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ON THE LANDAU DAMPING AND DECOHERENCE OF TRANSVERSE DIPOLE OSCILLATIONS IN COLLIDING BEAMS

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Coherent transverse dipole oscillations in colliding head-on non-rigid bunches are studied using the Vlasov equation. The corresponding eigenvalue problem is solved numerically in the case of round Gaussian bunches of equal size but with not necessarily equal intensities. Transition from the weak–strong to the strong–strong cases is found at the intensity ratio of about 60% when a discrete π-mode frequency emerges from continuum of eigenfrequencies related to the beam–beam tunespread in the weaker bunch.

In the strong–strong case the large coherent beam–beam tuneshift dominates over interchange processes between coherent and incoherent motion; it can switch off Landau damping of dipole transverse oscillations, slows down incoherent emittance growth due to external kicks on the beams. The consequences for the transverse feedback operation in collision are discussed.

Keywords: Colliding beams

1 INTRODUCTION

Suppression of coherent oscillations and emittance growth in colliding beams is essential for achievement of highest luminosities in large hadron colliders like LHC.[1] Therefore it is important to understand the effect of beam–beam interaction on decoherence and Landau damping of coherent transverse oscillations. The previous analytical and numerical studies[2,3] were carried out for the weak–strong case whereas

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in the future LHC a truly strong–strong regime of collisions is envisaged.

An adequate approach to the strong–strong case was developed by Yokoya *et al.* in Reference 4 where an eigenvalue problem for the Vlasov equation was formulated and studied in the case of beams that collide head-on. Spectrum of the π-component of dipole oscillations was shown to consist of a discrete line shifted from the single particle tune by 1.2ξ (for round beams) and continuum $(0, \xi)$ corresponding to the incoherent beam–beam tunespread, ξ being the linear beam–beam parameter.

Due to the gap between coherent and incoherent tuneshifts the beam–beam interaction in the strong–strong case not only fails to produce Landau damping by itself,[†] but at sufficiently large values of ξ can switch off the stabilizing effect of momentum spread and the machine nonlinearity. As the consequence even very weak transverse instabilities may show up.

A question may arise why this effect has not been observed in the existing hadron colliders (Tevatron, SPS). To answer it one should examine transition from the weak–strong to the strong–strong case. This is done in Section 3 where the discrete π-mode frequency is found to emerge from the continuum of eigenfrequencies at the intensity ratio of about 60% which may be considered as the boundary value. This value normally is not surpassed in the existing machines.

Presence of the discrete π-mode in the strong–strong case drastically changes the process of decoherence of dipole oscillations. As shown in Section 5 only about 18% of the energy received from a kick at one of the beams is imparted into the continuum of eigenmodes leading to irreversible emittance growth. The other 82% are carried by persistent Σ- and π-modes which may decohere only on a much longer time scale due to nonlinear mode coupling (the Σ-mode can be damped also by non-Gaussian tail particles).

The approach developed is used in subsequent sections in analysis of the colliding beams emittance growth due to external noise and the transverse feedback operating in different regimes.

[†] The absence of the Landau damping of coherent beam–beam oscillations was first discussed by R. Talman (see Reference 5) basing on observations in CESR and numerical simulations for flat beams.

2 EQUILIBRIUM STATE

Let us make a number of simplifying assumptions:

(a) betatron tune spreads due to chromaticity and nonlinearity of the machine magnetic elements are negligible as compared to the beam–beam tune spread;
(b) motions in x and y planes are uncoupled, with exception for nonlinear coupling via the beam–beam force, the emittances being equal $\varepsilon_x = \varepsilon_y = \varepsilon_0$;
(c) beams collide head-on and at only one interaction point (IP) in the ring;
(d) the non-perturbed beams are round at the IP with equal r.m.s. radii σ^*;
(e) the working point on the tune diagram is chosen sufficiently far from low order resonances so that invariant tori are not destroyed by the beam–beam interaction

First we introduce normalized to ε_0 action (J_x, J_y) and angle (φ_x, φ_y) variables via the standard relations:

$$u = \sqrt{2\beta_u\varepsilon_0 J_u}\sin[\phi_u(\theta) - \nu_{u0}\theta + \varphi_u],$$

$$p_u = \sqrt{\frac{2\varepsilon_0 J_u}{\beta_u}}\{\cos[\phi_u(\theta) - \nu_{u0}\theta + \varphi_u] - \alpha_u \sin[\phi_u(\theta) - \nu_{u0}\theta + \varphi_u]\},$$

$$\phi_u(\theta) = R\int_0^\theta \frac{\mathrm{d}\theta'}{\beta_u(\theta')}, \quad \theta = \frac{s}{R}, \quad u = x, y. \tag{1}$$

Here α_u, β_u are the Twiss parameters, ν_{x0}, ν_{y0} are betatron tunes in absence of collisions, R is the average machine radius.

The next step is to solve nonlinear dynamics in colliding (but stationary) beams. Due to assumption (e) new canonical variables (I_u, ψ_u) can be found in which the unperturbed Hamiltonian acquires the normal form

$$H_0^{(k)} = \nu_{x0}I_x + \nu_{y0}I_y + V^{(k)}(I_x, I_y), \tag{2}$$

where index $k = 1, 2$ refers to either of the two beams. Then I_x, I_y are the constants of motion, which can be employed in construction of the equilibrium distribution function, which we presume to be Gaussian (and normalized to unity):

$$F_0 = \frac{1}{(2\pi)^2} \exp(-I_x - I_y). \tag{3}$$

To the first order in the beam–beam parameter

$$\xi_u^{(k)} = -\frac{N_{(3-k)} r_p \beta_u^*}{2\pi\gamma\sigma_u^*(\sigma_x^* + \sigma_y^*)}, \quad k = 1, 2, \ u = x, y \tag{4}$$

betatron tunes are given by expressions

$$\nu_u^{(k)} = \frac{\partial H_0^{(k)}}{\partial I_u} = \nu_{u0} + \xi_u^{(k)} \cdot Q_u(I_x, I_y). \tag{5}$$

There are various representations of the function $Q_u(I_x, I_y)$ (see Reference 4 for example), here we will present without derivation one more formula that is useful in practical calculations

$$Q_x(I_x, I_y) = \frac{1+r}{2} \int_0^1 \frac{\mathrm{d}t}{\sqrt{1 + (r^2 - 1)t}} \exp\left[-\frac{t}{2}\left(I_x + I_y \frac{r^2}{1 + (r^2 - 1)t}\right)\right]$$
$$\times I_0\left[\frac{t}{2} I_y \frac{r^2}{1 + (r^2 - 1)t}\right] \cdot \left[I_0\left(\frac{t}{2} I_x\right) - I_1\left(\frac{t}{2} I_x\right)\right], \tag{6}$$

where $r = \sigma_y^*/\sigma_x^*$ is the beam aspect ratio (in the following $r = 1$), $I_n(x)$ is the modified Bessel function of order n.

3 EIGENMODES OF TWO COLLIDING BUNCHES COHERENT OSCILLATIONS

Now let us introduce some perturbation of particle distribution and expand everything in series w.r.t. its amplitude so that for the kth beam

$$F^{(k)} = F_0 + \sum_{n=1}^{\infty} F_n^{(k)}, \qquad H^{(k)} = \sum_{n=0}^{\infty} H_n^{(k)}.$$

Limiting our consideration to the first order in ξ (and in the perturbation as well) we can use (1) with $J_u = I_u$, $\varphi_u = \psi_u$ in calculation of the perturbative part of the Hamiltonian due to beam–beam interaction:

$$H_1^{(k)} = 2\pi\xi_x^{(k)}(1+r)\sum_n \delta(\theta - 2\pi n)\int \ln\Big[(\sqrt{2I_x}\sin\psi_x - \sqrt{2I_x'}\sin\psi_x')^2 + r^2(\sqrt{2I_y}\sin\psi_y - \sqrt{2I_y'}\sin\psi_y')^2\Big] F_1^{(3-k)}\,\mathrm{d}\psi_x'\,\mathrm{d}\psi_y'\,\mathrm{d}I_x'\,\mathrm{d}I_y'. \quad (7)$$

For a while we will ignore other coherent forces created by impedances and feedback.

Generally the solution of the linearized Liouville equation

$$\frac{\partial F_1^{(k)}}{\partial\theta} + \nu_x^{(k)}\frac{\partial F_1^{(k)}}{\partial\psi_x} + \nu_y^{(k)}\frac{\partial F_1^{(k)}}{\partial\psi_y} - \frac{\partial H_1^{(k)}}{\partial\psi_x}\frac{\partial F_0}{\partial I_x} - \frac{\partial H_1^{(k)}}{\partial\psi_y}\frac{\partial F_0}{\partial I_y} = 0 \quad (8)$$

can be sought as expansion in the Fourier series

$$F_1^{(k)} = \mathrm{Re}\left\{\mathrm{e}^{-(I_x+I_y)/2}\sum_{m_x,m_y}\mathrm{e}^{im_x(\psi_x-\nu_{x0}\theta)+im_y(\psi_y-\nu_{y0}\theta)}f_{m_x,m_y}^{(k)}(I_x,I_y,\theta)\right\}, \quad (9)$$

where the factor $\exp[-(I_x+I_y)/2]$ was taken out in order to symmetrize the resulting integral equation. But up to the first order in ξ there is no coupling between terms with different m_x, m_y. Also, the assumption (e) rules out the possibility of a higher order term to become large due to small resonant denominator (such a case was considered in Reference 6). Therefore we may retain in the sum (9) only one term, namely that with $m_x = 1$, $m_y = 0$, since we are interested in the horizontal dipole oscillations. These indices will be omitted in the following.

The made assumptions also permit to average the periodic δ-function in (7) replacing it with $1/2\pi$.

Now taking average in (8) over betatron phases, introducing the integral operator

$$G\circ f = \int G(I_x, I_y, I_x', I_y')f(I_x', I_y')\,\mathrm{d}I_x'\,\mathrm{d}I_y' \quad (10)$$

with the kernel defined in the Appendix and assuming without loss of generality the first beam to be the weaker one so that $|\xi_x^{(1)}| \geq |\xi_x^{(2)}|$, we obtain the system of integro-differential equations

$$\mathrm{i}\frac{\partial}{\partial\theta}\boldsymbol{f} = \xi_x^{(1)}\hat{A}\boldsymbol{f} \tag{11}$$

where

$$\boldsymbol{f} = \begin{pmatrix} \sqrt{r_\xi}f^{(1)} \\ f^{(2)} \end{pmatrix}, \quad \hat{A} = \begin{pmatrix} Q_x & -\sqrt{r_\xi}G\circ \\ -\sqrt{r_\xi}G\circ & r_\xi Q_x \end{pmatrix}, \quad r_\xi = \frac{\xi_x^{(2)}}{\xi_x^{(1)}} = \frac{N_1}{N_2} \leq 1, \tag{12}$$

the function Q_x is given by (6). Assuming $\boldsymbol{f} \sim \exp(-\mathrm{i}\xi_x^{(1)}\lambda\theta)$ we finally arrive at the eigenvalue problem formulated in Reference 4:

$$\lambda\boldsymbol{f} = \hat{A}\boldsymbol{f}. \tag{13}$$

The operator $\hat{A}$ acts in space $\boldsymbol{D}_A$ of 2-tuples $\boldsymbol{X} = (X_1, X_2)^{\mathrm{T}}$ whose components are functions of the action variables of the corresponding beam. The scalar product defines a metric on this space:

$$(\boldsymbol{X}, \boldsymbol{Y}) = \int (X_1^* Y_1 + X_2^* Y_2)\,\mathrm{d}I_x\,\mathrm{d}I_y. \tag{14}$$

Some general properties of the operator $\hat{A}$ allow making conclusions concerning its spectrum. This operator is self-conjugate and bounded (but not compact owing to the multiplicative Q-part) so that its eigenvalues are real, bounded and form a continuous set (possibly with a discrete addition).

One particular solution of (13) can be found analytically. It can be verified (see Reference 4) that the function

$$\Psi_0(I_x, I_y) = \sqrt{I_x}\mathrm{e}^{-(I_x+I_y)/2} \tag{15}$$

satisfies the integral equation

$$Q_x\Psi_0 = G\circ\Psi_0. \tag{16}$$

Accordingly, (13) has a solution

$$f^{(1)} = f^{(2)} = \sqrt{\frac{2}{1+r_\xi}}\Psi_0(I_x, I_y) \tag{17}$$

with $\lambda = 0$. This eigenvalue belongs to the discrete part of the spectrum since the corresponding eigenfunction has a finite norm (we have chosen $\sqrt{2}$). Physically it corresponds to the rigid Σ-mode in which the beams oscillate in phase at the IP without changing their shape.

The other solutions can be found numerically. Let us start with a simpler case of equal intensities, $r_\xi = 1$, when due to the symmetry between the beams space $\boldsymbol{D}_A$ splits into an orthogonal direct sum of two invariant subspaces corresponding to Σ-modes $f^{(1)} = f^{(2)} = f^{(+)}$ and π-modes $f^{(1)} = -f^{(2)} = f^{(-)}$. Defining projecting matrices

$$\hat{P}_+ = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad \hat{P}_- = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \tag{18}$$

we can present operator $\hat{A}$ in the form

$$\hat{A} = \hat{P}_+(Q_x - G\circ) + \hat{P}_-(Q_x + G\circ). \tag{19}$$

Accordingly, system (13) is reduced to decoupled eigenvalue problems for Σ- and π-modes:

$$\lambda f^{(\pm)} = Q_x f^{(\pm)} \mp G \circ f^{(\pm)}. \tag{20}$$

Owing to the Q-term each of Eqs. (20) has a solution for any $\lambda \in (0,1)$. To get a notion of the form of the corresponding eigenfunctions it is convenient to introduce new variables $q = Q_x(I_x, I_y)$, $\chi = \arctan(I_x/I_y)$. It is obvious then that an arbitrary function $h(\chi)$ will generate a pair of eigenfunctions satisfying the equations

$$\Psi_\lambda^{(\pm)} = \mp \mathrm{p.v.} \frac{1}{\lambda - q} \cdot G \circ \Psi_\lambda^{(\pm)} + h(\chi) \cdot \delta(\lambda - q). \tag{21}$$

Therefore every eigenvalue from the continuum $\lambda \in (0,1)$ has an infinite multiplicity. Choosing an appropriate set of functions $\{h_n(\chi)\}$, where $n = 1, 2, \ldots$ is the number of nodes, we can construct two families of eigenfunctions satisfying the orthonormality condition

$$\int \Psi_{\lambda n}^{(\pm)}(I_x, I_y) \Psi_{\lambda' n'}^{(\pm)}(I_x, I_y)\, \mathrm{d}I_x\, \mathrm{d}I_y = \delta_{nn'} \delta(\lambda - \lambda'). \tag{22}$$

The physical meaning of these eigenmodes can be understood on the analogy of the Schottky noise. The term with $h(\chi)$ in the r.h.s. of (21) gives some prime perturbation of particles with a particular tune while the first term describes collective response of the other particles. So these modes are incoherent in their origin.

As found in Reference 4 there is a discrete eigenvalue,‡ $\lambda = \lambda_0 \approx 1.214$ in the case of round beams ($r = 1$), for the π-oscillations as well which corresponds to a truly coherent motion.¶ To understand the character of this mode let us introduce function $d(x, y)$ which describes non-rigidity of bunch oscillations. With its help the charge density of the perturbed beam can be expressed through the equilibrium density as

$$\tilde{\rho}(x, y) = \rho_0(x - x_c d(x, y), y),$$

where x_c is displacement of the beam barycenter. For the beam shifted as a whole $d(x, y) \equiv 1$. Figure 1 shows function $d(x, y)$ for the discrete π-mode obtained by the Fourier–Laguerre expansion method of Reference 4. The maximum value is $d(0, 0) = 3.27$.

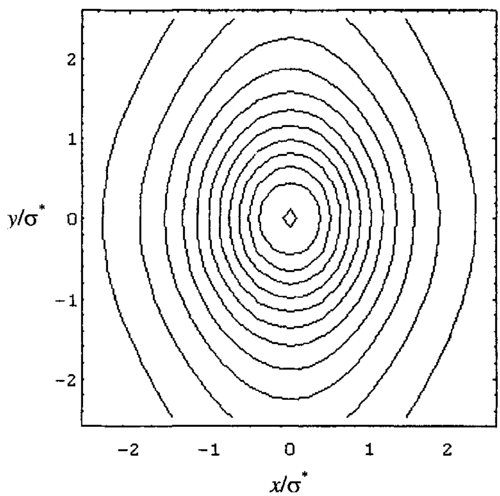

FIGURE 1 Contour plot of function $d(x, y)$ of the discrete π-mode. Shown are levels with a step 0.25 starting from the value 0.5.

‡ In principle there could have been a larger (but finite) number of discrete eigenvalues.

¶ Tuneshifts of $(1.2-1.3)\xi$ were obtained for individual particles in simulation described in Reference 5 but attributed to the incoherent oscillations. In fact each particle participates in both incoherent and coherent motion (especially particles with small amplitudes) so spectrum of its oscillations contains information on both tunes.

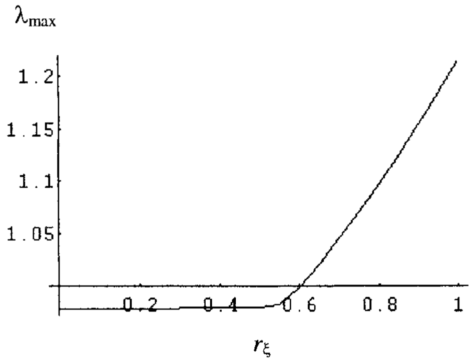

FIGURE 2 Largest eigenvalue λ_{max} vs. intensity ratio r_ξ.

As can be seen in Figure 1 in this mode of oscillations mainly particles with small incoherent betatron amplitudes participate which are strongly affected by the movements of the opposing beam. This explains the large value of the coherent tune shift.

It is clear that in the weak–strong case ($r_\xi \rightarrow 0$) such a mode does not exist (in contradistinction to the discrete Σ-mode which exists at any value of the intensity ratio r_ξ). Therefore it seems interesting to trace at what r_ξ the discrete eigenvalue emerges from the continuum $\lambda = (0, 1)$ which corresponds to the incoherent tune spread in the weaker beam. Figure 2 shows dependence on r_ξ of the largest eigenvalue λ_{max} found by numerical integration of (13). The trapezoid rule was used for integration with the number of points in (I_x, I_y)-plane equal to $N_p = 17 \times 18 = 306$ (the number of points in I_x-direction was less by one since it had been possible to exclude points with $I_x = 0$ where all eigenfunctions tend to zero).

Nascence of the discrete eigenvalue is clearly seen at $0.55 < r_\xi < 0.6$. Starting from the value $\lambda_{max} = 0.978$ corresponding to the maximum Q_x value in the mesh points with the chosen N_p, λ_{max} keeps practically constant[§] until $r_\xi \approx 0.55$ where a steep rise begins. Transition of λ_{max} from continuum to point spectrum can be confirmed by the dependence on N_p of the scalar product of the corresponding eigenfunction with a

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[§] In contrast to what was found in Reference 4. The difference may be a consequence of a slow convergence of the Laguerre–Fourier series used in Reference 4 for the continuum modes.

well-behaved function, e.g. the 2-tuple $(\Psi_0, -\Psi_0)$. For the continuum modes this product should behave approximately as $N_p^{-1/2}$, whereas for a discrete mode it should be practically independent of N_p. According to this criterion the discrete eigenvalue appears at $r_\xi \approx 0.6$.

4 SPECTRAL DECOMPOSITION

Since the operator $\hat{A}$ is not degenerate its eigenfunctions form a complete basis in $\boldsymbol{D}_A$. We will limit the following analysis to the case $r_\xi = 1$ only. In this case the eigenmodes split into Σ- and π-families with spectrum of each family comprising continuum $\lambda \in (0,1)$ and one discrete eigenvalue, $\lambda = 0$ for Σ-modes and $\lambda = \lambda_0 \approx 1.214$ for π-modes. Every eigenvalue from the continuum has infinite but countable multiplicity. Correspondingly, the spectral decomposition of operator $\hat{A}$ (and its powers including the identity operator $\hat{I}$) is the Stieltjes integral

$$\begin{aligned} \hat{A}^n &= \sum_{\alpha=+,-} \hat{P}_\alpha \int \mathrm{d}w_\alpha(\lambda) \cdot \lambda^n E_\lambda^{(\alpha)} \circ, \quad n = 1, 2, \ldots, \\ \hat{I} &= \sum_{\alpha=+,-} \hat{P}_\alpha \int \mathrm{d}w_\alpha(\lambda) \cdot E_\lambda^{(\alpha)} \circ, \end{aligned} \tag{23}$$

where the weight functions

$$w_+(\lambda) = \begin{cases} 0, & \lambda < 0, \\ 1+\lambda, & 0 \le \lambda < 1, \\ 2, & 1 \le \lambda, \end{cases} \qquad w_-(\lambda) = \begin{cases} 0, & \lambda < 0, \\ \lambda, & 0 \le \lambda < 1, \\ 1, & 1 \le \lambda < \lambda_0, \\ 2, & \lambda_0 \le \lambda \end{cases} \tag{24}$$

and the projecting integral operators

$$E_\lambda^{(\pm)} \circ f = \sum_n \Psi_{\lambda n}^{(\pm)}(I_x, I_y) \int \Psi_{\lambda n}^{(\pm)}(I'_x, I'_y) f(I'_x, I'_y)\, \mathrm{d}I'_x \mathrm{d}I'_y \tag{25}$$

were introduced. The sum in (25) is reduced to one term if λ belongs to the point spectrum.

Using representation (23) we can perform expansion in terms of the operator $\hat{A}$ eigenfunctions:

$$\boldsymbol{f}=\sum_{\alpha=+,-}\boldsymbol{u}_\alpha\int \mathrm{d}w_\alpha(\lambda)\cdot\sum_n a_{\lambda n}^{(\alpha)}(\theta)\Psi_{\lambda n}^{(\alpha)},\quad a_{\lambda n}^{(\alpha)}(\theta)=\tfrac{1}{2}(\boldsymbol{u}_\alpha\Psi_{\lambda n}^{(\alpha)},\boldsymbol{f}),\tag{26}$$

where the scalar product is defined by (14) and

$$\boldsymbol{u}_+=\begin{pmatrix}1\\1\end{pmatrix},\qquad \boldsymbol{u}_-=\begin{pmatrix}1\\-1\end{pmatrix}\tag{27}$$

are eigenvectors of the projecting matrices (18). Solution of the initial-value problem for (11) is then given by (26) and

$$a_{\lambda n}^{(\pm)}(\theta)=\mathrm{e}^{-\mathrm{i}\xi\lambda\theta}a_{\lambda n}^{(\pm)}(0),\qquad \xi\equiv\xi_x^{(1)}=\xi_x^{(2)}.\tag{28}$$

As a rule it is not the distribution function itself, which presents the most interest but some integral characteristics of the beams, such as the barycenter displacement, emittance etc.

To describe the barycenter motion let us introduce the complexified Courant–Snyder variable

$$\eta=\frac{x+\mathrm{i}(\beta_x p_x+\alpha_x x)}{\sqrt{\beta_x\varepsilon_0}}.\tag{29}$$

Making use of (1), (9), (15), we obtain the following expression, correct to the first order in ξ, for the centroid of the kth beam:

$$\begin{aligned}\eta_c^{(k)}&=\int\eta F_1^{(k)}\mathrm{d}\psi_x\mathrm{d}\psi_y\mathrm{d}I_x\mathrm{d}I_y=2\sqrt{2}\pi^2\mathrm{i}\mathrm{e}^{-\mathrm{i}\phi_x(\theta)}\\&\quad\times\int\Psi_0(I_x,I_y)f^{(k)}(I_x,I_y,\theta)\,\mathrm{d}I_x\,\mathrm{d}I_y\\&=2\sqrt{2}\pi^2\mathrm{i}\mathrm{e}^{-\mathrm{i}\phi_x(\theta)}\left[a_0^{(+)}(\theta)-(-1)^k\int\mathrm{d}w_-(\lambda)\cdot\sum_n c_n(\lambda)a_{\lambda n}^{(-)}(\theta)\right]\end{aligned}\tag{30}$$

where

$$c_n(\lambda)=\int\Psi_0(I_x,I_y)\Psi_{\lambda n}^{(-)}(I_x,I_y)\,\mathrm{d}I_x\,\mathrm{d}I_y.\tag{31}$$

Since Ψ_0 is the eigenfunction corresponding to the discrete Σ-mode, the other (continuum) Σ-modes being orthogonal to Ψ_0 do not enter (30).

The coefficients (31) play the key role in the subsequent analysis. It will be shown that the sum

$$s(\lambda) = \sum_n c_n^2(\lambda) \tag{32}$$

describes spectral density of dipole oscillations. With the use of the particular property (16) of function Ψ_0 a few moments of $s(\lambda)$ can be found analytically

$$\begin{aligned} \int s(\lambda)\mathrm{d}w_-(\lambda) &= \int \Psi_0^2 \,\mathrm{d}I_x\,\mathrm{d}I_y = 1, \\ \int \lambda s(\lambda)\mathrm{d}w_-(\lambda) &= 2\int Q_x\Psi_0^2\,\mathrm{d}I_x\,\mathrm{d}I_y = 1, \\ \int \lambda^2 s(\lambda)\mathrm{d}w_-(\lambda) &= 4\int Q_x^2\Psi_0^2\,\mathrm{d}I_x\,\mathrm{d}I_y \approx 1.09907, \end{aligned} \tag{33}$$

(the first one being just the Parseval identity) to serve for the accuracy control of numeric calculations.

When eigenfunctions are found numerically, their coefficients exhibit chaotic dependence on the eigenvalue (see Figure 3) since eigenfunctions for close but different λ should describe all variations in χ that are possible with the given number of mesh points N_p. Hence to obtain a

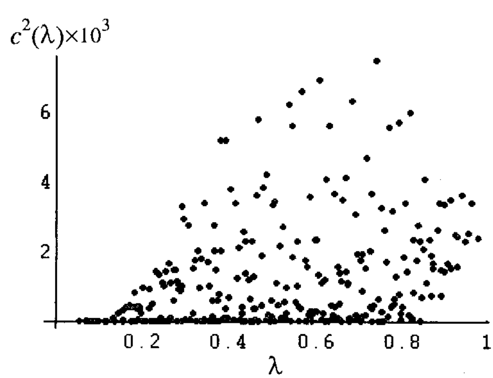

FIGURE 3 Eigenmode coefficients vs. eigenvalues from the continuum range (0, 1) obtained with $N_p = 306$.

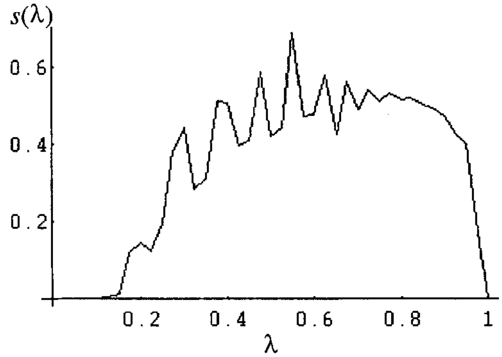

FIGURE 4 Spectral density of dipole oscillations.

smooth function $s(\lambda)$ one should perform summing over sufficiently large intervals $\Delta\lambda$.

Figure 4 shows function $s(\lambda)$ in the continuum range which was obtained with $\Delta\lambda = 0.025$ from the numerical data presented in Figure 3 in such a way that integration of functions $s(\lambda)$, $\lambda s(\lambda)$, $\lambda^2 s(\lambda)$ by the Simpson rule gives locally the same result with direct summation of coefficients over each paired step $2\Delta\lambda$. The total values of moments (33) found numerically with $N_p = 306$ were 0.9995, 0.9993, 1.0980. Oscillations of $s(\lambda)$ in Figure 4 have no physical implication and can be reduced by increasing the number of points N_p and/or integration intervals $\Delta\lambda$.

For the discrete π-mode **

$$s_0 \equiv s(\lambda_0) = c^2(\lambda_0) \approx 0.645. \tag{34}$$

5 BEAM RESPONSE TO A KICK

Let us consider the effect of a kick received by one of the beams (the first one for certainty) at $\theta = \theta_0$ assuming its magnitude, δp_x, to be independent of particle position. Extracting the part linear in δp_x from

** The cited value is specific for round beams. The coefficient s_0 is the weight with which the discrete π-mode is excited by an external force. Its knowledge is essential for interpretation of the observed in Reference[5] spectra of the beam–beam oscillations.

the perturbed distribution function taken just after the kick

$$
\begin{aligned}
F^{(1)}(x, y, p_x, p_y\, \theta_0) &= F_0(x, y, p_x - \delta p_x, p_y\, \theta_0) \\
&= F_0(I_x, I_y) - \delta p_x \sqrt{\frac{2\beta_x(\theta_0) I_x}{\varepsilon_0}} \cos[\phi_x(\theta_0) - \nu_{x0}\theta_0 + \psi_x] \\
&\quad \times \frac{\mathrm{d}F_0}{\mathrm{d}I_x}(I_x, I_y) + O[(\delta p_x)^2], \qquad (35)
\end{aligned}
$$

we obtain the associated jump in the normalized first order distribution function which we present in vector notations to make provision for kick on the other beam:

$$
\delta f(\theta_0) = \frac{\vec{\Delta}}{2\sqrt{2}\pi^2} \mathrm{e}^{\mathrm{i}\phi_x(\theta_0)} \Psi_0(I_x, I_y), \quad \vec{\Delta} = \begin{pmatrix} \Delta_1 \\ \Delta_2 \end{pmatrix} \qquad (36)
$$

where Ψ_0 is given by (15). For the particular perturbation

$$
\Delta_1 = \delta p_x \sqrt{\frac{\beta_x(\theta_0)}{\varepsilon_0}}, \qquad \Delta_2 = 0.
$$

It is obvious that Δ is just excited by the kick oscillation amplitude taken in the beam σ's.

The corresponding variation in the expansion coefficients is

$$
\begin{aligned}
\delta a_{\lambda n}^{(+)}(\theta_0) &= \frac{\boldsymbol{u}_+ \cdot \vec{\Delta}}{4\sqrt{2}\pi^2} \mathrm{e}^{\mathrm{i}\phi_x(\theta_0)} \times \begin{cases} 1, & \lambda = 0, \\ 0, & \lambda \neq 0, \end{cases} \\
\delta a_{\lambda n}^{(-)}(\theta_0) &= \frac{\boldsymbol{u}_- \cdot \vec{\Delta}}{4\sqrt{2}\pi^2} \mathrm{e}^{\mathrm{i}\phi_x(\theta_0)} c_n(\lambda)
\end{aligned} \qquad (37)
$$

with $c_n(\lambda)$ defined by (31). For the beam barycenter motion from (30), (32) and (37) follows

$$
\eta_c^{(k)}(\theta) = \mathrm{i}\frac{\Delta_1}{2} \mathrm{e}^{-\mathrm{i}[\phi_x(\theta) - \phi_x(\theta_0)]} \left[1 - (-1)^k \int \mathrm{e}^{-\mathrm{i}\xi\lambda(\theta - \theta_0)} s(\lambda) \mathrm{d}w_-(\lambda)\right], \qquad (38)
$$

where the first term in the square brackets corresponds to the discrete (rigid) Σ-mode and the second one describes contribution from all

π-modes. Figure 5 shows envelope (absolute value) of the π-modes contribution (and separately contribution from the continuum modes only) as a function of $|\xi|N$ where N is the number of turns $N = \theta/2\pi$. Contribution from the continuum modes smears out in $N \approx 1/|\xi|$ turns leaving the discrete π-mode with amplitude $c^2(\lambda_0)\Delta_1/2$. Envelopes of the total centroid displacements shown in Figure 6 exhibit beatings due to tune-split between the discrete π- and Σ-modes.

To find emittances of perturbed beams let us first note that in the considered case of horizontal dipole oscillations the first order

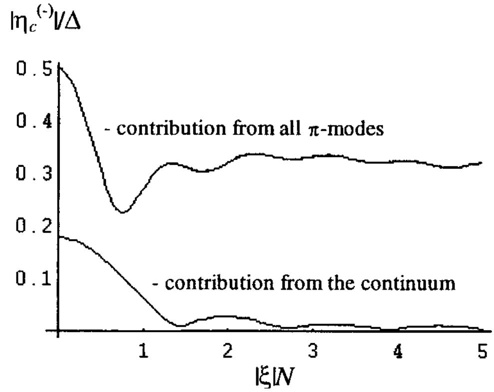

FIGURE 5 Envelope of the π-component of the beam centroid oscillations.

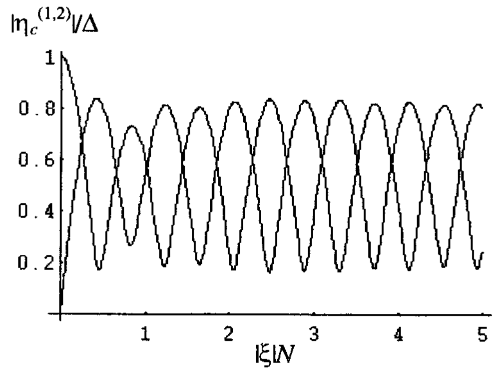

FIGURE 6 Envelopes of the total centroid displacements after a kick at one of the beams.

Liouville equation can be rewritten as

$$\frac{\partial H_1^{(k)}}{\partial \psi_x} = -\frac{1}{F_0}\left(\frac{\partial F_1^{(k)}}{\partial \theta} + \nu_x \frac{\partial F_1^{(k)}}{\partial \psi_x}\right) - \frac{\partial H_1^{(k)}}{\partial \psi_y}, \tag{39}$$

where F_0 is the equilibrium distribution (3). Now we have up to the second order in ξ

$$\begin{aligned}\frac{1}{\varepsilon_0}\frac{\mathrm{d}\varepsilon_x^{(k)}}{\mathrm{d}\theta} &= \int (F_0 + F_1^{(k)})\frac{\mathrm{d}I_x}{\mathrm{d}\theta}\,\mathrm{d}\Omega = -\int F_1^{(k)}\frac{\partial}{\partial \psi_x}H_1^{(k)}\,\mathrm{d}\Omega \\ &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\theta}\int \frac{1}{F_0}(F_1^{(k)})^2\mathrm{d}\Omega = 4\pi^4\frac{\mathrm{d}}{\mathrm{d}\theta}\int |f^{(k)}|^2\,\mathrm{d}I_x\,\mathrm{d}I_y,\end{aligned} \tag{40}$$

where $\mathrm{d}\Omega = \mathrm{d}\psi_x \mathrm{d}\psi_y \mathrm{d}I_x \mathrm{d}I_y$. In the particular case of initial conditions (37)

$$\frac{\varepsilon_x^{(1,2)}}{\varepsilon_0} = 1 + \frac{\Delta_1^2}{8}\left\{1 \pm 2\int \cos[\xi\lambda(\theta-\theta_0)]s(\lambda)\,\mathrm{d}w_-(\lambda) + \int s(\lambda)\mathrm{d}w_-(\lambda)\right\}. \tag{41}$$

The first and the third terms in curly brackets (equal due to the Parseval identity (33)) describe relative partition of energy between Σ- and π-modes, the second term being the interference term which cancels out in the sum for two beams. The total increment of emittance is $\Delta_1^2/2$. As follows from (41) only a small fraction of energy, namely $(1-s_0)/2 \approx$ 18%, is imparted into the continuum modes leading to the irreversible emittance growth, the other 82% are carried by discrete modes which in principle can be damped by a feedback system.

6 LANDAU DAMPING

Now let us include in the consideration linear elements reacting on the barycenter motion of the beams, assuming them to be identical in both rings so that Σ- and π-modes remain uncoupled. Also, for the present purposes we may uniformly distribute these elements over the ring circumference and write for the elementary kicks produced by them

$$\vec{\Delta}(\theta) = -\mathrm{i}\zeta\vec{\eta}_c\,\mathrm{d}\theta, \tag{42}$$

where ζ is a complex parameter related to integrated transverse impedance $\Sigma\beta_x Z_\perp$ and/or feedback gain factor.

The Liouville equation will now include a term associated with these kicks

$$\frac{\partial f}{\partial\theta} = -\mathrm{i}\xi\hat{A}f + \frac{\delta f}{\mathrm{d}\theta}, \tag{43}$$

where δf is of the form (36) with $\vec{\Delta}$ given by (42). Expanding (43) in the eigenmodes we obtain for the coefficients

$$\begin{aligned} a_0^{(+)}(\theta) &= a_0^{(+)}(0) - \mathrm{i}\frac{\zeta}{2\sqrt{2}\pi^2}\int_0^\theta b_+(\theta')\,\mathrm{d}\theta', \\ a_{\lambda n}^{(-)}(\theta) &= \mathrm{e}^{-\mathrm{i}\xi\lambda\theta}a_{\lambda n}^{(-)}(0) - \mathrm{i}\frac{\zeta c_n(\lambda)}{2\sqrt{2}\pi^2}\int_0^\theta \mathrm{e}^{-\mathrm{i}\xi\lambda(\theta-\theta')}b_-(\theta')\,\mathrm{d}\theta', \end{aligned} \tag{44}$$

where the slow varying coherent amplitudes were introduced

$$b_\pm(\theta) = \tfrac{1}{2}\mathrm{e}^{\mathrm{i}\phi_x(\theta)}\boldsymbol{u}_\pm\cdot\vec{\eta}_c(\theta). \tag{45}$$

From (30), (44) follow integral equations for coherent amplitudes (45). Solution for the rigid Σ-mode is simply

$$a_0^{(+)}(\theta) = a_0^{(+)}(0)\mathrm{e}^{\zeta\theta}, \qquad b_+(\theta) = 2\sqrt{2}\pi^2\mathrm{i}a_0^{(+)}(\theta), \tag{46}$$

so that ζ is just multiplied by $-\mathrm{i}$ single-beam coherent tune shift. Equation for the π-component of barycenter motion can be solved using the Laplace transformation:

$$\begin{aligned} b_-(\theta) &= \frac{1}{2\pi\mathrm{i}}\int_{\sigma-\mathrm{i}\infty}^{\sigma+\mathrm{i}\infty}\mathrm{e}^{p\theta}b_-(p)\,\mathrm{d}p, \\ b_-(p) &= \frac{2\sqrt{2}\pi^2\mathrm{i}}{D(p,\zeta)}\int\frac{\sum c_n(\lambda)a_{\lambda n}^{(-)}(0)}{p+\mathrm{i}\xi\lambda}\,\mathrm{d}w_-(\lambda), \end{aligned} \tag{47}$$

where the dispersion function was introduced:

$$D(p,\zeta) = 1 - \zeta \int \frac{s(\lambda)}{p + \mathrm{i}\xi\lambda}\, \mathrm{d}w_-(\lambda). \tag{48}$$

This function is analytical in the complex domain of p with exception of the point $p = -\mathrm{i}\xi\lambda_0$ where it has the first order pole, and the cut on imaginary axis $p \in (0, \mathrm{i}|\xi|)$. Zeros of the dispersion function (if any) give tune shifts (generally complex) of free π-oscillations in colliding beams. In the limiting case $|\zeta| \ll |\xi| \ll 1$ there is the unique solution

$$p_0 \approx -\mathrm{i}\xi\lambda_0 + \zeta s_0 \tag{49}$$

which shows some 35% reduction in the effect of external elements on the π-mode in comparison with that on the Σ-mode (and a single beam oscillations as well). This reduction is merely the consequence of partition of energy delivered by elementary kicks (42) between the discrete and continuum π-modes and is not a form of the Landau damping.

It is important to note that although the continuum eigenmodes receive about 35% of energy from every elementary kick, in the case of instability ($\mathrm{Re}\,\zeta > 0$) there is no appreciable build up of energy in these modes since the kicks are not in phase due to the large (compared to $|\zeta|$) gap between the discrete π-mode tune and the boundary of continuum. In the limit $\theta \to \infty$ from (44), (47) follows for the ratio of expansion coefficients

$$\frac{a_{\lambda n}^{(-)}(\theta)}{a_0^{(-)}(\theta)} \to \mathrm{i}\frac{\zeta c_n(\lambda) c(\lambda_0)}{\xi(\lambda_0 - \lambda)}. \tag{50}$$

Correspondingly, in the considered limiting case $|\zeta| \ll |\xi|$ contribution of the continuum modes to the beam emittance growth (40) is negligible, the latter being completely determined by the discrete mode amplitude which testifies once more the absence of the Landau damping.

As follows from the above discussion the beam–beam tune spread does not provide the Landau damping up to the first order in ξ. In a real beam, however, the Σ-mode can be damped by non-Gaussian tail particles if there are other sources of tune spread, i.e. the lattice nonlinearity and chromaticity. This additional tune spread is of the order of 10^{-4} in machines like LHC, which is marginally sufficient for suppression of the transverse instabilities at the top energy. But it is insufficient to span the gap between the discrete π-mode and incoherent tunes which has the order of 10^{-3}. As the consequence the discrete π-mode can become unstable when beams are put into collision. The possibility of damping this mode due to nonlinear coupling to the continuum modes is yet to be studied.

In conclusion of this section let us consider a hypothetical situation when interaction with some external elements (e.g. reactive feedback) produce sufficiently large positive coherent tune shift, $\Delta\nu = -\zeta'' \equiv -\mathrm{Im}\,\zeta > (\lambda_0 - 1)|\xi|$, in order to bring the coherent tune within the continuum range. Looking for the solution of the dispersion relation $D(p,\zeta)=0$ in the form $p=\alpha - \mathrm{i}\xi\mu$ and making use of the Sohotsky formula

$$\begin{aligned} D^{\pm}(-\mathrm{i}\xi\mu,\zeta) &\equiv D(-\mathrm{i}\xi\mu \pm 0,\zeta) \\ &= 1 - \mathrm{i}\frac{\zeta}{|\xi|}\left[\frac{s_0}{\lambda_0-\mu} + \mathrm{p.v.}\int_0^1 \frac{s(\lambda)\mathrm{d}\lambda}{\lambda-\mu} \mp \pi \mathrm{i} s(\mu)\right] \end{aligned} \tag{51}$$

we obtain in the limit $|\alpha| \ll |\xi|$ for imaginary and real parts of the dispersion relation

$$\begin{aligned} &\frac{s_0}{\lambda_0-\mu} + \mathrm{p.v.}\int_0^1 \frac{s(\lambda)\mathrm{d}\lambda}{\lambda-\mu} = \frac{|\xi|\zeta''}{|\zeta|^2}, \\ &\alpha = \frac{(\lambda_0-\mu)^2}{s_0}\left[\frac{\xi^2\zeta'}{|\zeta|^2} - \pi|\xi|s(\mu)\,\mathrm{sgn}\,\alpha\right], \end{aligned} \tag{52}$$

where it was assumed that μ defined by the first equation falls within the range (0,1). For $\zeta' < \pi s(\mu)|\zeta|^2/|\xi|$ the second equation (hence the dispersion equation on the whole) has no solution which means that the π-mode is completely Landau damped. But one should realize that large positive coherent tune shift due to external elements would switch off the Landau damping of the Σ-mode (if there had been any).

7 EMITTANCE GROWTH IN PRESENCE OF LOW GAIN LINEAR FEEDBACK

The developed formalism can be employed in analysis of emittance growth in collision regime due to noise and its suppression by a feedback system. The damping effect of a low gain linear feedback on a single beam in absence of collisions can be described by simply putting $\zeta = -g/4\pi$ in (42), where g is the feedback gain factor. We will assume that both rings have independent feedback systems with equal gain factors.

Let us first consider the evolution of the modes after a kick. The dependence of the expansion coefficients on time can be found from (44)–(48) with initial conditions given by (37). Solution for the Σ-mode is just exponential fall-off. The Laplace transform of the π-mode coefficient is

$$a_{\lambda n}^{(-)}(p) = e^{i\phi_x(\theta_0) - p\theta_0} \cdot \frac{\boldsymbol{u}_- \cdot \vec{\Delta}}{4\sqrt{2}\pi^2} \cdot \frac{c_n(\lambda)}{(p + i\xi\lambda) D(p, -g/4\pi)}. \tag{53}$$

For all λ including the discrete eigenvalue λ_0 it has a pole in the left half-plane, Re $p < 0$, corresponding to zero of the dispersion function p_0. For λ from the continuum (0,1) there is also a pole at $p_\lambda = -i\xi\lambda$ lying strictly on the cut (see Figure 7), whereas for $\lambda = \lambda_0$ there is no additional pole since the denominator in (53) does not vanish at $p \to -i\xi\lambda_0$. Therefore in the limit $\theta \to \infty$ only the continuum modes persist, both Σ- and π-discrete modes are damped to zero.

To determine the asymptotic behavior of the continuum modes at $\theta \to \infty$ let us deform the path of integration in the complex p-plane as

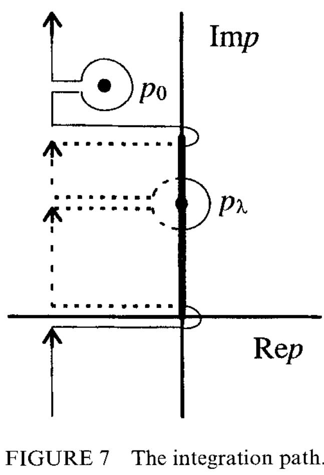

FIGURE 7 The integration path.

shown in Figure 7, threading it into and out of the cut and encircling the pole at $p_\lambda = -i\xi\lambda$. For $D(p,\zeta)$ inside the cut we must take its analytic continuation from the right side of the cut where it is given by the Sohotsky formula (51) with the upper sign. In the limit $\theta \to \infty$ contribution from the parts of the contour lying in the left halfplane vanish leaving us with the residue in the pole p_λ,

$$a_{\lambda n}^{(-)}(\theta) \to e^{-i\xi\lambda(\theta-\theta_0)+i\phi_x(\theta_0)} \cdot \frac{\boldsymbol{u}_- \cdot \vec{\Delta}}{4\sqrt{2}\pi^2} \cdot \frac{c_n(\lambda)}{D^+(-i\xi\lambda, -g/4\pi)}. \tag{54}$$

Now with the help of (26), (40) we can calculate the final emittance values after the kick, which turn out to be equal for both beams no matter which one was kicked (the first beam assumed beneath). The emittance increment can be written in the form

$$\frac{\Delta\varepsilon_x^{(1,2)}}{\varepsilon_0} \to 4\pi^4 \lim_{\theta\to\infty} \int_0^1 \sum_n \left| a_{\lambda n}^{(-)}(\theta) \right|^2 d\lambda = \frac{\Delta_1^2}{8}(1-s_0) S(g/2\pi|\xi|), \tag{55}$$

where

$$S(x) = \frac{1}{1-s_0} \int_0^1 \frac{s(\lambda) d\lambda}{[1+(\pi x/2)s(\lambda)]^2 + (x^2/4)\left[s_0/(\lambda_0-\lambda) + \text{p.v.} \int_0^1 (s(\mu)\, d\mu)/(\mu-\lambda)\right]^2}. \tag{56}$$

Let us explain the factors in the r.h.s. of (55). The two beams share the total energy imparted by the kick, which makes $\Delta_1^2/4$ for each beam (on average over the beatings period). This value is divided equally between Σ- and π-modes. Due to the feedback with whatever small but finite gain factor the discrete Σ- and π-modes are damped so that only the continuum π-modes can contribute to the emittance growth; their relative share in the kick energy being initially equal to $(1-s_0)/2$. The function $S(g/2\pi|\xi|)$ which graphics is shown in Figure 8 describes the effect of the feedback on the continuum modes. With an accuracy of better than 18% at all values of x the following approximation is valid

$$S(x) \approx \frac{1}{(1+x)^2}. \tag{57}$$

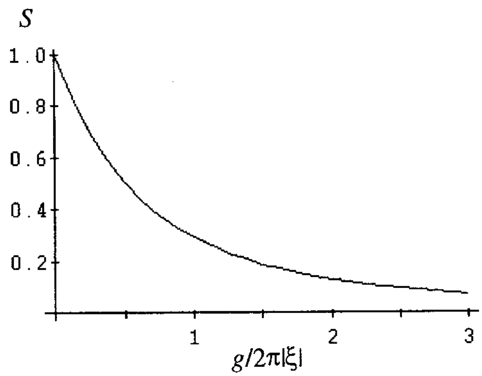

FIGURE 8 Continuum mode suppression factor due to feedback.

These results can be compared with the weak–strong case formulas of Reference 2, which in the present notations look as

$$\frac{\Delta\varepsilon_x^{(\text{weak})}}{\varepsilon_0} = \frac{\Delta_1^2}{2} S_{\text{w-s}}(g/2\pi|\xi|),$$
$$\Delta\varepsilon_x^{(\text{strong})} = 0,\ S_{\text{w-s}}(x)|_{x^2\gg 1} \approx \frac{4}{x^2}\overline{Q_x^2} \approx \frac{0.12}{x^2}, \tag{58}$$

where the bar denotes averaging over the weak beam. It can be seen that the feedback system in the strong–strong case is by an order of magnitude less efficient in suppression of the continuum modes so that in the limit $g \gg 2\pi|\xi|$ the emittance growth in each beam appears to be almost as high as that in the weak beam of the weak–strong pair. This lack of the feedback efficiency is caused by interference from the discrete π-mode which drastically increases the effective tune spread.

The present analysis can be extended on the case of multiple kicks received by both beams. Then the Laplace transform of the mode expansion coefficients will be given by a superposition of terms of the form (53) with the corresponding values for θ_0 and $\vec{\Delta}$. If the noise is a continuous process then the discrete modes being sustained by successive kicks do not vanish but remain bounded whereas the continuum modes may grow until nonlinear effects come into force.

Let us consider the growing modes limiting ourselves to the case when the noise is introduced by a single short element located at $\theta = \theta_0$ in one of the rings. Denoting by $\Delta_1^{(k)}$ the normalized kick magnitude (see (36) for definition) received at the $(k+1)$th turn we obtain for the growing part of the expansion coefficient (54)

$$a_{\lambda n}^{(-)}(\theta) \to \mathrm{e}^{-\mathrm{i}\xi\lambda(\theta-\theta_0)+\mathrm{i}\phi_x(\theta_0)} \cdot \frac{c_n(\lambda)}{4\sqrt{2}\pi^2 D^+} \sum_{k=0}^{N-1} \Delta_1^{(k)} \mathrm{e}^{2\pi\mathrm{i}(\nu_{x0}+\xi\lambda)k}, \tag{59}$$

where $N = \mathrm{Integer}[(\theta-\theta_0)/2\pi]+1$ is the number of passages through the noisy element. We will proceed further in the assumption that the noise can be described as a stationary stochastic process with the normalized correlation function $R(\theta)$:

$$\langle \Delta_1^{(k)} \Delta_1^{(l)} \rangle = \Delta^2 R[2\pi(k-l)], \qquad R(0) = 1, \tag{60}$$

where brackets mean averaging over realizations. Introducing the noise spectral density $\Pi(\nu)$ by the relations

$$R(\theta) = \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\nu\theta}\Pi(\nu)\,\mathrm{d}\nu, \quad \Pi(\nu) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\nu\theta} R(\theta)\,\mathrm{d}\theta \tag{61}$$

we get from (59)

$$4\pi^4 \langle |a_{\lambda n}^{(-)}(\theta)|^2 \rangle \to \frac{\Delta^2 c_n^2(\lambda)}{8|D^+|^2} \int_{-\infty}^{\infty} \frac{\sin^2[\pi(\nu-\nu_{x0}-\xi\lambda)N]}{\sin^2[\pi(\nu-\nu_{x0}-\xi\lambda)]} \Pi(\nu)\,\mathrm{d}\nu. \tag{62}$$

Making use of the formula for periodic δ-function

$$\lim_{N\to\infty} \frac{1}{N} \frac{\sin^2(\pi x N)}{\sin^2(\pi x)} = \sum_{k=-\infty}^{\infty} \delta(x+k) \tag{63}$$

we obtain for the average rate of the emittance growth

$$\frac{1}{\varepsilon_0}\frac{\mathrm{d}\varepsilon_x^{(1,2)}}{\mathrm{d}N} = \frac{\Delta^2}{8}\int_0^1 \frac{s(\lambda)}{|D^+(-\mathrm{i}\xi\lambda,-g/4\pi)|^2} \sum_{k=-\infty}^{\infty} \Pi(\nu_{x0}+\xi\lambda-k)\cdot \mathrm{d}\lambda. \tag{64}$$

Therefore only noise in narrow bands around combination frequencies contribute to the emittance growth. When the noise is due to the ground motion only the term with k closest to the betatron tune may be retained in the sum since the spectral density rapidly falls off with the frequency as $\nu^{-2.5}$ (see Reference 7 and references therein).

In the case of the "white" noise

$$R(\theta) = \lim_{\tau \to 0} \mathrm{e}^{-|\theta|/\tau}, \qquad \Pi(\nu) = \lim_{\tau \to 0} \frac{1}{\pi} \frac{\tau}{1 + \nu^2 \tau^2} \tag{65}$$

all terms in the sum of (64) must be retained. Their summation with subsequent passage to the limit lead to the result

$$\frac{1}{\varepsilon_0} \frac{\mathrm{d}\varepsilon_x^{(1,2)}}{\mathrm{d}N} = \frac{\Delta^2}{8} (1 - s_0) S(g/2\pi|\xi|) \tag{66}$$

which can be obtained directly noticing that in this case the correlation function over n turns is just the Kronecker symbol: $R(2\pi n) = \delta_{n0}$.

When there are uncorrelated noise sources of equal intensity in both rings the growth rate (66) should be doubled. If a common element introduces the noise to both beams element the correlation should be taken into account (keeping in mind that the same θ_0 means for the two beams different, mirror symmetric points).

An important source of noise is the feedback system itself. This noise originates mainly from random errors in measurement of the beam position, δ_{BPM}, which is transferred into the feedback kicker error

$$\delta p_x = g \frac{\delta_{\mathrm{BPM}}}{\sqrt{\beta_{\mathrm{kicker}} \beta_{\mathrm{BPM}}}}. \tag{67}$$

Assuming the noise sources in both rings to be uncorrelated and equal in strength and adding the feedback noise due to the BPM errors with the normalized dispersion

$$\Delta_{\mathrm{BPM}}^2 = \frac{1}{\sigma_x^2} \langle \delta_{\mathrm{BPM}}^2 \rangle, \quad \sigma_x = \sqrt{\varepsilon_0 \beta_{\mathrm{BPM}}}, \tag{68}$$

we can finally write for the emittance growth rate

$$\frac{1}{\varepsilon_0} \frac{\mathrm{d}\varepsilon_x^{(1,2)}}{\mathrm{d}N} = \frac{1 - s_0}{4} (\Delta^2 + g^2 \Delta_{\mathrm{BPM}}^2) S(g/2\pi|\xi|). \tag{69}$$

Let us take LHC for numerical example. There is a number of reasons which make transverse feedback indispensable in the collision mode. The first one is the lack of the Landau damping discussed in Section 6 which leaves undamped slow instabilities, such as the resistive wall transverse instability. Its rise time at the top energy can be estimated from data of Reference 8 as $\tau_{\mathrm{r.w.}} \approx 0.2\,\mathrm{s}$. Another reason arises from the necessity to put the so-called PACMAN bunches (see Reference 9 for definition) into the common orbit with the help of a pulsed system which will introduce noise due to pulse-to-pulse jitter.

The total beam–beam parameter for two head-on and a number of long-range collisions can be as high as $|\xi| = 0.01$. For the feedback gain factor let us take the typical value $g = 0.2$. Imposing then the requirement on the emittance growth to be limited by a factor of two in 8 h ($3.24 \cdot 10^8$ turns) and allowing the feedback system to make an equal contribution with the other sources of noise we get the limitations $\Delta \leq 5 \cdot 10^{-4}$, $\Delta_{\mathrm{BPM}} \leq 2.5 \cdot 10^{-3}$. With $\varepsilon_0 = 5 \cdot 10^{-10}\,\mathrm{m}$, $\beta_{\mathrm{BPM}} = 200\,\mathrm{m}$ ($\sigma_x = 0.316\,\mathrm{mm}$) these correspond to the absolute r.m.s. values of the betatron amplitude excited by the external noise and the BPM error

$$\delta x \leq 0.16\,\mu\mathrm{m}, \qquad \delta_{\mathrm{BPM}} \leq 0.8\,\mu\mathrm{m}. \tag{70}$$

For the sake of completeness let us assess the contribution from the discrete modes into the beam emittance. Amplitude of the Σ-mode can be easily found with the help of (37) and (46) with $\zeta = -g/4\pi$, that of the π-mode is given by the residue of the superposition of coefficients (53) in the pole p_0 (see Figure 7). For the figures from the above example

$$\left(\frac{\Delta\varepsilon_x^{(1,2)}}{\varepsilon_0}\right)_{\mathrm{d.m.}} \approx \frac{1+s_0}{4g}\left(\Delta^2 + g^2\Delta_{\mathrm{BPM}}^2\right) \leq 10^{-6}$$

which is completely negligible.

8 FEEDBACK WITH A STEPWISE TRANSFER FUNCTION

A rather stringent limitation (70) on the BPM resolution in the case of linear feedback revived interest to the idea proposed in Reference 10 to damp the beam oscillations with kicks of a fixed amplitude which are applied when the beam center-of-mass displacement exceeds a certain threshold, x_{th}. This would allow holding the coherent amplitude within the specified limit without introducing the incessant noise.

Let us examine emittance growth with such a feedback in the collision mode considering the two different mechanisms of the coherent oscillations growth: (i) some slow instability when the elementary external kicks are correlated over a period of time much longer than the decoherence time and (ii) the white noise when the kicks are completely uncorrelated.

As was emphasized in Section 6 in the case of a slow instability there is no appreciable build-up of energy in the continuum modes hence no irreversible emittance growth, the latter being caused mainly by the stabilizing kicks. We will consider this case with simplifying assumptions that:

(a) a single bunch motion is unstable with the instability rise time $\tau_0 \approx 0.2$ s (ignoring the fact that the resistive wall instability is really a multibunch effect);
(b) only the π-modes are excited (which requires the kickers in both rings to be fired simultaneously);
(c) the feedback threshold is much larger than the BPM resolution error, $x_{\text{th}} \gg \delta_{\text{BPM}}$.

Figure 9 illustrates the damping scenario. When the barycenter amplitude reaches the threshold, x_{th}, the kickers are actuated putting it down to zero with a small error due to assumption (c). What is important is the mode contents of the beam motion before and after the kick. Before the kick (let us choose its moment for $\theta = 0$) the barycenter

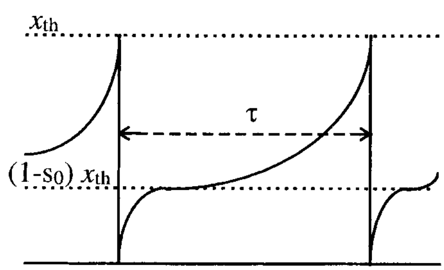

FIGURE 9 Damping scenario in the slow instability case.

motion is determined mainly by the discrete π-mode.[§§] Taking into account $\pi/2$ phase advance from the BPM to the kicker we can derive from (30) the mode expansion coefficient

$$a_0^{(-)}(-0) = -\mathrm{i}\frac{b_-(-0)}{2\sqrt{2}\pi^2 c_0}, \quad b_-(-0) = -\mathrm{i}\frac{x_{\text{th}}}{\sigma_x}, \tag{71}$$

where $\sigma_x = (\varepsilon_0 \beta_{\text{BPM}})^{1/2}$ is the r.m.s. beam size at the BPM location, the barycenter amplitude $b\_(\theta)$ being defined by (45). The normalized kick amplitude necessary to put the beams into their equilibrium orbits is just $\Delta_1 = \mathrm{i}b\_(-0)$, $\Delta_2 = -\Delta_1$. The corresponding jump of the π-modes expansion coefficients can be calculated with the help of (37):

$$\delta a_{\lambda n}^{(-)} = \frac{\Delta_1 - \Delta_2}{4\sqrt{2}\pi^2} c_n(\lambda) = -c_n(\lambda) c_0 a_0^{(-)}(-0). \tag{72}$$

Accordingly, for the barycenter motion after the kick we have

$$b_-(\theta) = b_-(-0)\left[(1 - s_0)\mathrm{e}^{-\mathrm{i}\xi\lambda_0\theta} - \int_0^1 \mathrm{e}^{-\mathrm{i}\xi\lambda\theta} s(\lambda)\,\mathrm{d}\lambda\right], \tag{73}$$

so that when the continuum decoheres we are left with $(1 - s_0) \approx 35\%$ of the threshold amplitude. Due to the instability the threshold will be reached again in the period of time equal (with account of the growth rate reduction (49)) to

$$\tau = \tau_0 \frac{1}{s_0} \ln \frac{1}{1 - s_0}. \tag{74}$$

Due to the BPM error there will be some jitter around this value, which should destroy phase correlation between consecutive jumps of the mode coefficients (72). Adding them up quadratically we will obtain from (40) the average emittance growth rate

$$\frac{1}{\varepsilon_0}\frac{\mathrm{d}\varepsilon_x^{(1,2)}}{\mathrm{d}t} \approx \frac{(1 - s_0)x_{\text{th}}^2}{2\tau\sigma_x^2}. \tag{75}$$

[§§] In the general case the Σ-mode will also contribute.

Having required again no more than doubling emittance in 8 h we obtain from (74), (75) with $\tau_0 \approx 0.2$ s the following limitation on the threshold amplitude

$$x_{\rm th} \leq 8 \cdot 10^{-3} \sigma_x = 2.5\,\mu\text{m}. \tag{76}$$

One might conclude from the present consideration that the period τ could be substantially increased and the emittance growth rate lowered by raising the kick amplitude by a factor of $1/s_0$, so that the discrete π-mode expansion coefficient were cancelled rather than the beam displacement. However, the Σ-mode which is present in the real situation would be overdamped then. So we must accept limitation (76) which, together with the assumption (c) of this section, implies that requirement (70) to the BPM resolution cannot be significantly alleviated.

Let us consider now the white noise case assuming each beam to receive a kick every turn with normalized r.m.s. magnitude Δ. Since the kicks are not correlated, the squared absolute values of the mode coefficients grow on average linearly with the number of turns N. From (37) follow relation between the π- and Σ-mode coefficients growth rate

$$\frac{\mathrm{d}}{\mathrm{d}N}|a_{\lambda n}^{(-)}(N)|^2 = c_n^2(\lambda)\frac{\mathrm{d}}{\mathrm{d}N}|a_0^{(+)}(N)|^2, \quad |a_0^{(+)}(N)|^2 = |a_0^{(+)}(0)|^2 + \frac{\Delta^2}{16\pi^4}N, \tag{77}$$

where $a_0^{(+)}(0)$ is the Σ-mode coefficient value left after the preceding damping kick.

With the first of (77) we can find the rate of the continuous emittance growth due to noise. It is complemented by the emittance growth due to damping kicks. Let us find their repetition rate. These kicks occur when the discrete π- and Σ-modes contribute to the barycenter displacement with either the same or the opposite phases rendering one of the beams displacement maximum equal to $x_{\rm th}$. Neglecting the continuum contribution we have from (30) just before the kicker actuation

$$|a_0^{(+)}(N)| + c(\lambda_0)|a_0^{(-)}(N)| = \frac{x_{\rm th}}{2\sqrt{2}\pi^2\sigma_x}. \tag{78}$$

The jump in the mode coefficients due to a damping kick is given by (37) with $\Delta_1 = -x_{\rm th}/\sigma_{\rm x}$, $\Delta_2 = 0$ (or vice versa if the second kicker was actuated). Noticing now that at all times the approximate equality $|a_0^{(-)}| \approx c(\lambda_0)|a_0^{(+)}|$ holds we obtain for the maximum and minimum amplitudes

$$\begin{aligned} \left|a_0^{(+)}(N)\right| &\approx \frac{1}{1+s_0}\frac{x_{\mathrm{th}}}{2\sqrt{2}\pi^2\sigma_x}, \\ \left|a_0^{(+)}(0)\right| = \left|a_0^{(+)}(N)\right| - \frac{x_{\mathrm{th}}}{4\sqrt{2}\pi^2\sigma_x} &\approx \frac{1-s_0}{1+s_0}\frac{x_{\mathrm{th}}}{4\sqrt{2}\pi^2\sigma_x} \end{aligned} \tag{79}$$

and for the number of turns between consecutive damping kicks

$$N \approx \frac{3-s_0}{2(1+s_0)}\left(\frac{x_{\mathrm{th}}}{\Delta\sigma_x}\right)^2. \tag{80}$$

Since the damping kicks have random phases for the continuum modes they add up quadratically to the emittance growth almost doubling its rate

$$\frac{1}{\varepsilon_0}\frac{\mathrm{d}\varepsilon_x^{(1,2)}}{\mathrm{d}N} \approx \frac{1-s_0}{4}\Delta^2 + \frac{1-s_0}{8N}\frac{x_{\mathrm{th}}^2}{\sigma_x^2} = \frac{1-s_0}{3-s_0}\Delta^2. \tag{81}$$

As the consequence in the present case limitation on the noise amplitude is more stringent than with a linear feedback, in the LHC example $\Delta \leq 1.4 \cdot 10^{-4}$.

9 SUMMARY

The major results obtained in the present paper can be summarized as follows.

- A natural criterion of transition from the weak–strong to the strong–strong case is established which consists in emergence of the discrete spectral line of dipole oscillations; for round beams of equal sizes at the interaction point it takes place at the intensity ratio of about 60%.
- Large beam–beam tunespread fails to provide the Landau damping of the coherent dipole oscillations in the strong–strong case;

moreover, the beam–beam interaction can switch off stabilizing effect of other tunespreads.

- In a perturbation caused by an external kick the discrete modes get about 82% of the delivered energy and only the remaining 18% is imparted into the continuum modes leading to the irreversible emittance growth due to decoherence of these modes.
- The discrete π- and Σ-modes, being unaffected by the decoherence process, can be damped by a linear feedback system with a small gain factor and practically do not contribute to the emittance growth. However, the feedback system is less efficient in damping the continuum modes, which makes the emittance growth rate almost as high as in the weak–strong case under the same conditions.
- Feedback with a stepwise transfer function does not alleviate limitation on the BPM resolution in comparison with the linear case. Moreover, it allows smaller external noise intensity not only being unable to damp the continuum modes but even increasing their growth by the stabilizing kicks.

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APPENDIX. REPRESENTATION OF THE GREEN FUNCTION

Making use of (7) for the perturbed Hamiltonian and performing averaging in the Liouville (8) one can obtain the integral operator (10) kernel in the form

$$G(I_x, I_y, I'_x, I'_y) = (1+r)\cdot \mathrm{e}^{-(I_x+I_y+I'_x+I'_y)/2} B(I_x, I_y, I'_x, I'_y), \tag{A.1}$$

$$\begin{aligned} B = -\frac{1}{(2\pi)^4}\int \ln\Big[\Big(\sqrt{2I_x}\sin\psi_x - \sqrt{2I'}_x\sin\psi'_x\Big)^2 \\ + r^2\Big(\sqrt{2I_y}\sin\psi_y - \sqrt{2I'_y}\sin\psi'_y\Big)^2\Big]\sin\psi_x\sin\psi'_x\,\mathrm{d}\psi_x\,\mathrm{d}\psi'_x\,\mathrm{d}\psi_y\,\mathrm{d}\psi'_y, \end{aligned} \tag{A.2}$$

where integration over period 2π by all angle variables is implied. By performing integration by parts the kernel can be brought into the form presented in Reference 4.

Without loss of generality we may assume that $I_x \leq I'_x$ and introduce notations

$$\begin{gathered} a_x = \sqrt{I_x/I'_x}, \quad a_y = r\sqrt{I_y/I'_x}, \quad a'_y = r\sqrt{I'_y/I'_x}, \\ b = a_y\sin\psi_y - a'_y\sin\psi'_y. \end{gathered} \tag{A.3}$$

Integrating by parts in (A.2) by ψ_x we can present B in the form

$$B = \frac{a_x}{8\pi^4}\mathrm{Re}\int\frac{\cos^2\psi_x\sin\psi'_x}{\sin\psi'_x - a_x\sin\psi_x - \mathrm{i}b}\,\mathrm{d}\psi_x\,\mathrm{d}\psi'_x\,\mathrm{d}\psi_y\,\mathrm{d}\psi'_y. \tag{A.4}$$

One integration in (A.4) (that by ψ'_x being the most convenient) can be performed analytically by transition to the contour integral in the domain of complex variable $z = |z|\exp(\mathrm{i}\psi'_x)$ leading to the result

$$B = a_x\left[1 + \frac{1}{4\pi^3}\mathrm{Im}\int\sqrt{1-(a_x\sin\psi_x + \mathrm{i}b)^2}\sin\psi_x\,\mathrm{d}\psi_x\,\mathrm{d}\psi_y\,\mathrm{d}\psi'_y\right], \tag{A.5}$$

where the sign of the radical should be chosen so that its real part be of the same sign with b. The triple integral in (A.5) can be evaluated either by numerical integration or via the asymptotic expansion:

$$B = a_x\left[1 - \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(n+1)!}\left(\frac{a_x}{2}\right)^{2n}\sum_{m=0}^{\infty}\frac{1}{(m!)^2}\left(\frac{a_{y<}}{2}\right)^{2m}\left(U_{2n+2m+1}(a_{y>})\right.\right.$$
$$\left.\left. - \sum_{l=1}^{m}(-1)^{n+l}\frac{(2n+2l-3)!!(2n+2l-1)!![(2m-2l-1)!!]^2}{a_{y>}^{2m-2l+1}}\right)\right], \tag{A.6}$$

where $a_{y<} = \min[a_y, a'_y]$, $a_{y>} = \max[a_y, a'_y]$ and

$$U_n(a) = \int_0^{\pi/2} R_n(a\sin\Psi)\mathrm{d}\Psi, \quad R_n(x) = \frac{\mathrm{d}^n}{\mathrm{d}x^n}\sqrt{1+x^2}, \tag{A.7}$$

A few first of the functions $U_n(a)$ found with the help of ***Mathematica*** are

U1(a) = ArcTan[a];
U3(a) = − a*(3 + a^2)/(1 + a^2)^2;
U5(a) = a*(45 + 5*a^2 + 11*a^4 + 3*a^6)/(1 + a^2)^4;
U7(a) = − 3*a*(525 − 525*a^2 + 378*a^4 + 222*a^6 + 89*a^8
+ 15*a^10)/(1 + a^2)^6;
U9(a) = − 9*a*(−11025 + 33075*a^2 − 32193*a^4 − 10629*a^6
− 9659*a^8 − 4863*a^10 − 1395*a^12 − 175*a^14)/(1 + a^2)^8;
U11(a) = − 45*a*(218295 − 1285515*a^2 + 2192652*a^4
− 136620*a^6 + 571010*a^8 + 459350*a^10 + 263100*a^12
+ 98884*a^14 + 22015*a^16 + 2205*a^18)/(1 + a^2)^10;
U13(a) = − 675*a*(−2081079 + 20117097*a^2 − 56189133*a^4
+ 30791475*a^6 − 18419830*a^8 − 13164918*a^10
− 11456106*a^12 − 7172650*a^14 − 3192195*a^16
− 958755*a^18 − 174489*a^20 − 14553*a^22)/(1 + a^2)^12.