BULL. AUSTRAL. MATH. SOC. MOS 2092, 2093

VOL. 1 (1969), 231-235.

# On the lattice of congruences on a regular semigroup

T. E. Hall

A result of Reilly and Scheiblich for inverse semigroups is proved true also for regular semigroups. For any regular semigroup $S$ the relation $\Theta$ is defined on the lattice, $\Lambda(S)$ , of congruences on $S$ by: $(\rho,\tau) \in \Theta$ if $\rho$ and $\tau$ induce the same partition of the idempotents of $S$ . Then $\Theta$ is a congruence on $\Lambda(S)$ , $\Lambda(S)/\Theta$ is complete and the natural homomorphism of $\Lambda(S)$ onto $\Lambda(S)/\Theta$ is a complete lattice homomorphism.

## 1. Introduction and summary

Let $S$ be a semigroup, $E$ its set of idempotents, $\Lambda(S)$ its lattice of congruences, and define on $\Lambda(S)$ the relation

$$\Theta = \{(\rho,\sigma) \in \Lambda(S) \times \Lambda(S) : \rho \cap (E\times E) = \sigma \cap (E\times E)\} .$$

Using the work of Munn [4] and Lallement [2], Reilly and Scheiblich [5] have proved the following

THEOREM 3.4 of [5]. *If $S$ is a regular semigroup then*

*(i) $\Theta$ is a meet compatible equivalence on $\Lambda(S)$ ;*

*(ii) each $\Theta$-class is a complete modular sublattice of $\Lambda(S)$ .*

For inverse semigroups they prove considerably more in Theorem 5.1 [5]. We consider whether or not Theorem 5.1 [5] holds true also for regular semigroups. We answer this in the affirmative by proving the following

Received 21 March 1969. Received by J. Austral. Math. Soc. 6 December 1968. Communicated by G.B. Preston. This research was carried out at Monash University while the author held a Commonwealth Postgraduate Award. The author wishes to thank his supervisor, Professor G.B. Preston, for an inspiring discussion and further invaluable help.



https://doi.org/10.1017/S0004972700041472 Published online by Cambridge University Press

theorem, which reduces to Theorem 5.1 [5] when $S$ is an inverse semigroup.

**MAIN THEOREM.** *Let $S$ be a regular semigroup, $E$ the set of idempotents of $S$, and $\Lambda(S)$ the lattice of congruences on $S$. Define on $\Lambda(S)$ the relation*

$$\Theta = \{(\rho,\sigma) \in \Lambda(S) \times \Lambda(S) : \rho \cap (E\times E) = \sigma \cap (E\times E)\} .$$

*Then*

*(i) $\Theta$ is a congruence on $\Lambda(S)$;*

*(ii) each $\Theta$-class is a complete modular sublattice of $\Lambda(S)$;*

*(iii) the quotient lattice $\Lambda(S)/\Theta$ is complete and the natural homomorphism $\Theta^{\natural}$ of $\Lambda(S)$ onto $\Lambda(S)/\Theta$ is a complete lattice homomorphism.*

We use wherever possible, and often without comment, the notations and conventions of Clifford and Preston [1]. For any equivalence $E$ on $S$, we shall often denote the equivalence $E \cap (E\times E)$ by $E|E$. We shall use the following well-known result, which one may readily verify.

**RESULT 1.** *Let $L$ be a complete lattice and $\Theta$ a congruence on $L$ satisfying the following condition.*

*(A) If $\{a_i : i \in I\}$ and $\{b_i : i \in I\}$ ($I$ is some index set) are any two subsets of $L$ such that $(a_i,b_i) \in \Theta$ for each $i \in I$, then $\left(\bigvee_{i\in I} a_i , \bigvee_{i\in I} b_i\right) \in \Theta$ and $\left(\bigwedge_{i\in I} a_i , \bigwedge_{i\in I} b_i\right) \in \Theta$.*

*Then the quotient lattice $L/\Theta$ is a complete lattice and the natural homomorphism $\Theta^{\natural}$ of $L$ onto $L/\Theta$ is a complete lattice homomorphism.*

## 2. $K(\sigma)$-regularity

This section gives a routine generalization of Lallement's Theorem 3.6 [3] (see also Theorem 2.3 [2]).

Let $S$ be a semigroup and $\sigma$ any congruence on $S$. Let $K$ be any one of Green's relations $R$, $L$, $H$, $D$, $J$ on $S$ (see [1]). To indicate that we are considering one of Green's relations on a different semigroup $T$ say, we shall use, for example, the phrase "$K$ *on* $T$". Define now $K(\sigma) = \{(a,b) \in S\times S : a\sigma$ and $b\sigma$ are $K$-related in $S/\sigma\}$. Then clearly

https://doi.org/10.1017/S0004972700041472 Published online by Cambridge University Press

$K(\sigma)$ is an equivalence relation on $S$, $K \subseteq K(\sigma)$, and $\sigma \subseteq K(\sigma)$. Also clearly $H(\sigma) = R(\sigma) \cap L(\sigma)$ and $D(\sigma) = R(\sigma) \circ L(\sigma) = L(\sigma) \circ R(\sigma) = L(\sigma) \vee R(\sigma)$.

For any element $a$ in $S$, let $K(\sigma)_a$ $[K_a]$ denote the $K(\sigma)$-class [$K$-class] of $S$ containing the element $a$, and let $K_{a\sigma}$ denote the $K$-class of $S/\sigma$ containing the element $a\sigma$. Clearly there is a one-to-one mapping, $\psi$ say, from the set of $K(\sigma)$-classes of $S$ onto the set of $K$-classes of $S/\sigma$ such that, for any element $a$ in $S$, $K(\sigma)_a\psi = K_{a\sigma}$.

Now for $K \neq D$, there is a natural ordering $\leq$ (see [1] or [2]) on the set of $K$-classes of any semigroup. We define $\leq$ on the set of $K(\sigma)$-classes of $S$ (for $K \neq D$), as follows. For any two elements $a$, $b$ in $S$, $K(\sigma)_a \leq K(\sigma)_b$ if and only if $K_{a\sigma} \leq K_{b\sigma}$. Since $\psi$ is a one-to-one mapping, $\leq$ is a partial ordering. Note, for example, that $R(\sigma)_a \leq R(\sigma)_b$ if and only if

$$\{a\sigma^{\natural}\} \cup (a\sigma^{\natural})(S\sigma^{\natural}) \subseteq \{b\sigma^{\natural}\} \cup (b\sigma^{\natural})(S\sigma^{\natural}) .$$

For any two elements $a$, $b$ in $S$, the following are clear.

(i) If $K_a = K_b$, then $K(\sigma)_a = K(\sigma)_b$ (since $K \subseteq K(\sigma)$).

(ii) For $K \neq D$, if $K_a \leq K_b$, then $K(\sigma)_a \leq K(\sigma)_b$.

DEFINITION (from [2]) *Let* $E$ *be any equivalence relation on* $S$. *Then* $S$ *is called* $E$*-regular if, for any congruence* $\rho$ *on* $S$, $\rho|E \subseteq E$ *implies* $\rho \subseteq E$.

THEOREM 1. *If* $S$ *is a regular semigroup, then* $S$ *is* $K(\sigma)$*-regular.*

Proof. With minor alterations (involving (i) and (ii) above), Lallement's proof of Theorem 3.6 [3] is sufficient (see also Theorem 2.3 [2]). The alterations consist mainly in replacing, for example, $R_a$ by $R(\sigma)_a$.

## 3. The lattice of congruences

LEMMA 1. *Let* $S$ *be a regular semigroup. For each element* $i$ *of some index set* $I$, *let* $\rho_i$, $\rho_i'$ *be congruences on* $S$ *such that*

https://doi.org/10.1017/S0004972700041472 Published online by Cambridge University Press

$(\rho_i, \rho_i') \in \Theta$ . *Then*

$$\left(\bigvee_{i\in I} \rho_i , \bigvee_{i\in I} \rho_i'\right) \in \Theta .$$

*In particular,* $\Theta$ *is a join compatible equivalence relation.*

Proof. For each $i \in I$ , let $\alpha_i$ and $\beta_i$ be the least and greatest element respectively of $\rho_i\Theta$ and put $\sigma = \bigvee_{i\in I} \alpha_i$ . Define $H(\sigma)$ as in Section 2. Clearly $H(\sigma)|E = \sigma|E$ . Then for each $i \in I$ ,

$$\beta_i|E = \alpha_i|E \subseteq \sigma|E = H(\sigma)|E ,$$

and from Theorem 1, $\beta_i \subseteq H(\sigma)$ . Hence $\bigvee_{i\in I} \beta_i \subseteq H(\sigma)$ . Therefore

$$\left(\bigvee_{i\in I} \alpha_i\right)|E \subseteq \left(\bigvee_{i\in I} \beta_i\right)|E \subseteq H(\sigma)|E = \sigma|E = \left(\bigvee_{i\in I} \alpha_i\right)|E ,$$

whence $\left(\bigvee_{i\in I} \alpha_i\right)|E = \left(\bigvee_{i\in I} \beta_i\right)|E$ . Since, for each $i \in I$ , we have $\alpha_i \subseteq \rho_i \subseteq \beta_i$ and $\alpha_i \subseteq \rho_i' \subseteq \beta_i$ we easily obtain that $\left(\bigvee_{i\in I} \rho_i\right)|E = \left(\bigvee_{i\in I} \rho_i'\right)|E = \left(\bigvee_{i\in I} \alpha_i\right)|E$ , giving the required result.

Following Reilly and Scheiblich [5], we see that

$$\left(\bigcap_{i\in I} \rho_i\right) \cap (E\times E) = \bigcap_{i\in I} [\rho_i \cap (E\times E)] = \bigcap_{i\in I} [\rho_i' \cap (E\times E)] = \left(\bigcap_{i\in I} \rho_i'\right) \cap (E\times E) .$$

From this and Lemma 1, it follows that $\Theta$ is a congruence on $\Lambda(S)$ satisfying condition *(A)* of Result 1. The proof of the main theorem is now complete.

COROLLARY 1. *For each element* $\rho$ *in* $\Lambda(S)$ *, let* $M(\rho)$ *be the greatest element of the* $\Theta$*-class* $\rho\Theta$ . *Then for any two congruences* $\rho , \sigma$ *on* $S$ *, if* $\rho \subseteq \sigma$ *then* $M(\rho) \subseteq M(\sigma)$ .

Proof. Now from Lemma 1, $M(\rho) \vee M(\sigma) \in (\rho \vee \sigma)\Theta$ , and so $M(\rho) \subseteq M(\rho) \vee M(\sigma) \subseteq M(\rho \vee \sigma) = M(\sigma)$ .

REMARK 1. If $m(\rho)$ denotes the least element of $\rho\Theta$ , then it is clear that $\rho \subseteq \sigma$ implies $m(\rho) \subseteq m(\sigma)$ , since $m(\rho) = [\rho \cap (E\times E)]^*$ , the congruence on $S$ generated by the relation $\rho \cap (E\times E)$ .

REMARK 2. For any $\rho \in \Lambda(S)$ , the $\Theta$-class $\rho\Theta$ is isomorphic to the lattice of idempotent separating congruences on $S/m(\rho)$ (see the proof of (ii) Theorem 3.4 [5]) and if $S$ is an inverse semigroup then $\Lambda(S)/\Theta$ is (isomorphic to) a sublattice of the lattice of congruences on the semilattice $E$ ; for let $\rho$ , $\sigma$ be any elements of $\Lambda(S)$ . Then $(\rho \cap \sigma)|E = (\rho|E) \cap (\sigma|E)$ and $(\rho \vee \sigma)|E = (\rho|E) \vee (\sigma|E)$ (the proof of Theorem 5.1 [5] shows that $(\rho_1 \vee \rho_3)|E = (\rho_1|E) \vee (\rho_3|E))$ . Let us call $\{\rho \cap (E \times E) : \rho \in \Lambda(S)\}$ the set of normal congruences on $E$ , and denote it by $N(E)$ . Then $N(E)$ is a sublattice of the lattice of congruences on $E$ and is isomorphic to $\Lambda(S)/\Theta$ . By considering any Brandt semigroup with three or more idempotents, we see that $N(E)$ is not always equal to the lattice of all congruences on $E$ . We note that the normal congruences on $E$ are characterized as follows (due to Reilly and Scheiblich; see Definition 4.1 [5]): a congruence $\zeta$ on $E$ is normal if and only if for every pair $(e,f) \in \zeta$ and every element $a \in S$ , we have $(aea^{-1}, afa^{-1}) \in \zeta$ .

## References

[1] A.H. Clifford and G.B. Preston, *The algebraic theory of semigroups*, vols. 1 and 2, (Amer. Math. Soc. Mathematical Surveys 7, Providence, 1961 and 1967).

[2] G. Lallement, "Congruences et équivalences de Green sur un demi-groupe régulier", *C.R. Acad. Sc. Paris*, Série A, 262 (1966), 613-616.

[3] Gérard Lallement, "Demi-groupes réguliers" (Doctoral dissertation), *Ann. Mat. Pura Appl.* (IV), Vol. 77, (1967), 47-130.

[4] W.D. Munn, "A certain sublattice of the lattice of congruences on a regular semigroup", *Proc. Cambridge Philos. Soc.* 60 (1964), 385-391.

[5] N.R. Reilly and H.E. Scheiblich, "Congruences on regular semigroups", *Pacific J. Math.* 23 (1967), 349-360.

Department of Mathematics,
Monash University,
Clayton, Victoria.

https://doi.org/10.1017/S0004972700041472 Published online by Cambridge University Press