

On the Largest Prime Divisor of an Odd Perfect Number. II

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Abstract. It is proved here that every odd perfect number has a prime factor greater than 100110.

If n is an element of the (possibly empty) set of odd perfect numbers, then it is well known that

$$(1) \quad n = p_0^{\alpha_0} \cdot p_1^{\alpha_1} \cdot \cdots \cdot p_t^{\alpha_t},$$

where the p_i are distinct primes, $p_0 \equiv \alpha_0 \equiv 1 \pmod{4}$, and $2|\alpha_i$ if $i > 0$. In [2], it was proved that at least one of the p_i exceeds 11200. Our purpose here is to improve this bound by proving the following:

THEOREM. *If n is odd and perfect, then n has a prime factor which exceeds 100110.*

The method of proof is similar to that employed in [2], and we shall not give the details here. We shall, however, explain our strategy and exemplify the arguments which are used. The complete proof [1] has been deposited in the UMT file.

The proof is by *reductio ad absurdum*. Thus, we assume that $p_i < 100110$ for every p_i in (1) and show that this assumption is untenable. Since n is perfect $\sigma(n) = 2n$, and since $\sigma(n)$ is multiplicative,

$$(2) \quad 2n = \prod_{i=0}^t \sigma(p_i^{\alpha_i}) = \prod_{i=0}^t \prod_d F_d(p_i).$$

Here F_d is the d th cyclotomic polynomial, and d runs over the divisors of $\alpha_i + 1$ which exceed 1. d assumes the value 2 if and only if $i = 0$. We see immediately that the set of p_i in (1) is identical with the set of odd prime divisors of the $F_d(p_i)$ in (2). In particular, recalling our assumption, we note that all the prime factors of each $F_d(p_i)$ must be less than 100110.

For a given odd prime p we shall say that the prime Q is $(p; 100110)$ -acceptable or simply p -acceptable if every prime divisor of $F_Q(p)$ is less than 100110. According to a result of Kanold [3, (21) Satz], if $Q > 50053$, then Q is unacceptable for every odd prime. We shall say that p is *inadmissible* if no Q is p -acceptable. ($Q = 2$ is taken into consideration only if it is possible that $p = p_0$.)

Our proof is in two stages, and we show first that n is not divisible by certain "small" primes.

LEMMA. *If every prime in the factorization of the odd perfect number n is less*

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than 100110, then n is not divisible by any prime in the set V where $V = \{3, 5, 7, 11, 13, 19, 23, 31, 37, 61, 127, 131, 151, 1093\}$.

The proof of this lemma goes as follows: Assuming that $p|n$ (which we wish to disprove), we find all p -acceptable primes and then factor $F_Q(p)$; from (2) $F_Q(p)|2n$ for at least one p -acceptable prime Q and each odd prime divisor of $F_Q(p)$ divides n ; for each acceptable $F_Q(p)$, a single prime divisor is selected and its acceptable primes are determined; this procedure is iterated and a finite tree is generated (*finite*, since each prime on which we branch is less than 100110 and its acceptable primes are bounded by 50053); each path through the tree terminates at a node corresponding to either an inadmissible prime or some other contradiction so that $p \nmid n$. *A priori*, a third type of *terminal* node might be encountered—one corresponding to an admissible prime r such that every odd prime divisor of each r -acceptable cyclotomic number has already been branched upon on the path joining p to r , in which case our procedure fails. We encountered no such nodes, and fortunately most of the trees generated were small. We illustrate by showing that neither 1093 nor 151 divides n , and begin by proving:

(A) If $613|n$, then $613 = p_0$. The only odd 613-acceptable primes are 3 and 5 and $F_3(613) = 3 \cdot 7 \cdot 17923$, $F_5(613) = 131 \cdot 20161 \cdot 53551$. Therefore, if $613|n$ and $613 \neq p_0$ then $17923|n$ or $53551|n$. Since both 17923 and 53551 are inadmissible, our result follows.

(B) $1093 \nmid n$. Only 2 is 1093-acceptable and $F_2(1093) = 2 \cdot 547$. Therefore, if $1093|n$, then $547|n$ also. Only 3 is 547-acceptable and $F_3(547) = 3 \cdot 163 \cdot 613$. Therefore, $1093 = p_0$ and (from (A)) $613 = p_0$. We have reached a contradiction.

(C) $151 \nmid n$. For, only 3 is 151-acceptable and $F_3(151) = 3 \cdot 7 \cdot 1093$, so that if $151|n$, then $1093|n$, which contradicts (B).

To describe the second stage of our proof, we need several more definitions. Let q be the smallest prime divisor of n and let $W(q)$ denote the set of primes which are not less than q . For a given prime p , we shall say that the prime Q is $(p; q; 100110)$ -feasible or simply (p, q) -feasible if Q is p -acceptable and if every odd prime divisor of $F_Q(p)$ belongs to the set $W(q) \cap V'$ where V' denotes the complement of V with respect to the set of all primes. (Of course, for each p_i in (2), each prime divisor of $\alpha_i + 1$ must be (p_i, q) -feasible.) If p cannot be p_0 , we omit $Q = 2$ from consideration. If no Q is (p, q) -feasible, we shall say that p is q -impossible.

Now, according to the table in [4], $q < 307$ since otherwise n would have a prime factor which exceeds 100549. But (see [1]) except for the elements of the set $T = \{17, 41, 59, 67, 71, 79, 89, 101, 149, 167, 173, 197, 293\}$ every odd prime r less than 307 is either r -impossible or belongs to V , so that $q \in T$. Using basically the method described above for the proof of our lemma, we complete the proof of our theorem by showing that no prime in T is q . We illustrate by proving:

(a) $q \neq 17$. For, only 3 and 5 are $(17, 17)$ -feasible. But $F_3(17) = 307$, only 5 is $(307, 17)$ -feasible, $1051|F_5(307)$, and 1051 is 17-impossible. $F_5(17) = 88741$, only 2 is $(88741, 17)$ -feasible, $44371|F_2(88714)$, and 44371 is 17-impossible.

Concluding Remarks. If P is the largest prime divisor of the odd perfect number n , then a “good” bound on P is very helpful if one is investigating such questions as

“How large is n ?” or “How many prime divisors does n have?”. This is the motivation for the present paper. It is obvious that by modifying appropriately the definitions of p -acceptable, (p, q) -feasible, etc., and expending the requisite effort and computer time, one could very probably improve our lower bound on P . The present investigation consumed approximately 6.5 hours of CDC 6400 time, most of which was devoted to verifying that, for each prime on which we branched, almost all $Q \leq 50053$ were unacceptable. The complete factorizations of all p -acceptable $F_Q(p)$ encountered are given in Table I in [1]. We do not intend to pursue this research further and would hope that if someone else does that he aim for a lower bound on P of at least 10^6 .

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