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# ON THE LAPLACIAN ESTRADA INDEX OF A GRAPH 

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Abstract Let $G$ be a graph of order $n$. Let $\lambda_{1}, \lambda_{2}, \ldots, \lambda_{n}$ be the eigenvalues of the adjacency matrix of $G$, and let $\mu_{1}, \mu_{2}, \ldots, \mu_{n}$ be the eigenvalues of the Laplacian matrix of $G$. Much studied Estrada index of the graph $G$ is defined as $E E=E E(G)=\sum_{i=1}^{n} e^{\lambda_{i}}$. We define and investigate the Laplacian Estrada index of the graph $G, L E E=L E E(G)=\sum_{i=1}^{n} e^{\left(\mu_{i}-\frac{2 m}{n}\right)}$. Bounds for $L E E$ are obtained, as well as some relations between $L E E$ and graph Laplacian energy.

## 1. INTRODUCTION

Let $G=(V, E)$ be a graph without loops and multiple edges. Let $n$ and $m$ be the number of vertices and edges of $G$, respectively. Such a graph will be referred to as an $(n, m)$-graph. For $v \in V(G)$, let $d(v)$ be the degree of $v$.

In this paper, we are concerned only with undirected simple graphs (loops and multiple edges are not allowed). Let $G$ be a graph with n vertices and the adjacency matrix $A(G)$. Let $D(G)$ be a diagonal matrix with degrees of the corresponding vertices of $G$ on the main diagonal and zero elsewhere. The matrix $L(G)=D(G)-A(G)$ is called the Laplacian matrix of $G$. Since $A(G)$ and $L(G)$ are real symmetric matrices, their eigenvalues are real numbers. So we can assume that $\lambda_{1} \geq \lambda_{2} \geq \cdots \geq \lambda_{n}$, and $\mu_{1} \geq \mu_{2} \geq \cdots \geq \mu_{n}=0$ are the adjacency and the Laplacian eigenvalues of G , respectively. The multiset of eigenvalues of $\mathrm{A}(\mathrm{G})(\mathrm{L}(\mathrm{G}))$ is called the adjacency (Laplacian) spectrum of $G$. Other undefined notations may be referred to [1].

The basic properties of the eigenvalues and Laplacian eigenvalues of the graph can be found in the book [2].

[^0]The energy of the graph $G$ is defined in $[\mathbf{8}, \mathbf{9}]$ as:

$$
\begin{equation*}
E=E(G)=\sum_{i=1}^{n}\left|\lambda_{i}\right| \tag{1}
\end{equation*}
$$

The Estrada index of the graph $G$ is defined in $[\mathbf{4 - 6}]$ as:

$$
\begin{equation*}
E E=E E(G)=\sum_{i=1}^{n} e^{\lambda_{i}} \tag{2}
\end{equation*}
$$

Denoting by $M_{k}=M_{k}(G)$ the $k$-th spectral moment of the graph $G$ ([4] equal to the number of closed walks of length $k$ of the graph $G$ ),

$$
M_{k}=M_{k}(G)=\sum_{i=1}^{n}\left(\lambda_{i}\right)^{k}
$$

and bearing in mind the power-series expansion of $e^{x}$, we have

$$
\begin{equation*}
E E=\sum_{k=0}^{\infty} \frac{M_{k}}{k!} \tag{3}
\end{equation*}
$$

The quantity defined by (2) or (3) appears in Physics and Chemistry; for details see the surveys $[\mathbf{4 - 6}]$. Recently much work on the Estrada index of the graph appeared also in the mathematical literature (see, for instance, $[\mathbf{3}, \mathbf{1 0}]$ ).

It is evident from (3) that if graph $G$ can be transformed to another graph $G^{\prime}$, such that $M_{k}\left(G^{\prime}\right) \geq M_{k}(G)$ holds for all values of $k$, and $M_{k}\left(G^{\prime}\right)>M_{k}(G)$ holds for at least some values of $k$, then $E E\left(G^{\prime}\right)>E E(G)$. We assume that graph transformation $G \rightarrow G^{\prime}$, where $G^{\prime}=G+e$ be the graph obtained from $G$ by adding a new edge $e$ into $G$. By adding a new edge $e$ into $G$, the number of closed walks of length $k$ will certainly not decrease, and in some cases (e.g., for $k=2$ ) will strictly increase. Bearing that in mind, we conclude that the $n$-vertex with as few as possible and as many as possible edges has the minimum and the maximum $E E$, respectively. From (2), we find that $E E\left(\overline{K_{n}}\right)=n$ and $E E\left(K_{n}\right)=e^{n-1}+(n-1) \frac{1}{e}$. Hence, for any graph $G$ of order $n$, different from the complete graph $K_{n}$ and from its (edgeless) complement $\overline{K_{n}}$, we have

$$
n=E E\left(\overline{K_{n}}\right)<E E(G)<E E\left(K_{n}\right)=e^{n-1}+(n-1) \frac{1}{e}
$$

Recently, J. A. de la Peña et al. [3] established lower and upper bounds for $E E$ in terms of the number of vertices and number of edges, and also obtained some inequalities between $E E$ and the energy of $G$. Their results are as follows.

Theorem 1. [3] Let $G$ be an $(n, m)$-graph. Then the Estrada index of $G$ is bounded as

$$
\begin{equation*}
\sqrt{n^{2}+4 m} \leq E E(G) \leq n-1+e^{\sqrt{2 m}} \tag{4}
\end{equation*}
$$

Equality on both sides of (4) is attained if and only if $G \cong \overline{K_{n}}$.

Theorem 2. [3] Let $G$ be a regular graph of degree $r$ and of order $n$. Then its Estrada index is bounded as

$$
\begin{aligned}
e^{r} & +\sqrt{n+2 n r-\left(2 r^{2}+2 r+1\right)+(n-1)(n-2) e^{-2 r /(n-1)}} \\
& \leq E E(G) \leq n-2+e^{r}+e^{\sqrt{r(n-r)}}
\end{aligned}
$$

Theorem 3. [3] Let $G$ be an ( $n, m$ )-graph. Then

$$
\begin{equation*}
E E(G)-E(G) \leq n-1-\sqrt{2 m}+e^{\sqrt{2 m}} \tag{5}
\end{equation*}
$$

Or

$$
\begin{equation*}
E E(G) \leq n-1+e^{E(G)} \tag{6}
\end{equation*}
$$

Equality (5) or (6) is attained if and only if $G \cong \overline{K_{n}}$.
Theorem 4. [3] Let $G$ be a regular graph of degree $r$ and of order $n$. Then

$$
E E(G)-E(G) \leq n-2+e^{r}-r-\sqrt{r(n-r)}+e^{\sqrt{r(n-r)}}
$$

In this paper, we define and investigate the Laplacian Estrada index of $G$ as $L E E=L E E(G)=\sum_{i=1}^{n} e^{\left(\mu_{i}-\frac{2 m}{n}\right)}$, and get some analogy between the properties of $E E(G)$ and $\operatorname{LEE}(G)$, but also some significant differences.

## 2. THE LAPLACIAN ESTRADA INDEX CONCEPT

The first Zagreb index of $G$ is defined in [7] as $M=M(G)=\sum_{u \in V(G)} d^{2}(u)$.
Lemma 1. [12] Let $G$ be an $(n, m)$-graph. Then,

$$
M \leq m\left(\frac{2 m}{n-1}+n-2\right)
$$

with equality if and only if $G$ is $S_{n}$ or $K_{n}$.
Lemma 2. [13] If $G$ is a triangle-free and a quadrangle-free graph, then

$$
M \leq n(n-1)
$$

with equality if and only if $G$ is the star or a Moore graph of diameter 2.
Lemma 3. Let $G$ be an $(n, m)$-graph with maximum degree $\Delta$ and minimum degree $\delta$, then

$$
\begin{equation*}
\frac{4 m^{2}}{n} \leq M \leq 2 m(\Delta+\delta)-n \delta \Delta \tag{7}
\end{equation*}
$$

Equality on both sides of (7) is attained if and only if $G$ is regular.
Proof. By the Cauchy-Schwarz inequality, we have

$$
M=\sum_{u \in V(G)} d^{2}(u) \geq \frac{\left(\sum_{u \in V(G)} d(u)\right)^{2}}{n}=\frac{4 m^{2}}{n}
$$

Equality holds if and only if $G$ is regular.
Summing the inequality $(d(u)-\delta)(d(u)-\Delta) \leq 0$ for every $u \in V(G)$, we have that

$$
\sum_{u \in V(G)} d^{2}(u)-(\Delta+\delta) \sum_{u \in V(G)} d(u)+n \delta \Delta \leq 0
$$

Then,

$$
M \leq 2 m(\Delta+\delta)-n \delta \Delta
$$

Equality holds if and only if $G$ is regular. This ends the proof.
The Laplacian energy of the graph $G$ is defined in [11] as:

$$
\begin{equation*}
L E=L E(G)=\sum_{i=1}^{n}\left|\mu_{i}-\frac{2 m}{n}\right| \tag{8}
\end{equation*}
$$

Definition. If $G$ is an ( $n, m$ )-graph, and its Laplacian eigenvalues are $\mu_{1} \geq \mu_{2} \geq$ $\cdots \geq \mu_{n}=0$, then the Laplacian Estrada index of $G$, denoted by $\operatorname{LEE}(G)$, is equal to

$$
\begin{equation*}
L E E=L E E(G)=\sum_{i=1}^{n} e^{\left(\mu_{i}-\frac{2 m}{n}\right)} \tag{9}
\end{equation*}
$$

And let

$$
M_{k}^{\prime}=\sum_{i=1}^{n}\left(\mu_{i}-\frac{2 m}{n}\right)^{k}
$$

Then $M_{0}^{\prime}=n ; M_{1}^{\prime}=0 ; M_{2}^{\prime}=M+2 m\left(1-\frac{2 m}{n}\right)$ (where $M$ is the first Zagreb index of $G$ as above). And bearing in mind the power-series expansion of $e^{x}$, we have

$$
\begin{equation*}
L E E=\sum_{k=0}^{\infty} \frac{M_{k}^{\prime}}{k!} \tag{10}
\end{equation*}
$$

Lemma 4. If the graph $G$ is regular, then $M_{2 k}(G)=M_{2 k}^{\prime}(k \in \mathbb{Z}), L E(G)=E(G)$, $L E E(G)=\sum_{i=1}^{n} e^{-\lambda_{i}}$. Further, if the regular graph $G$ is bipartite, then $M_{k}(G)=M_{k}^{\prime}$, $\operatorname{LEE}(G)=E E(G)$.

Proof. If an $(n, m)$-graph is regular of degree $r$, then $r=2 m / n$ and [2], we have

$$
\mu_{i}-r=-\lambda_{n-i+1}, \quad i=1,2, \ldots, n
$$

and from the definition of $M_{k}, M_{k}^{\prime}, E, L E, E E$ and $L E E$ of the graph $G$, respectively. We have $M_{2 k}(G)=M_{2 k}^{\prime}(k \in \mathbb{Z}), L E(G)=E(G), L E E(G)=\sum_{i=1}^{n} e^{-\lambda_{i}}$.

If the regular graph $G$ is a bipartite graph, from [2], we have

$$
\lambda_{i}=-\lambda_{n+1-i}, \quad i=1,2, \ldots, n
$$

Then, $\sum_{i=1}^{n} \lambda_{i}^{k}=\sum_{i=1}^{n}\left(-\lambda_{i}\right)^{k}$ and $\sum_{i=1}^{n} e^{-\lambda_{i}}=\sum_{i=1}^{n} e^{\lambda_{i}}$. We have $M_{k}(G)=M_{k}^{\prime}, \operatorname{LEE}(G)=$ $E E(G)$.

Theorem 5. Let $G$ be an ( $n, m$ )-graph with maximum degree $\Delta$ and minimum degree $\delta$, then the Laplacian Estrada index of $G$ is bounded as

$$
\begin{equation*}
\sqrt{n^{2}+4 m} \leq L E E(G) \leq n-1+e^{\sqrt{2 m\left(\Delta+\delta+1-\frac{2 m}{n}\right)-n \delta \Delta}} \tag{11}
\end{equation*}
$$

Equality on both sides of (11) is attained if and only if $G \cong \overline{K_{n}}$.
Proof. Lower bound. Directly from (9), we get

$$
\begin{equation*}
L E E^{2}(G)=\sum_{i=1}^{n} e^{2\left(\mu_{i}-\frac{2 m}{n}\right)}+2 \sum_{i<j} e^{\left(\mu_{i}-\frac{2 m}{n}\right)} e^{\left(\mu_{j}-\frac{2 m}{n}\right)} \tag{12}
\end{equation*}
$$

In view of the inequality between the arithmetic and geometric means,

$$
\begin{align*}
2 \sum_{i<j} e^{\left(\mu_{i}-\frac{2 m}{n}\right)} e^{\left(\mu_{j}-\frac{2 m}{n}\right)} & \geq n(n-1)\left(\prod_{i<j} e^{\left(\mu_{i}-\frac{2 m}{n}\right)} e^{\left(\mu_{j}-\frac{2 m}{n}\right)}\right)^{2 /[n(n-1)]} \\
& =n(n-1)\left[\left(\prod_{i=1}^{n} e^{\left(\mu_{i}-\frac{2 m}{n}\right)}\right)^{n-1}\right]^{2 /[n(n-1)]} \\
& =n(n-1)\left(e^{M_{1}^{\prime}}\right)^{2 / n} \\
& =n(n-1) \tag{13}
\end{align*}
$$

By means of a power-series expansion, and $M_{0}^{\prime}=n, M_{1}^{\prime}=0$ and $M_{2}^{\prime}=$ $M+2\left(1-\frac{2 m}{n}\right)$, we get

$$
\begin{aligned}
\sum_{i=1}^{n} e^{2\left(\mu_{i}-\frac{2 m}{n}\right)} & =\sum_{i=1}^{n} \sum_{k \geq 0} \frac{\left[2\left(\mu_{i}-\frac{2 m}{n}\right)\right]^{k}}{k!} \\
& =n+2\left[2 m\left(1-\frac{2 m}{n}\right)+M\right]+\sum_{i=1}^{n} \sum_{k \geq 3} \frac{\left[2\left(\mu_{i}-\frac{2 m}{n}\right)\right]^{k}}{k!}
\end{aligned}
$$

We use a multiplier $r \in[0,8]$, as to arrive at,

$$
\begin{aligned}
\sum_{i=1}^{n} e^{2\left(\mu_{i}-\frac{2 m}{n}\right)} & \geq n+2\left[2 m\left(1-\frac{2 m}{n}\right)+M\right]+r \sum_{i=1}^{n} \sum_{k \geq 3} \frac{\left(\mu_{i}-\frac{2 m}{n}\right)^{k}}{k!} \\
& =(1-r) n+(4-r)\left[m\left(1-\frac{2 m}{n}\right)+\frac{M}{2}\right]+r L E E
\end{aligned}
$$

Further, by Lemma 7, we get

$$
\sum_{i=1}^{n} e^{2\left(\mu_{i}-\frac{2 m}{n}\right)} \geq(1-r) n+(4-r)\left[m\left(1-\frac{2 m}{n}\right)+\frac{2 m^{2}}{n}\right]+r L E E
$$

i.e.,

$$
\begin{equation*}
\sum_{i=1}^{n} e^{2\left(\mu_{i}-\frac{2 m}{n}\right)} \geq(1-r) n+(4-r) m+r L E E \tag{14}
\end{equation*}
$$

By substituting (13) and (14) back into (12), and solving for $L E E$, we have

$$
L E E \geq \frac{r}{2}+\sqrt{\left(n-\frac{r}{2}\right)^{2}+(4-r) m}
$$

It is elementary to show that for $n \geq 2$ and $m \geq 1$ the function

$$
f(x):=\frac{x}{2}+\sqrt{\left(n-\frac{x}{2}\right)^{2}+(4-r) m}
$$

monotonically decreases in the interval $[0,8]$. Consequently, the best lower bound for $L E E$ is attained for $r=0$. Then we arrive at the first half of Theorem 5 .
Upper bound. Starting from the following inequality, we get

$$
\begin{aligned}
L E E=n+\sum_{i=1}^{n} \sum_{k \geq 1} \frac{\left(\mu_{i}-\frac{2 m}{n}\right)^{k}}{k!} & \leq n+\sum_{i=1}^{n} \sum_{k \geq 1} \frac{\left|\mu_{i}-\frac{2 m}{n}\right|^{k}}{k!} \\
& =n+\sum_{k \geq 1} \frac{1}{k!} \sum_{i=1}^{n}\left[\left(\mu_{i}-\frac{2 m}{n}\right)^{2}\right]^{k / 2} \\
& \leq n+\sum_{k \geq 1} \frac{1}{k!}\left[\sum_{i=1}^{n}\left(\mu_{i}-\frac{2 m}{n}\right)^{2}\right]^{k / 2} \\
& =n+\sum_{k \geq 1} \frac{1}{k!}\left[M+2 m\left(1-\frac{2 m}{n}\right)\right]^{k / 2} \\
& =n-1+\sum_{k \geq 0} \frac{\left(\sqrt{M+2 m\left(1-\frac{2 m}{n}\right)}\right)^{k}}{k!} \\
& =n-1+e^{\sqrt{M+2 m\left(1-\frac{2 m}{n}\right)}} .
\end{aligned}
$$

By Lemma 3, we have

$$
L E E \leq n-1+e^{\sqrt{2 m\left(\Delta+\delta+1-\frac{2 m}{n}\right)-n \delta \Delta}}
$$

which directly leads to the right-hand side inequality in (11).
From the derivation of (11) it is evident that equality will be attained if and only if the graph $G$ has all zero eigenvalues. This happens only in the case of the edgeless graph $\overline{K_{n}},[\mathbf{2}]$.

The proof is completed.
Remark. If in inequality (15) we utilize the upper bound of $M$ in Lemma 1, we have the upper bound for $L E E$ in terms of the number of vertices and number of edges as follows,

$$
L E E \leq n-1+e^{\sqrt{m\left(n+\frac{2 m}{n-1}-\frac{4 m}{n}\right)}} .
$$

Further, if the $(n, m)$-graph is a triangle-free and a quadrangle-free graph, by Lemma 2 , we have,

$$
L E E \leq n-1+e^{\sqrt{n(n-1)+2 m\left(1-\frac{2 m}{n}\right)}} .
$$

Since $\mu_{n}=0$, if we consider $L E E-e^{2 m / n}=\sum_{i=1}^{n-1} e^{\left(\mu_{i}-2 m / n\right)}$ in the same way as in Theorem 5, we have the following bounds of $\operatorname{LEE}(G)$ :

Theorem 6. Let $G$ be an $(n, m)$-graph with maximal degree $\Delta$ and minimal degree $\delta$, then the Laplacian Estrada index of $G$ is bounded as

$$
\begin{align*}
e^{-2 m / n} & +\sqrt{(n-1)\left[1+(n-2) e^{\frac{4 m}{n(n-1)}}\right]+4 m\left(1+\frac{1}{n}-\frac{2 m}{n^{2}}\right)} \\
& \leq L E E(G) \leq n-2+e^{\frac{-2 m}{n}}+e^{\sqrt{2 m\left(1+\Delta+\delta-\frac{2 m}{n}-\frac{2 m}{n^{2}}\right)-n \Delta \delta}} \tag{16}
\end{align*}
$$

Equality on both sides of (16) is attained if and only if $G \cong \overline{K_{n}}$.
If $G$ is regular, by Lemma 4, Theorem 2 and Theorem 6 , the following result is obviously.

Theorem 7. Let $G$ be a regular graph of degree $r$ and of order $n$. Then its Laplacian Estrada index is bounded as

$$
\begin{aligned}
e^{-r} & +\sqrt{n+2 n r-\left(2 r^{2}-2 r+1\right)+(n-1)(n-2) e^{2 r /(n-1)}} \\
& \leq \operatorname{LEE}(G) \leq n-2+e^{-r}+e^{\sqrt{r(n-r)}} .
\end{aligned}
$$

Further, if the regular graph $G$ is a bipartite graph, then its Laplacian Estrada index is bounded as

$$
\begin{aligned}
e^{r} & +\sqrt{n+2 n r-\left(2 r^{2}+2 r+1\right)+(n-1)(n-2) e^{-2 r /(n-1)}} \\
& \leq L E E(G) \leq n-2+e^{r}+e^{\sqrt{r(n-r)}} .
\end{aligned}
$$

## 3. BOUNDS FOR THE LAPLACIAN ESTRADA INDEX INVOLVING GRAPH LAPLACIAN ENERGY

Theorem 8. Let $G$ be an ( $n, m$ )-graph with maximum degree $\Delta$ and minimum degree $\delta$, then

$$
\begin{equation*}
L E E-L E \leq n-1-\sqrt{2 m\left(\Delta+\delta+1-\frac{2 m}{n}\right)-n \delta \Delta}+e^{\sqrt{2 m\left(\Delta+\delta+1-\frac{2 m}{n}\right)-n \delta \Delta}} \tag{17}
\end{equation*}
$$ or

$$
\begin{equation*}
L E E(G) \leq n-1+e^{L E(G)} \tag{18}
\end{equation*}
$$

Equality (17) or (18) is attained if and only if $G \cong \overline{K_{n}}$.
Proof. In the proof of Theorem 5, we have the following inequality,

$$
L E E=n+\sum_{i=1}^{n} \sum_{k \geq 1} \frac{\left(\mu_{i}-\frac{2 m}{n}\right)^{k}}{k!} \leq n+\sum_{i=1}^{n} \sum_{k \geq 1} \frac{\left|\mu_{i}-\frac{2 m}{n}\right|^{k}}{k!} .
$$

Taking into account the definition of graph Laplacian energy (8), we have

$$
L E E \leq n+L E+\sum_{i=1}^{n} \sum_{k \geq 2} \frac{\left|\mu_{i}-\frac{2 m}{n}\right|^{k}}{k!},
$$

which, as in Theorem 5, leads to

$$
\begin{align*}
L E E-L E & \leq n+\sum_{i=1}^{n} \sum_{k \geq 2} \frac{\left|\mu_{i}-\frac{2 m}{n}\right|^{k}}{k!} \\
& \leq n-1-\sqrt{M+2 m\left(1-\frac{2 m}{n}\right)}+e^{\sqrt{M+2 m\left(1-\frac{2 m}{n}\right)}} . \tag{19}
\end{align*}
$$

It is elementary to show that the function $f(x):=e^{x}-x$ monotonically increases in the interval $[0,+\infty]$. Consequently, the best upper bound for $L E E-L E$ is attained for $M=2 m(\Delta+\delta)-n \delta \Delta$ by Lemma 3 . Then we have

$$
L E E-L E \leq n-1-\sqrt{2 m\left(\Delta+\delta+1-\frac{2 m}{n}\right)-n \delta \Delta}+e^{\sqrt{2 m\left(\Delta+\delta+1-\frac{2 m}{n}\right)-n \delta \Delta}} .
$$

This inequality holds for all $(n, m)$-graphs. Equality is attained if and only if $G \cong \overline{K_{n}}$.

Another route to connect $L E E$ and $L E$, is the following:

$$
\begin{aligned}
L E E \leq n+\sum_{i=1}^{n} \sum_{k \geq 1} \frac{\left|\mu_{i}-\frac{2 m}{n}\right|^{k}}{k!} & \leq n+\sum_{k \geq 1} \frac{1}{k!}\left(\sum_{i=1}^{n}\left|\mu_{i}-\frac{2 m}{n}\right|\right)^{k} \\
& =n+\sum_{k \geq 1} \frac{L E^{k}}{k!}=n-1+\sum_{k \geq 0} \frac{L E^{k}}{k!}
\end{aligned}
$$

implying, $\operatorname{LEE}(G) \leq n-1+e^{L E(G)}$.
Also on this formula equality occurs if and only if $G \cong \overline{K_{n}}$.
Remark. If in inequality (19) we utilize the upper bound of $M$ in Lemma 1, we have the upper bound for $L E E-L E$ in terms of the number of vertices and number of edges as follows,

$$
L E E-L E \leq n-1-m\left(n+\frac{2 m}{n-1}-\frac{4 m}{n}\right)+e^{\sqrt{m\left(n+\frac{2 m}{n-1}-\frac{4 m}{n}\right)}}
$$

Further, if the $(n, m)$-graph is a triangle-free and a quadrangle-free graph, by Lemma 6 , we have,

$$
L E E-L E \leq n-1-n(n-1)+2 m\left(1-\frac{2 m}{n}\right)+e^{\sqrt{n(n-1)+2 m\left(1-\frac{2 m}{n}\right)}}
$$

Since $\mu_{n}=0$, if we consider $L E E-e^{2 m / n}=\sum_{i=1}^{n-1} e^{\left(\mu_{i}-2 m / n\right)}$ in the same way as in Theorem 8, we have the following results:

Theorem 9. Let $G$ be an ( $n, m$ )-graph with maximum degree $\Delta$ and minimum degree $\delta$, then

$$
\begin{align*}
L E E-L E \leq n-2 & +e^{-\frac{2 m}{n}-\frac{2 m}{n}-\sqrt{2 m\left(\Delta+\delta+1-\frac{2 m}{n}-\frac{2 m}{n^{2}}\right)-n \delta \Delta}} \\
& +e^{\sqrt{2 m\left(\Delta+\delta+1-\frac{2 m}{n}-\frac{2 m}{n^{2}}\right)-n \delta \Delta}} \tag{20}
\end{align*}
$$

or

$$
\begin{equation*}
L E E(G) \leq n-2+e^{-\frac{2 m}{n}}\left(1+e^{L E(G)}\right) \tag{21}
\end{equation*}
$$

Equality (20) or (21) is attained if and only if $G \cong \overline{K_{n}}$.
By Lemma 4 and Theorem 9, a similar formula is deduced for regular graphs,
Theorem 10. Let $G$ be a regular graph of degree $r$ and of order $n$. Then

$$
L E E-L E \leq n-2+e^{-r}-r-\sqrt{r(n-r)}+e^{\sqrt{r(n-r)}}
$$

or

$$
\operatorname{LEE}(G) \leq n-2+e^{-r}\left(1+e^{L E(G)}\right)
$$

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## REFERENCES

1. J. Bondy, U. Murty: Graph theory with applications. New York, MacMillan, 1976.
2. D. Cvetković, M. Doob, H. Sachs: Spectra of Graphs-Theory and Application. Academic Press, New York, 1980.
3. J. A. de la Peña, I. Gutman, J. Rada: Estimating the estrada index. Linear Algebra Appl., 427 (2007), 70-76.
4. E. Estrada, J. A. Rodríguez-Velásquez: Subgraph centrality in complex network. Phys. Rev., E71 (2005), 056103.
5. E. Estrada, J. A. Rodríguez-Velásquez: Spectral measures of bipartivity in complex network. Phys. Rev., E72 (2005), 046105.
6. E. Estrada, J. A. Rodríguez-Velásquez, M. Randić: Atomic branching in molecules. Int. J. Quantum Chem., 106 (2006), 823-832.
7. I. Gutman, N. Trinajstić: Graph theory and molecular orbitals. Total-electron energy of alternant hydrocarbons. Chem. Phys., 17 (1972), 535-538.
8. I. Gutman: Acyclic conjugated molecules, trees and their energies. J. Math. Chem., 1 (1987), 123-143.
9. I. Gutman: The energy of a graph: Old and new results, in: A. Kohnert, R. Laue, A. Wassermann (Eds.). Algebraic Combinatorcs and Application. Springer-Verlag, Berlin, 2001, pp. 196-211.
10. I. Gutman, E. Estrada, J. A. Rodríguez-Velásquez: On a Graph-Spectrum-Based structure descriptor. Croat. Chem. Acta., 80(2) (2006), 151-154.
11. I. Gutman, B. Zhou: Laplacian energy of a graph. Linear Algebra Appl., 414 (2006), 29-37.
12. J. Li, Y. PAN: de Caen's inequality and bounds on the largest Laplacian eigenvalue of a graph. Linear Algebra Appl., 328 (2001), 153-160.
13. B. Zhou, D. Stevanović: A note on Zagreb indices. MATCH Commun. Math. Comput. Chem., 56 (2006), 571-578.

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