho}\Gamma^{\rho}_{\nu\lambda})\Big\}. \quad (3)$$

Here, g, $\Gamma^{\rho}_{\mu\nu}$, m_a and z_a denote det $(g_{\mu\nu})$, the Christoffel symbol, the rest mass and the coordinate of a-th body, respectively, and $z^0_{a,0}=1$. In the expression (3), the metric tensor $g_{\mu\nu}$ and $\Gamma^{\rho}_{\mu\nu}$ should be given as functions of the coordinates (z_a's) and the velocities ($\dot{z}_a$'s) of the bodies.

The formal proof of this action principle was given by Infeld and Plebanski[6] under the assumption that one surface integral should vanish. They showed that the equation derived from the variation (2) was equivalent to the equation of a geodesic in general relativity. We used this action to obtain the Lagrangian in papers [I] and [II]. In order to evaluate the Lagrangian L^*, we first had to solve Einstein equations and obtain the explicit expression for the metric tensor as a function of the z_a's and $\dot{z}_a$'s. In this process, we found that in the coordinate conditions used in the literature the metric tensor to the PPN order of approximation was divergent at spatial infinity ($r\to\infty$). In that case, the Fokker action principle no longer holds because of the divergence of the surface integral. Then, we tried to find the coordinate conditions under which the metric tensor became Minkowskian at spatial infinity. We found a class of coordinate conditions which led to well-behaved metric tensors.[1]

However, we encountered one serious difficulty. When the Lagrangian is evaluated by substituting the explicit expression of the metric tensor on the right-hand side of Eq. (3), the terms with acceleration appear in the Lagrangian L^*. This Lagrangian $L^*(z_a, \dot{z}_a, \ddot{z}_a)$ is not only beyond the framework of the ordinary description of dynamical system, but also outside the framework of the Fokker action principle. The generalized Euler-Lagrange equation derived from $L^*(z_a, \dot{z}_a, \ddot{z}_a)$ is no longer equivalent to the equation of the geodesic in general relativity. Even if the action principle itself could be improved to incorporate the acceleration terms, it would not be applicable to our problem.

Next we made a conjecture on what happened when the acceleration $\ddot{z}_a$ in L^* was rewritten by the use of the equations of motion in the Newtonian order of approximation, $\ddot{z}_a=-\sum Gm_b\boldsymbol{n}_{ab}/r^2_{ab}$ ($r_{ab}=|z_a-z_b|$, $\boldsymbol{n}_{ab}=(z_a-z_b)/r_{ab}$). The equations of motion derived from the modified Lagrangian $L(z_a, \dot{z}_a)$ thus obtained, are not equivalent to the generalized equations derived from the original Lagrangian $L^*(z_a, \dot{z}_a, \ddot{z}_a)$. Of course, there is no reason to justify this substitution in the Lagrangian. However, we found that the Hamiltonian derived from this modified Lagrangian $L(z_a, \dot{z}_a)$ was identical with the Hamiltonian obtained in the canonical formalism.

It was in paper [III] that we confirmed the correct expression of the higher order gravitational potential for many-body system. An essential point of the problem is in the relation between the coordinate condition and the potential. From this point of view, we adopted the canonical formalism of ADM[7] as another method to get the potential. This formalism is best suited to investigate the relation mentioned above. We found in this formalism that under a certain class of coordinate conditions, the Hamiltonian was given as a simple three space integral of the metric tensor, and that it contained only the coordinates and the momenta of the bodies. Namely, we have only to calculate the integral,

$$H=-\int d^3x\Delta h^T, \quad (4)$$

where

$$h^T=h_{ii}-\frac{1}{\Delta}h_{ij,ij},\ h_{ij}=g_{ij}-\delta_{ij},\quad (i,j=1,2,3) \quad (5)$$

and the metric tensor must, of course, be Minkowskian at spatial infinity. Hamilton's equation for the bodies derived from the Hamiltonian (4) is equivalent to Infeld's equation in general relativity.

In the most simple coordinate condition (which Damour and Schäfer named the "ADM gauge"), we obtained the explicit expression of the Hamiltonian (4) for many-body systems to the PPN order of approximation. The result was identical with the Hamiltonian derived from the modified Lagrangian $L(z_a, \dot{z}_a)$ in paper [II].

In § 5 of paper [III], we investigated in detail how this coincidence occurred. We examined the relation between the coordinate condition and the Hamiltonian in each method and found

$$H^*(z_a, \boldsymbol{p}_a)=H(z_a, \boldsymbol{p}_a)-F(z_a, \boldsymbol{p}_a),_0$$

Downloaded from https://academic.oup.com/ptp/article/76/1/329/1909026 by guest on 21 August 2022

$$+\sum(\dot{\boldsymbol{p}}_a+\sum\frac{Gm_am_b}{r_{ab}^2}\boldsymbol{n}_{ab})(\nabla A(\boldsymbol{x}, \boldsymbol{z}_a, \boldsymbol{p}_a))_{x=z_a}, \tag{6}$$

where $H(\boldsymbol{z}_a, \boldsymbol{p}_a)$ is the Hamiltonian given in paper [III] (and also derived from the modified Lagrangian), and $H^*(\boldsymbol{z}_a, \boldsymbol{p}_a)$ is the Hamiltonian formally evaluated by using the metric tensor in paper [I]. The explicit forms of the functions F and A are also given in paper [III]. This relation (6) clearly shows the reason why the replacement of the acceleration in the Lagrangian L led us to the correct potential. Since the explicit form of H^* includes not only $\boldsymbol{z}_a$ and $\boldsymbol{p}_a$ but also $\dot{\boldsymbol{p}}_a$, it cannot be regarded as an adequate Hamiltonian. The dynamics of the bodies is governed by the Hamiltonian $H(\boldsymbol{z}_a, \boldsymbol{p}_a)$. The relation between the coordinate conditions in two methods is easily understood from Eqs. (5·1) and (5·2) in paper [III]. We had eliminated, in paper [III], the third term on the right-hand side of the abeve relation (6) by the use of the equations of motion.

The criticism of our approach by Damour and Schäfer is concentrated on the substitution of the acceleration in the Lagrangian by means of the equations of motion. They conjectured that we insisted, without justification, for the substitution. As discussed above, however, we used the substitution only as a shortcut to find the right answer. The confirmation of our result is given in paper [III].

Our original plan of getting the potential in the S-matrix approach was carried out in paper [IV], where we made use of a quantized field theory for the system of scalar particles and gravitons. In order to obtain the potential to the PPN order of approximation, we had to calculate S-matrix elements for scatterings of two, three and four particles. Since we needed the contributions which survive in the limit $h\to 0$, we had only to retain tree diagrams.

In paper [IV], we calculated the PPN potential, namely, G^3, G^2v^2 and Gv^4 potentials. In the calculation of the G^3 potential, we were faced with the difficulty that the potential from the S-matrix was different from that obtained in the canonical method. Later, this difficulty was resolved by Yokoya et al.[8] They found that the origin of this difficulty was in the process of subtraction in the S-matrix element.

In the PPN order, the G^3-static potential is derived from S-matrix element for four-particle scattering, where in several diagrams, we need double subtractions. Yokoya et al. pointed out that there remained one arbitrary parameter α in the potential through the process of this double subtraction. It is caused when the energy of internal particle is expressed with physical quantities of external particles. The four-body static potential from S-matrix is

$$(12-27\alpha)\,V_A-\frac{3}{4}V_B+V^{\mathrm{TT}}, \tag{7}$$

where V^{TT} is the contribution from the transverse-traceless part of the graviton. The symbols V_A and V_B denote the potentials in the form

$$\sum\frac{G^3m_am_bm_cm_d}{r_{ab}r_{ac}r_{ad}} \quad \text{and} \quad \sum\frac{G^3m_am_bm_cm_d}{r_{ab}r_{bc}r_{cd}},$$

respectively, where r_{ab} is the distance between a-th and b-th particles. From a physical point of view, one of the present authors clarified that the parameter α had to take the value $1/2$.[9] Then the four-body static potential from the S-matrix coincided with the result in the canonical formalism. By performing the consistent subtraction in all orders of the diagrams, we can derive in the S-matrix method the potential given in paper [III]. In paper [IV], we had chosen an inappropriate value $\alpha=0$.

The important point of the third method is as follows. Regarding the physical quantities of the external particles, only the masses and the momenta appear in the S-matrix element. The coordinates of the particles are introduced through the Fourier-integration on the momenta of internal gravitons. Then the final result of the potential is expressed with the masses, momenta and coordinates of the particeles. There is no place into which the acceleration of the particle enters. This method leads to the Hamiltonian which is a function of the coordinates and the momenta. From a particle physicist's viewpoint, this is quite a simple and useful method to obtain the potential.

Practically speaking, we are most interested in the higher order effects for the two-body system. In paper [II], we presented the explicit expression of the Hamiltonian $H(1, 2)$ for the two-body system. There was, however, one error in summing up the coefficients of several terms in the calculation of the integral $U^{\mathrm{TT}}(1, 2)$ which

Downloaded from https://academic.oup.com/ptp/article/76/1/329/1909026 by guest on 21 August 2022

represented the contribution of the transverse-traceless part of the graviton to the two-body potential. The coefficient of the integral I_2 in Eq. (5·6) in paper [II] was not $-3/2$ but -1. This was pointed out by Damour and Schäfer.[5] We acknowledge their effort of having checked many troublesome integrations. They calculated the integral I_2 by dividing its integrand into four small parts i_1, i_2, i_3 and i_4, and gave each integral of them the finite value. Strictly speaking, a part of their result is not correct, because the integrals of i_3 and i_4 are divergent. One must calculate the whole integral $\int d^3x(i_3+i_4)/4$ which takes the finite value $1/r^3$. The correct value of U^{TT} is therefore

$$U^{TT}(1,2)=-\frac{G^3}{2}\frac{m_1^2m_2^2}{r^3}, \tag{8}$$

and the Hamiltonian for the two-body system is

$$\begin{aligned}
H(1,2)=&m_1+m_2+\frac{\boldsymbol{p}_1^2}{2m_1}+\frac{\boldsymbol{p}_2^2}{2m_2}-\frac{1}{8}\left\{m_1\left(\frac{\boldsymbol{p}_1^2}{m_1^2}\right)^2+m_2\left(\frac{\boldsymbol{p}_2^2}{m_2^2}\right)^2\right\}+\frac{1}{16}\left\{m_1\left(\frac{\boldsymbol{p}_1^2}{m_1^2}\right)^3+m_2\left(\frac{\boldsymbol{p}_2^2}{m_2^2}\right)^3\right\}\\
&-\frac{Gm_1m_2}{r}+\frac{G^2}{2}\cdot\frac{m_1m_2(m_1+m_2)}{r^2}-\frac{G^3}{4}\cdot\frac{m_1m_2(m_1^2+m_2^2+5m_1m_2)}{r^3}\\
&+\frac{G^2}{4}\cdot\frac{m_1m_2^2}{r^2}\left(10\frac{\boldsymbol{p}_1^2}{m_1^2}+19\frac{\boldsymbol{p}_2^2}{m_2^2}\right)+\frac{G^2}{4}\cdot\frac{m_1^2m_2}{r^2}\left(19\frac{\boldsymbol{p}_1^2}{m_1^2}+10\frac{\boldsymbol{p}_2^2}{m_2^2}\right)\\
&-\frac{G^2}{4}\cdot\frac{m_1m_2(m_1+m_2)}{r^2}\left\{27\frac{(\boldsymbol{p}_1\cdot\boldsymbol{p}_2)}{m_1m_2}+6\frac{(\boldsymbol{n}\cdot\boldsymbol{p}_1)(\boldsymbol{n}\cdot\boldsymbol{p}_2)}{m_1m_2}\right\}\\
&+\frac{G}{8}\cdot\frac{m_1m_2}{r}\left[-12\left(\frac{\boldsymbol{p}_1^2}{m_1^2}+\frac{\boldsymbol{p}_2^2}{m_2^2}\right)+28\frac{(\boldsymbol{p}_1\cdot\boldsymbol{p}_2)}{m_1m_2}+4\frac{(\boldsymbol{n}\cdot\boldsymbol{p}_1)(\boldsymbol{n}\cdot\boldsymbol{p}_2)}{m_1m_2}\right.\\
&+5\left\{\left(\frac{\boldsymbol{p}_1^2}{m_1^2}\right)^2+\left(\frac{\boldsymbol{p}_2^2}{m_2^2}\right)^2\right\}-11\frac{\boldsymbol{p}_1^2\boldsymbol{p}_2^2}{m_1^2m_2^2}-2\frac{(\boldsymbol{p}_1\cdot\boldsymbol{p}_2)^2}{m_1^2m_2^2}\\
&\left.+5\frac{\boldsymbol{p}_1^2(\boldsymbol{n}\cdot\boldsymbol{p}_2)^2+(\boldsymbol{n}\cdot\boldsymbol{p}_1)^2\boldsymbol{p}_2^2}{m_1^2m_2^2}-12\frac{(\boldsymbol{p}_1\cdot\boldsymbol{p}_2)(\boldsymbol{n}\cdot\boldsymbol{p}_1)(\boldsymbol{n}\cdot\boldsymbol{p}_2)}{m_1^2m_2^2}-3\frac{(\boldsymbol{n}\cdot\boldsymbol{p}_1)^2(\boldsymbol{n}\cdot\boldsymbol{p}_2)^2}{m_1^2m_2^2}\right],
\end{aligned} \tag{9}$$

where $\boldsymbol{n}=\boldsymbol{n}_{12}$.

The authors are very grateful to Professor F. I. Cooperstock for careful reading of the manuscripts and precious comments, and also to Professor Kip. S. Thorne for useful suggestions.

1) T. Ohta, H. Okamura, T. Kimura and K. Hiida, Prog. Theor. Phys. **50** (1973), 492. (referred to as [I])
2) T. Ohta, H. Okamura, T. Kimura and K. Hiida, Prog. Theor. Phys. **51** (1974), 1220. (referred to as [II])
3) T. Ohta, H. Okamura, T. Kimura and K. Hiida, Prog. Theor. Phys. **51** (1974), 1598. (referred to as [III])
4) H. Okamura, T. Ohta, T. Kimura and K. Hiida, Prog. Theor. Phys. **50** (1973), 2066. (referred to as [IV])
5) T. Damour and G. Schäfer, Gen. Relat. Gravit. J. **17** (1985), 879.
6) L. Infeld and J. Plebanski, *Motion and Relativity* (Pergamon Press, 1960), Chap. IV.
7) R. Arnowitt, S. Deser and C. W. Misner, Phys. Rev. **116** (1959), 1322; **117** (1960), 1959; **118** (1960), 1100.
8) K. Yokoya, A. Shimizu, H. Kato and K. Hiida, Phys. Rev. **D16** (1977), 2655.
9) T. Kimura, Prog. Theor. Phys. **57** (1977), 1437.

Downloaded from https://academic.oup.com/ptp/article/76/1/329/1909026 by guest on 21 August 2022