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ON THE LAW OF THE ITERATED LOGARITHM FOR INDEPENDENT BANACH SPACE VALUED RANDOM VARIABLES[1]

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In this paper we establish some general forms of the law of the iterated logarithm for independent random variables (X_n) with Banach space values, where (X_n) is not necessarily identically distributed. Our results include the Kolmogorov law of the iterated logarithm (LIL) in both finite and infinite dimensional cases, and they improve the Wittmann LIL as well as extend it to the vector setting. The Ledoux–Talagrand LIL for an i.i.d. sequence is also a simple corollary of our results.

1. Introduction. Let B denote a separable Banach space with topological dual B^* and the norm $\|\cdot\|$. For a B-valued random variable X, we write $X \in WM_0^2$ if for all $f \in B^*$, we have $Ef(X) = 0$ and $Ef^2(X) < +\infty$. Throughout, $\{X_n\}$ are independent random variables with values in B and as usual $S_n = X_1 + \cdots + X_n (n \geq 1)$. Write

$$s_n \equiv \sup_{f \in B_1^*} \left\{ \sum_{j=1}^{n} Ef^2(X_j) \right\}^{1/2}, \qquad n \geq 1, \tag{1.1}$$

where B_1^* is the unit ball of B^*. By Lemma 2.1 of Goodman, Kuelbs and Zinn (1981), we have $s_n < +\infty$ if $X_n \in WM_0^2$ for each n. We write $L_2 x$ to denote the function

$$\log \max\{e, \log x\}, \qquad x > 0.$$

When $B = R$, the almost sure limit behavior of $\{S_n / \sqrt{2 s_n^2 L_2 s_n^2}\}_{n \geq 1}$ has been studied extensively. It is known, under some assumptions, that

$$\limsup_{n \to \infty} \frac{S_n}{\sqrt{2 s_n^2 L_2 s_n^2}} = 1 \quad \text{a.s.} \tag{1.2}$$

This is the "law of the iterated logarithm" (LIL).

In his famous paper, Wittmann (1985) established a generalization of the LIL. Based on the convergence estimate in the central limit theorem,

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Wittmann's theorem states that (1.2) holds if the following conditions are fulfilled:

$$EX_n = 0 \quad \text{and} \quad EX_n^2 < +\infty, \qquad n \geq 1, \tag{1.3}$$

$$\sum_{n=1}^{\infty} \frac{|X_n|^p}{\left(2s_n^2 L_2 s_n^2\right)^{p/2}} < +\infty \quad \text{for some } 2 < p \leq 3, \tag{1.4}$$

[or $EX_n^3 = 0$, $n \geq 1$, and (1.4) holds for some $3 < p \leq 4$],

$$\lim_{n\to\infty} s_n = +\infty \quad \text{and} \quad \limsup_{h\to\infty} \frac{s_{n+1}}{s_n} < +\infty. \tag{1.5}$$

According to Wittmann, the classical result of Hartman and Wintner (1941) is just a simple corollary of his theorem. That is, if $\{X, X_n;\ n \geq 1\}$ is a sequence of i.i.d. random variables, then

$$\limsup_{n\to\infty} \frac{S_n}{\sqrt{2nL_2 n}} = \{EX^2\}^{1/2} \quad \text{a.s.}, \tag{1.6}$$

holds if and only if

$$EX = 0 \quad \text{and} \quad EX^2 < +\infty.$$

Now, turn to the infinite-dimensional case. Some nice results in this subject were achieved by Ledoux and Talagrand (1988, 1990, 1991). Ledoux and Talagrand (1988) gave a characterization for i.i.d. random variables satisfying the LIL that led to the complete extension of the Hartman–Wintner LIL to Banach space valued random variables. In their remarkable book, Ledoux and Talagrand [(1991), page 197. Theorem 8.2] established an LIL result of Kolmogorov type, which states that

$$1 \leq \limsup_{n\to\infty} \frac{\|S_n\|}{\sqrt{2s_n^2 L_2 s_n^2}} \leq \Gamma \quad \text{a.s. for some } \Gamma > 0, \tag{1.7}$$

if the following conditions are fulfilled:

$$X_n \in WM_0^2 \quad \text{and} \quad \|X_n\| \leq \eta_n \sqrt{s_n^2 / L_2 s_n^2} \qquad \text{a.s. for each } n, \tag{1.8}$$

for some sequence $\{\eta_n\}$ of positive numbers tending to 0,

$$\lim_{n\to\infty} s_n = +\infty, \tag{1.9}$$

$$\left\{ \frac{S_n}{\sqrt{2s_n^2 L_2 s_n^2}} \right\}_{n\geq 1} \quad \text{is bounded in probability.} \tag{1.10}$$

Furthermore,

$$\limsup_{n\to\infty} \frac{\|S_n\|}{\sqrt{2s_n^2 L_2 s_n^2}} = 1 \quad \text{a.s.}, \tag{1.11}$$

whenever the condition (1.10) is strengthened to

$$\frac{S_n}{\sqrt{2s_n^2 L_2 s_n^2}} \to 0 \quad \text{in probability.} \tag{1.12}$$

However, three important problems remained unanswered.

1. Does Wittmann's theorem also hold when $p > 4$?
2. Even on the real line, the Kolmogorov LIL is not, according to Wittmann, contained in Wittmann's theorem. Can the Kolmogorov LIL be contained in the LIL of Wittmann type by weakening the conditions in the theorem?
3. Can we obtain, in Banach spaces, some results which include the Ledoux–Talagrand LIL for i.i.d. random variables?

In another paper, Wittmann (1987) solved, on the real line, Problems 1 and 2 by giving some theorems of a different type. Concerning Problem 3, de Acosta (1983) gave a new proof for a Hartman–Wintner LIL as an application of the Kolmogorov LIL, where a clever truncation technique was used. However, his arguments do not work for Banach space valued random variables. It seems that the LIL of Kolmogrov type does not contain the Ledoux–Talagrand characterization for i.i.d. Banach space random variables satisfying the LIL.

In this article we try to solve these problems in Banach spaces. We now state the main results.

THEOREM 1.1. *Let $\{X_n\}$ be a sequence of independent random variables with values in B such that $X_n \in WM_0^2$, $n > 1$. Assume for every $\varepsilon > 0$ there exists $p \geq 2$ such that*

$$\sum_{n=1}^{\infty} \frac{E\left\{\|X_n\|^p I_{\{\varepsilon\sqrt{s_n^2/L_2 s_n^2} < \|X_n\| \leq \sqrt{2s_n^2 L_2 s_n^2}\}}\right\}}{\left(2s_n^2 L_2 s_n^2\right)^{p/2}} < +\infty \tag{1.13}$$

and

$$\lim_{n \to \infty} s_n = +\infty, \tag{1.14}$$

$$\left\{\frac{S_n}{\sqrt{2s_n^2 L_2 s_n^2}}\right\}_{n \geq 1} \quad \textit{is bounded in probability.} \tag{1.15}$$

Then,

$$\limsup_{n \to \infty} \frac{\|S_n\|}{\sqrt{2s_n^2 L_2 s_n^2}} \leq \Gamma \quad \textit{a.s. for some } \Gamma > 0. \tag{1.16}$$

Further,

$$\limsup_{n \to \infty} \frac{\|S_n\|}{\sqrt{2s_n^2 L_2 s_n^2}} \leq 1 \quad \textit{a.s.}, \tag{1.17}$$

whenever the condition (1.15) *is strengthened to*

$$\frac{S_n}{\sqrt{2s_n^2 L_2 s_n^2}} \to 0 \quad \textit{in probability}. \tag{1.18}$$

By strengthening the condition (1.13) we can give the lower bound in the conclusion.

THEOREM 1.2. *Let* $\{X_n\}$ *be a sequence of independent random variables with values in* B *such that* $X_n \in WM_0^2$. *Assume* (1.15) *holds*,

$$\lim_{n\to\infty} s_n = +\infty \quad \textit{and} \quad \limsup_{h\to\infty} \frac{s_{n+1}}{s_n} < +\infty, \tag{1.19}$$

and for every $\varepsilon > 0$ *there exists* $p > 2$ *such that*

$$\sum_{n=1}^{\infty} \frac{E\Big\{\|X_n\|^p I_{\{\|X_n\| > \varepsilon\sqrt{s_n^2/L_2 s_n^2}\}}\Big\}}{\left(2s_n^2 L_2 s_n^2\right)^{p/2}} < +\infty. \tag{1.20}$$

Then,

$$1 \le \limsup_{n\to\infty} \frac{\|S_n\|}{\sqrt{2s_n^2 L_2 s_n^2}} \le \Gamma \quad a.s. \textit{ for some } \Gamma > 0. \tag{1.21}$$

Further,

$$\limsup_{n\to\infty} \frac{\|S_n\|}{\sqrt{2s_n^2 L_2 s_n^2}} = 1 \quad a.s., \tag{1.22}$$

whenever (1.15) *is strengthened to* (1.18).

Now let us make the following comments. The condition (1.15) is of course necessary for (1.16). Hypotheses (1.15) and (1.18) are typical in the context of almost sure limit theorems for infinite-dimensional random variables, and hold automatically if B is a finite-dimensional space. The boundedness condition (1.8) in the Kolmogorov LIL implies $\limsup_{n\to\infty} s_{n+1}/s_n = 1$. Therefore, Theorem 1.2 can be viewed as a connection between Kolmogorov's and Wittmann's LIL. It is easily seen how Theorem 1.2 improves Wittmann's theorem as well as extends it to the vector setting.

We now try to show that our results contain all previous known results on the LIL for i.i.d. Banach space valued random variables (and therefore the Hartman–Wintner LIL on the real line). If $\{X, X_n;\ n \ge 1\}$ is a sequence of i.i.d. Banach space valued random variables we say X satisfies the bounded LIL (and write $X \in$ BLIL) if

$$\limsup_{n\to\infty} \frac{\|S_n\|}{\sqrt{2nL_2 n}} < +\infty \quad \text{a.s.} \tag{1.23}$$

and, we say X satisfies the compact LIL (and write $X \in \text{CLIL}$) if there exists a nonrandom compact set $K \subset B$ such that

$$\lim_{n \to \infty} d\left(\frac{S_n}{\sqrt{2nL_2n}}, K \right) = 0 \quad \text{a.s.,} \tag{1.24}$$

where $d(x, A) = \inf_{x \in A} \|x - y\|$. Ledoux and Talagrand (1988) have proven that $X \in \text{BLIL}$ if and only if

$$X \in WM_0^2, \tag{1.25}$$

$$E\left\{\|X\|^2 / L_2\|X\|^2\right\} < +\infty, \tag{1.26}$$

$$\left\{ \frac{S_n}{\sqrt{2nL_2n}} \right\}_{n \ge 1} \quad \text{is bounded in probability.} \tag{1.27}$$

As a simple corollary of the above result on the BLIL, a characterization for the CLIL was given in Ledoux and Talagrand (1988), which states that $X \in \text{CLIL}$ if and only if the conditions (1.25) and (1.26),

$$\left\{ f^2(X);\ f \in B_1^*;\ f \in B_1^* \right\} \quad \text{is uniformly integrable} \tag{1.28}$$

and

$$\frac{S_n}{\sqrt{2nL_2n}} \to 0 \quad \text{in probability.} \tag{1.29}$$

hold.

Further, Ledoux and Talagrand (1990) point out that

$$\limsup_{n \to \infty} \frac{\|S_n\|}{\sqrt{2nL_2n}} = \sup_{f \in B_1^*} \left\{ Ef^2(X) \right\}^{1/2} \quad \text{a.s.} \tag{1.30}$$

whenever (1.25), (1.26) and (1.27) hold.

The proof of the necessity parts of both the BLIL and CLIL is easy. [The arguments [Ledoux and Talagrand (1988)] rely partially on the Hartman–Wintner LIL which also is, as we will see in what follows, a simple corollary of our results.] We need only to show the sufficiency parts. If we observe that (1.26) implies

$$\sum_n \frac{E\left\{ \|X_n\|^3 I_{\{\|X_n\| \le \sqrt{2nL_2n}\}} \right\}}{(2nL_2n)^{3/2}} < +\infty, \tag{1.31}$$

it is easily seen, by Theorem 1.1, how the upper bound of the LIL can be well controlled. That is, if the conditions (1.25), (1.26) and (1.27) hold, we have $X \in \text{BLIL}$. (Therefore, as a simple corollary via the closed graph theorem [Ledoux and Talagrand (1988)], we have $X \in \text{CLIL}$ if the conditions (1.25), (1.26), (1.28) and (1.29) are fulfilled.) Further, if (1.27) is strengthened to (1.29), then

$$\limsup_{n\to\infty} \frac{\|S_n\|}{\sqrt{2nL_2 n}} \leq \sup_{f\in B_1^*} \{Ef^2(X)\}^{1/2} \quad \text{a.s.} \tag{1.32}$$

Before getting the lower bound, let us show how Theorem 1.2 contains the Hartman–Wintner LIL. If $B = R$ and X satisfies

$$E(X) = 0 \quad \text{and} \quad EX^2 < +\infty, \tag{1.33}$$

we write

$$Y_n = X_n I_{\{|X_n| \leq \sqrt{2nL_2 n}\}} - E\big(X_n I_{\{|X_n| \leq \sqrt{2nL_2 n}\}}\big),$$

$$Z_n = X_n I_{\{|X_n| > \sqrt{2nL_2 n}\}} - E\big(X_n I_{\{|X_n| > \sqrt{2nL_2 n}\}}\big),$$

$$B_n = \sum_{j=1}^{n} EY_j^2, \qquad n = 1, 2 \dots .$$

It is easily seen from (1.33) that

$$B_n \sim nEX^2, \tag{1.34}$$

$$\lim_{n\to\infty} \frac{1}{\sqrt{2nL_2 n}} \sum_{j=1}^{n} E\big(X_j I_{\{\|X_j\| > \sqrt{2nL_2 n}\}}\big) = 0, \tag{1.35}$$

$$\sum_n P\Big\{\|X\| > \sqrt{2nL_2 n}\Big\} < +\infty. \tag{1.36}$$

By the Borel–Cantelli lemma and (1.36),

$$\lim_{n\to\infty} \frac{1}{\sqrt{2nL_2 n}} \sum_{j=1}^{n} Z_j = 0 \quad \text{a.s.} \tag{1.37}$$

By (1.31) and (1.34) we have

$$\sum_n \frac{E|Y_n|^3}{(2B_n L_2 B_n)^{3/2}} < +\infty.$$

Applying Theorem 1.2, we have

$$\limsup_{n\to\infty} \frac{1}{\sqrt{2nL_2 n}} \sum_{j=1}^{n} Y_j = \{EX^2\}^{1/2} \quad \text{a.s.} \tag{1.38}$$

By (1.37) and (1.38), (1.6) is valid. Now, we can say that we have proven the Hartman–Wintner LIL, as the necessary part is just a simple corollary of the relation (1.6). (Indeed, the right-hand side of (1.6) is infinite whenever (1.33) is not true. See Stout [(1974), Theorem 5.35, page 297].)

Return to the vector setting. In order to obtain the lower bound, simply note that for every $f \in B_1^*$, $f(X)$ obeys the Hartman–Wintner LIL. Therefore

$$\limsup_{n\to\infty} \frac{\|S_n\|}{\sqrt{2nL_2n}} \geq \limsup_{n\to\infty} \frac{|f(S_n)|}{\sqrt{2nL_2n}} \tag{1.39}$$

$$= \{Ef^2(x)\}^{1/2} \quad \text{a.s. for every } f \in B_1^*.$$

Hence, (1.30) follows from (1.32) and (1.39).

The plan of our paper is as follows. In Section 2, we introduce the isoperimetric technique and the Hoffmann-Jørgensen inequality to prove Proposition 2.2, which is fundamental for the proof of Theorem 1.1, and also of independent interest. In Section 3 we deal with the proof of Theorem 1.1. The method we follow in this section relies on the randomization and isoperimetric technique, Sudakov minorization and Kolmogorov exponential estimation. In Section 4, we prove Theorem 1.2 by establishing the lower bound. In order to avoid some difficulties caused by the truncations, we apply the Lévy decomposition to our proof.

The following notation will be kept throughout the article. For a sequence $\{X_n\}$ of independent random variables, $\{X_n'\}$ will always denote an independent copy of $\{X_n\}$. We will also need a Rademacher sequence $\{\varepsilon_n\}$, and we make $\{\{X_n\},\{X_n'\},\{\varepsilon_n\}\}$ an independent system. We will therefore use the notation $E_X(P_X)$, $E_{X'}(P_{X'})$ and $E_\varepsilon(P_\varepsilon)$ to denote conditional expectations (probabilities) with respect to $\{X_n\}$, $\{X_n'\}$ and $\{\varepsilon_n\}$, respectively. For any family $\{a_f\}$ of numbers indexed by a subset U of B^*, we let

$$\|a_f\|_U \equiv \sup_{f\in U} |a_f|.$$

2. Some results on the upper bound. The isoperimetric technique plays an important role in the present article. The following lemma is a simple corollary of Ledoux and Talagrand [(1990), Proposition 1.1].

LEMMA 2.1. *Let $\{X_i\}_{i\leq n}$ be independent and symmetric random variables with values in B, U be a subset of B_1^*, m, q be integers with $m > q$ and $s > 0$, $t > 0$. Assume*

$$\|X_j\| \leq \frac{s}{m}, \qquad j = 1, 2, \dots, n. \tag{2.1}$$

Then

$$P\left\{\left\|\sum_{j=1}^n f(X_j)\right\|_U > t + 2s + 8Mq\right\}$$

$$\leq \left(\frac{K_0}{q}\right)^m + 4\exp\left\{-\frac{t^2}{64q\sigma^2}\right\} + 4\exp\left\{-\frac{mt^2}{768qMs}\right\},$$

where K_0 *is a universal constant,* $M = E\|\sum_{j=1}^{n} X_j\|$ *and* $\sigma^2 = \sup_{f\in U}\sum_{j=1}^{n} Ef^2(X_j)$.

PROOF. Equation (2.1) implies

$$\sum_{j=1}^{m} \|X_j\|^* \le s,$$

where $\{\|X_j\|^*\}_{j\le n}$ is the nonincreasing rearrangement of $\{\|X_j\|\}_{j<n}$. Therefore, our conclusion follows from Ledoux and Talagrand [(1990), Proposition 1.1] (with U instead of B_1^*). □

Now, consider a sequence $\{a_n\}$ of positive numbers such that $a_n \uparrow +\infty$. By Wittmann [(1985), Lemma 3.3], for any $\lambda > 1$, there exists a sequence $\{n_k\} \subset N$ with

$$\lambda a_{n_k} \le a_{n_{k+1}} \le \lambda^3 a_{n_k+1}. \tag{2.2}$$

PROPOSITION 2.2. *Let* $\{X_n\}$ *be a sequence of independent random variables with values in* B *such that* $X_n \in WM_0^2$, $\{a_n\}$ *be a sequence of positive numbers such that* $a_n \uparrow +\infty$, *and* $\{n_k\}$ *be integers satisfying* (2.2) *for some* $\lambda > 1$. *Write*

$$\sigma_k^2 = \sup_{f\in B_1^*}\sum_{j=n_k+1}^{n_{k+1}} Ef^2(X_j), \qquad k \ge 1.$$

Assume

$$\sum_{n=1}^{\infty}\frac{E\|X_n\|^p}{a_n^p} < +\infty \quad \textit{for some } p \ge 2. \tag{2.3}$$

Then,

$$\limsup_{n\to\infty}\frac{\|S_n\|}{a_n} \le \Gamma \quad a.s.\ \textit{for some } \Gamma > 0, \tag{2.4}$$

whenever the following two conditions are fulfilled:

$$\sum_{k=1}^{\infty}\exp\left\{-\frac{Ma_{n_{k+1}}^2}{\sigma_k^2}\right\} < +\infty \quad \textit{for some } M > 0, \tag{2.5}$$

$$\left\{\frac{S_n}{a_n}\right\}_{n\ge 1} \quad \textit{is bounded in probability}. \tag{2.6}$$

In addition,

$$\lim_{n\to\infty}\frac{\|S_n\|}{a_n} = 0 \quad a.s., \tag{2.7}$$

whenever the following two conditions are fulfilled:

$$\sum_{k=1}^{\infty} \exp\left\{-\frac{\delta a_{n_{k+1}}^2}{\sigma_k^2}\right\} < +\infty \quad \textit{for every } \delta > 0, \tag{2.8}$$

$$\frac{S_n}{a_n} \to 0 \quad \textit{in probability}. \tag{2.9}$$

Proposition 2.2 will be applied to the proof of Theorem 1.1. It appears to be of independent interest in the study of strong limit behaviour. The classical law of large numbers of Kolmogorov states that if $\{X_n\}$ are real independent random variables with mean 0, then $S_n/n \to 0$ almost surely whenever

$$\sum_n \frac{EX_n^2}{n^2} < +\infty.$$

In Banach space [Hoffman-Jørgensen (1976) and Kuelbs and Zinn (1979)], if

$$\sum_n \frac{E\|X_n\|^2}{n^2} < +\infty,$$

then $S_n/n \to 0$ almost surely if and only if $S_n/n \to 0$ in probability. To establish these results, simply note the following facts:

1. Conditions (2.6) and (2.9) are necessary for (2.4) and (2.7), respectively, and hold automatically in the finite dimensional setting.
2. Conditions (2.5) and (2.8) follow from (2.3) when $p = 2$, and are the best possible for our conclusion. For example, (2.5) and (2.8) become necessary if $a_n = n$ and $\{X_n\}$ satisfies some boundedness conditions [Ledoux and Talagrand (1990), Corollary 3.3].

Proof of Proposition 2.2. We only give the proof of (2.7), as the proof of (2.4) is analogous. Our arguments are based on the truncation method, the isoperimetric technique and the Hoffmann-Jørgensen inequality. By Ledoux and Talagrand [(1990), Lemma 2.1], it suffices to prove (2.7) under the assumption that $\{X_n\}$ is symmetric, so we do this. We write $I(k)$ to denote the set $\{n_k + 1, \dots, n_{k+1}\}$. By standard methods and symmetry, (2.7) follows if

$$\lim_{k\to\infty} \left\| \sum_{j\in I(k)} X_j \right\| \Big/ a_{n_{k+1}} = 0 \quad \text{a.s.} \tag{2.10}$$

Let $\delta_k = \sum_{j\in I(k)} E\|X_j\|^p / a_j^p$, $k = 1, 2\dots$ and $b_j = \delta_k^{1/2p} a_j$, $j \in I(k)$, $k = 1, 2\dots$. Define

$$Y_j = X_j I_{\{\|X_j\| \le b_j\}} \quad \text{and} \quad Z_j = X_j I_{\{\|X_j\| > b_j\}}, \qquad j = 1, 2\dots.$$

To show (2.10), it is enough to prove

$$\lim_{k\to\infty}\left\|\sum_{j\in I(k)} Z_j\right\|\Big/a_{n_{k+1}} = 0 \quad \text{a.s.} \tag{2.11}$$

and

$$\lim_{k\to\infty}\left\|\sum_{j\in I(k)} Y_j\right\|\Big/a_{n_{k+1}} = 0 \quad \text{a.s.} \tag{2.12}$$

By the Hoffmann-Jørgensen inequality [Hoffmann-Jørgensen (1974)], in order to establish (2.11), it is enough to show that

$$\sum_k P\left\{\max_{j\in I(k)}\|Z_j\| > \varepsilon a_{n_{k+1}}\right\} < +\infty \quad \text{for every } \varepsilon > 0 \tag{2.13}$$

and

$$\sum_k \left(P\left\{\left\|\sum_{j\in I(k)} Z_j\right\| > \varepsilon a_{n_{k+1}}\right\}\right)^2 < +\infty \quad \text{for every } \varepsilon > 0. \tag{2.14}$$

It is easily seen how (2.3) implies (2.13). Note that (2.3) implies

$$\sum_k \delta_k < +\infty \tag{2.15}$$

and

$$\begin{aligned} P&\left\{\left\|\sum_{j\in I(k)} Z_j\right\| > \varepsilon a_{n_{k+1}}\right\} \\ &\le P\left\{\max_{j\in I(k)}\|X_j\| > \delta_k^{1/2p} a_{n_k+1}\right\} \le \delta_k^{-1/2} a_{n_k+1}^{-p} \sum_{j\in I(k)} E\|X_j\|^p \\ &\le \lambda^{-3}\delta_k^{-1/2}\sum_{j\in I(k)} E\|X_j\|^p/a_j^p = \lambda^{-3}\delta_k^{1/2}, \qquad k = 1,2\dots. \end{aligned}$$

Therefore, (2.14) follows from (2.15).

Turn to the proof of (2.12). In order to apply Lemma 2.1, for each k and $\varepsilon > 0$, we take $m_k = [(1/4)\varepsilon\delta_k^{-1/2p}]$, $t = s = (\varepsilon/4)a_{n_k}$, $q = 2K_0$ and $U = B_1^*$. Then

$$\begin{aligned} P&\left\{\left\|\sum_{j\in I(k)} Y_j\right\| > \frac{3\varepsilon}{4}a_{n_{k+1}} + 16K_0M_k\right\} \\ &\le \left(\frac{1}{2}\right)^{m_k} + 4\exp\left\{-\frac{\varepsilon^2 a_{n_{k+1}}^2}{16\times 128K_0\sigma_k^2}\right\} + 4\exp\left\{-\frac{\varepsilon m_k a_{n_{k+1}}}{4\times 1536K_0M_k}\right\}, \end{aligned}$$

where $M_k = E\|\Sigma_{j\in I(k)}Y_j\|$. By standard arguments we can easily show that (2.9) implies

$$\lim_{k\to\infty} M_k/a_{n_{k+1}} = 0.$$

Therefore, for sufficient large k,

$$P\left\{\left\|\sum_{j\in I(k)} Y_j\right\| > \varepsilon a_{n_{k+1}}\right\} \tag{2.16}$$
$$\le \left(\frac{1}{2}\right)^{m_k} + 4\exp\left\{-\frac{\varepsilon^2 a^2_{n_{k+1}}}{16\times 128 K_0\sigma_k^2}\right\} + 4\exp\{-m_k\}.$$

It is easily seen how (2.15) implies

$$\sum_k \left\{\left(\tfrac{1}{2}\right)^{m_k} + 4\exp\{-m_k\}\right\} < +\infty.$$

By assumption (2.8)

$$\sum_k \exp\left\{-\frac{\varepsilon^2 a^2_{n_{k+1}}}{16\times 128 K_0\sigma_k^2}\right\} < +\infty.$$

Hence (2.16) implies

$$\sum_k P\left\{\left\|\sum_{j\in I(k)} Y_j\right\| > \varepsilon a_{n_{k+1}}\right\} < +\infty.$$

Therefore (2.12) holds. □

3. Proof of Theorem 1.1. Throughout this section, the assumptions of Theorem 1.1 will be assumed to hold, and we will use the notation in Section 1. Write

$$a_n = \sqrt{2s_n^2 L_2 s_n^2}, \qquad t_n = \sqrt{2L_2 s_n^2}, \qquad n \ge 1.$$

We only prove (1.17), as the proof of (1.16) is analogous. Before the technical details, we outline the main idea of the proof. First note that assumption (1.13) implies

$$C \equiv \sum_n P\{\|X_n\| > a_n\} < +\infty. \tag{3.1}$$

Therefore,

$$\lim_{n\to\infty}\left\|\sum_{j=1}^n X_j I_{\{\|X_j\| > a_j\}}\right\| \Big/ a_n = 0 \quad \text{a.s.}$$

Observe that

$$\begin{aligned}\frac{1}{a_n}\left\|\sum_{j=1}^n E\left(X_j I_{\{\|X_j\| > a_j\}}\right)\right\| &= \frac{1}{a_n}\sup_{f\in B_1^*}\sum_{j=1}^n Ef\left(X_j I_{\{\|X_j\| > a_j\}}\right)\\ &\le \frac{1}{a_n}\sup_{f\in B_1^*}\sum_{j=1}^n \left\{Ef^2(X_j)\right\}^{1/2}\left\{P\{\|X_j\| > a_j\}\right\}^{1/2}\\ &\le C^{1/2}\cdot\frac{s_n}{a_n}\to 0, \qquad n\to\infty.\end{aligned}$$

Hence

$$\lim_{n\to\infty}\left\|\sum_{j=1}^{n}\left(X_jI_{\{\|X_j\|>a_j\}}-E\left(X_jI_{\{\|X_j\|>a_j\}}\right)\right)\right\|\Big/a_n=0\quad\text{a.s.}$$

Thus, to prove Theorem 1.1, it suffices to show that

$$\limsup_{n\to\infty}\left\|\sum_{j=1}^{n}\left(X_jI_{\{\|X_j\|\le a_j\}}-E\left(X_jI_{\{\|X_j\|\le a_j\}}\right)\right)\right\|\Big/a_n\le 1\quad\text{a.s.}\tag{3.2}$$

To this aim, the usual truncation technique is used. That is, we write

$$\begin{cases}Y_j=X_jI_{\{\|X_j\|\le\alpha s_j/t_j\}}-E\left(X_jI_{\{\|X_j\|\le\alpha s_j/t_j\}}\right),\\ Z_j=X_jI_{\{\alpha s_j/t_j<\|X_j\|\le a_j\}}-E\left(X_jI_{\{\alpha s_j/t_j<\|X_j\|\le a_j\}}\right),\qquad j=1,2\dots,\end{cases}\tag{3.3}$$

where $\alpha>0$ is constant and to be specified. In order to prove (3.2), we need only to prove that

$$\lim_{n\to\infty}\left\|\sum_{j=1}^{n}Z_j\right\|\Big/a_n=0\quad\text{a.s.}\tag{3.4}$$

holds for every $\alpha>0$; and for every $\delta>0$, we can make $\alpha>0$ so small that

$$\limsup_{n\to\infty}\left\|\sum_{j=1}^{n}Y_j\right\|\Big/a_n\le 1+\delta\quad\text{a.s.}\tag{3.5}$$

We will see in what follows that (3.4) is a direct corollary of Proposition 2.2, and the proof of (3.5) relies on the Kolmogorov exponential estimate, Sudakov minorization and the isoperimetric inequality.

By assumption (1.13), there exists $p\ge 2$ such that

$$\sum_n\frac{E\|Z_n\|^p}{a_n^p}<+\infty.\tag{3.6}$$

It is easily seen, by standard arguments, that (1.18) implies

$$\sum_{j=1}^{n}Z_j/a_n\to 0\quad\text{in probability.}$$

Let $\{n_k\}$ satisfy (2.2) for some $\lambda>1$. To prove (3.4), by Proposition 2.2, we need only to verify that

$$\sum_k\exp\left\{-\varepsilon a_{n_{k+1}}^2\Big/\sup_{f\in B_1^*}\sum_{j\in I(k)}Ef^2(Z_j)\right\}<+\infty\quad\text{for every }\varepsilon>0.\tag{3.7}$$

Let

$$\mathbf{N}_1=\left\{k\in\mathbf{N};\ \sum_{j\in I(k)}\frac{E\|Z_j\|^p}{a_j^P}\le t_{n_{k+1}}^{-2p}\right\}.$$

It follows from (3.6) that

$$\sum_{k \in \mathbf{N} \setminus \mathbf{N}_1} t_{n_{k+1}}^{-2p} < +\infty. \tag{3.8}$$

For each $k \in \mathbf{N}_1$, we have

$$\begin{aligned} &\frac{1}{s_{n_{k+1}}^2} \sup_{f \in B_1^*} \sum_{j \in I(k)} Ef^2(Z_j) \\ &\leq \frac{2^{p-1}}{s_{n_{k+1}}^2} \sum_{j \in I(k)} E\|Z_j\|^p \left(\frac{t_j}{\alpha s_j} \right)^{p-2} \leq \frac{2^{2p-1}}{\alpha^{p-2}} \cdot t_{n_{k+1}}^{2p-2} \sum_{j \in I(k)} E\|Z_j\|^p / a_j^p \\ &\leq \frac{2^{2p-1}}{\alpha^{p-2}} \cdot \frac{1}{t_{n_{k+1}}^2} \to 0. \end{aligned} \tag{3.9}$$

It is easily seen that (3.9) implies

$$\sum_{k \in \mathbf{N}_1} \exp\left\{ -\varepsilon a_{n_{k+1}}^2 \Big/ \sup_{f \in B_1^*} \sum_{j \in I(k)} Ef^2(Z_j) \right\} < +\infty.$$

For each $k \in \mathbf{N} \setminus \mathbf{N}_1$,

$$\exp\left\{ -\varepsilon a_{n_{k+1}}^2 \Big/ \sup_{f \in B_1^*} \sum_{j \in I(k)} Ef^2(Z_j) \right\} \leq \exp\left\{ -\varepsilon t_{n_{k+1}}^2 \right\}.$$

Therefore, (3.8) implies that

$$\sum_{k \in \mathbf{N} \setminus \mathbf{N}_1} \exp\left\{ -\varepsilon a_{n_{k+1}}^2 \Big/ \sup_{f \in B_1^*} \sum_{j \in I(k)} Ef^2(Z_j) \right\} < +\infty.$$

Hence, (3.7) is valid.

Now we prove (3.5). By a standard procedure it suffices to show that for any $\delta > 0$ we can make $\alpha > 0$ sufficiently small that for any $\lambda > 1$ and $\{n_k\}$ satisfying (2.2),

$$\sum_k P\left\{ \left\| \sum_{j=1}^{n_k} Y_j \right\| > (1 + \delta) a_{n_k} \right\} < +\infty. \tag{3.10}$$

The idea of the proof can be found in Ledoux and Talagrand [(1991), page 197, Theorem 8.2]. We now make some observations. For each n we define a pseudometric $d_n(f, g)$ on B_1^*:

$$d_n(f,g) = \frac{1}{s_n} \left(\sum_{j=1}^{n} E(f-g)^2(Y_j) \right)^{1/2}, \qquad f, g \in B_1^*,$$

and write $N(\varepsilon, B_1^*, d_n)$, for every n and $\varepsilon > 0$, to denote the minimal number of open balls of radius $\varepsilon > 0$ in the pseudometric d_n which are necessary to cover B_1^*, that is,

$$N(\varepsilon, B_1^*, d_n) = \min\Big\{m; \text{there exist } f_1, \dots, f_m \in B_1^* \text{ such that } \min_{i \le m} d_n(f_i, f) < \varepsilon \text{ for every } f \in B_1^*\Big\}.$$

Let $V_k(\varepsilon)$ be the ε-net of the pseudometric space (B_1^*, d_{n_k}) such that

$$\operatorname{Card}(V_k(\varepsilon)) = N(\varepsilon, B_1^*, d_{n_k}) \tag{3.11}$$

and

$$U_k(\varepsilon) = \left\{ f \in B_1^*; \sum_{j=1}^{n_k} E f^2(Y_j) \le \frac{\varepsilon^2}{4} s_{n_k}^2 \right\}.$$

It is easily seen that for each k,

$$\|x\| \le \|f(x)\|_{V_k(\varepsilon)} + 2\|f(x)\|_{U_k(\varepsilon)}.$$

Therefore, it suffices to show that:

(A) For every $\varepsilon > 0$ and sufficiently small $\alpha > 0$,

$$\sum_k P\left\{ \left\| \sum_{j=1}^{n_k} f(Y_j) \right\|_{V_k(\varepsilon)} > (1+\delta) a_{n_k} \right\} < +\infty. \tag{3.12}$$

(B) For sufficiently small $\varepsilon > 0$ and $\alpha > 0$,

$$\sum_k P\left\{ \left\| \sum_{j=1}^{n_k} f(Y_j) \right\|_{U_k(\varepsilon)} > \delta a_{n_k} \right\} < +\infty. \tag{3.13}$$

First note that (3.11) implies

$$\begin{aligned} &P\left\{ \left\| \sum_{j=1}^{n_k} Y_j \right\|_{V_k(\varepsilon)} > (1+\delta) a_{n_k} \right\} \\ &\qquad \le N(\varepsilon, B_1^*, d_{n_k}) \sup_{f \in B_1^*} P\left\{ \left| \sum_{j=1}^{n_k} f(Y_j) \right| > (1+\delta) a_{n_k} \right\}. \end{aligned} \tag{3.14}$$

Naturally, in order to prove (3.12), we hope that $N(\varepsilon, B_1^*, d_n)$ can be well controlled. We claim that

$$\lim_{n\to\infty} \frac{1}{t_n^2} \log N(\varepsilon, B_1^*, d_n) = 0 \quad \text{for every } \varepsilon > 0. \tag{3.15}$$

The proof of (3.15) is based on arguments using an entropy estimate. Note that

$$\|Y_j\| \le 2\alpha \frac{s_j}{t_j}, \qquad j = 1, 2 \dots . \tag{3.16}$$

It is easily seen that assumption (1.18) implies

$$\lim_{n\to\infty} E\left\|\sum_{j=1}^{n}\varepsilon_j Y_j\right\|\Big/ a_n = 0. \tag{3.17}$$

By the triangle and Jensen inequalities we have

$$\begin{aligned} 2E\left\|\sum_{j=1}^{n}\varepsilon_j f^2(Y_j)\right\|_{B_1^*} &\geq E\left\|\sum_{j=1}^{n}\varepsilon_j\big(f^2(Y_j)-f^2(Y_j')\big)\right\|_{B_1^*} \\ &= E\left\|\sum_{j=1}^{n}\big(f^2(Y_j)-f^2(Y_j')\big)\right\|_{B_1^*} \\ &\geq E\left\|\sum_{j=1}^{n}\big(f^2(Y_j)-Ef^2(Y_j)\big)\right\|_{B_1^*}. \end{aligned}$$

Note (3.16) and (3.17). By standard arguments via a comparison theorem [Ledoux and Talagrand (1989), Theorem 5] we can easily show that

$$\lim_{n\to\infty}\frac{1}{s_n^2}E\left\|\sum_{j=1}^{n}\varepsilon_j f^2(Y_j)\right\|_{B_1^*} = 0.$$

Therefore,

$$\lim_{n\to\infty}\frac{1}{s_n^2}E\left\|\sum_{j=1}^{n}\big(f^2(Y_j)-Ef^2(Y_j)\big)\right\|_{B_1^*} = 0. \tag{3.18}$$

Now we apply, conditionally on the Y_j's, the Sudakov type minorization of Ledoux and Talagrand [(1991), page 114, Proposition 4.13]. Observe that, with high probability, for example, larger than $3/4$,

$$\frac{1}{s_n}E_\varepsilon\left\|\sum_{j=1}^{n}\varepsilon_j Y_j\right\| \leq \frac{\varepsilon^2}{4K}t_n \leq \frac{\varepsilon^2}{4K}\cdot\frac{1}{\max_{j\leq n}\|Y_j/s_n\|}\quad \text{for large } n,$$

where the last inequality follows from (3.16) with $\alpha < 1$. By (3.18), with probability larger than $3/4$,

$$\frac{1}{s_n^2}\sup_{f\in B_1^*}\left\|\sum_{j=1}^{n}\big(f^2(Y_j)-Ef^2(Y_j)\big)\right\|_{B_1^*} < \frac{\varepsilon^2}{16}\quad \text{for large } n.$$

By Ledoux and Talagrand [(1991), page 114, Proposition 4.13], with probability larger than $1/2$,

$$\frac{1}{s_n}E_\varepsilon\left\|\sum_{j=1}^{n}\varepsilon_j Y_j\right\| \geq \frac{\varepsilon}{4K}\big(\log N(\varepsilon, B_1^*, d_n)\big)^{1/2}.$$

Therefore,

$$\frac{1}{s_n}E\left\|\sum_{j=1}^{n}\varepsilon_j Y_j\right\| \geq \frac{\varepsilon}{8K}(\log N(\varepsilon, B_1^*, d_n))^{1/2}.$$

Thus, (3.15) follows from (3.17).

[I thank M. Ledoux for showing me the recent result on the minorization inequality for the Rademacher sequence that led to the simplification of the original proof for (3.15).]

We now prove (3.12). By the standard exponential estimate due to Kolmogorov, we can make $\alpha > 0$ so small that there exists $u > 0$ such that

$$\sup_{f\in B_1^*} P\left\{\left|\sum_{j=1}^{n_k} f(Y_j)\right| > (1+\delta)a_{n_k}\right\} \leq \exp\left\{-\frac{(1+2u)}{2}t_{n_k}^2\right\} \quad \text{for large } k.$$

By (3.15) we have

$$N(\varepsilon, B_1^*, d_k^n) \leq \exp\left\{\frac{u}{2}t_{n_k}^2\right\} \quad \text{for large } k.$$

It follows from (3.14) that

$$P\left\{\left\|\sum_{f=1}^{n_k} f(Y_j)\right\|_{V_k(\varepsilon)} > (1+\delta)a_{n_k}\right\} \leq \exp\left\{-\frac{1+u}{2}t_{n_k}^2\right\} \quad \text{for large } k.$$

Therefore, (3.12) is valid.

The proof of (3.13) is based on the isoperimetric inequality (Lemma 2.1). By Giné and Zinn [(1984), Lemma 2.5], we need only to show

$$\sum_k \left\{\left\|\sum_{f=1}^{n_k}\varepsilon_j f(Y_j)\right\|_{U_k(\varepsilon)} > \delta a_{n_k}\right\} < +\infty. \tag{3.19}$$

First note that

$$\left\|\sum_{f=1}^{n_k} Ef^2(Y_j)\right\|_{U_k(\varepsilon)} \leq \frac{\varepsilon^2}{4}s_{n_k}^2.$$

To apply Lemma 2.1, for given $\delta > 0$, we make $\alpha \leq \delta/8$ and $\varepsilon \leq \delta/64\sqrt{k_0}$. For each k, let $m = [t_{n_k}^2]$, $t = s = (\delta/4)a_{n_k}$ $q = 2K_0$ and $U = U_k(\varepsilon)$. By (3.16) and Lemma 2.1,

$$\begin{aligned} P&\left\{\left\|\sum_{f=1}^{n_k}\varepsilon_j f(Y_j)\right\|_{U_k(\varepsilon)} > \frac{3\delta}{4}a_{n_k} + 16K_0M_k\right\} \\ &\leq \left(\frac{1}{2}\right)^{[t_{n_k}^2]} + 4\exp\left\{-t_{n_k}^2\right\} + 4\exp\left\{-\frac{[t_{n_k}^2]a_{n_k}}{6144K_0M_k}\right\}, \qquad k \geq 1, \end{aligned}$$

where $M_k = E\|\Sigma_{j=1}^{n_k}\varepsilon_j Y_j\|$. By (3.17) we have

$$P\left\{\left\|\sum_{j=1}^{n_k}\varepsilon_j f(Y_j)\right\|_{U_k(\varepsilon)} > \delta a_{n_k}\right\}$$

$$\leq \left(\frac{1}{2}\right)^{[t_{n_k}^2]} + 4\exp\left\{-t_{n_k}^2\right\} + 4\exp\left\{-\left[t_{n_k}^2\right]\right\} \quad \text{for large } k.$$

This implies (3.19). □

4. Proof of Theorem 1.2. Since (1.20) implies (1.13), by Theorem 1.1 we need only to show that the lower bound holds, that is,

$$\limsup_{n\to\infty}\frac{\|S_n\|}{\sqrt{2s_n^2 L_2 s_n^2}} \geq 1 \quad \text{a.s.} \tag{4.1}$$

Let $\rho > D \equiv \limsup_{n\to\infty} s_{n+1}/s_n$, and for each k, let n_k be the smallest integer such that $s_n > \rho^k$. It is easily seen that

$$s_{n_k} \sim \rho^k \quad \text{and} \quad \frac{s_{n_{k+1}}}{s_{n_k}} \sim \rho. \tag{4.2}$$

We write $I(k)$ to denote the set $\{n_k + 1, \ldots, n_{k+1}\}$ and

$$B_k = \sup_{f\in B_1^*}\sum_{j\in I(k)} Ef^2(X_j), \qquad k \geq 1.$$

By the same arguments as the proof for the lower bound in Ledoux and Talagrand [(1989), Theorem 10], we need only to show that

$$P\left\{\left\|\sum_{j\in I(k)} X_j\right\| > (1-3\varepsilon)\sqrt{2B_k L_2 B_k}\ \text{i.o.}\right\} = 1 \quad \text{for every } \varepsilon > 0.$$

This is equivalent to

$$\sum_k P\left\{\left\|\sum_{j\in I(k)} X_j\right\| > (1-3\varepsilon)\sqrt{2B_k L_2 B_k}\right\} = +\infty \quad \text{for every } \varepsilon > 0.$$

To do this, it suffices to show that

$$\sum_k \sup_{f\in B_1^*} P\left\{\left|\sum_{j\in I(k)} f(X_j)\right| > (1-3\varepsilon)\sqrt{2B_k L_2 B_k}\right\} = +\infty. \tag{4.3}$$

In our proof, we use the exponential estimate of Kolmogorov for the lower bound. To do this, we do some truncations at the usual level $\delta\sqrt{B_k/L_2 B_k}$ for each X_j, $j \in I(k)$, where $\delta > 0$ is a constant and to be specified. However, instead of the complicated control for the random variables above the truncation level, which is the usual treatment in the literature, we write $\{X_j\}_{j\in I(k)}$ in terms of a Lévy decomposition. That is, for each k, if $\{\zeta_j, \eta_j, Y_j, Z_j\}_{j\in I(k)}$ are independent random variables such that $\{\zeta_j\}_{j\in I(k)}$ and $\{\eta_j\}_{j\in I(k)}$ are Bernoulli with

$$E(\zeta_j) = E(\eta_j) = P\left\{\|X_j\| \le \delta\sqrt{B_k/L_2B_k}\right\}, \qquad j \in I(k),$$

$$\mathscr{L}(Y_j)(A) = P\left\{\|X_j\| \le \delta\sqrt{B_k/L_2B_k}\,,\ X_j \in A\right\}/P\left\{\|X_j\| \le \delta\sqrt{B_k/L_2B_k}\right\},$$

and

$$\mathscr{L}(Z_j)(A) = P\left\{\|X_j\| > \delta\sqrt{B_k/L_2B_k}\,,\ X_j \in A\right\}/P\left\{\|X_j\| > \delta\sqrt{B_k/L_2B_k}\right\},$$

then

$$\mathscr{L}\left(X_j I_{\{\|X_j\| \le \delta\sqrt{B_k/L_2B_k}\}}\right) = \mathscr{L}(\eta_j Y_j),$$

$$\mathscr{L}\left(X_j I_{\{\|X_j\| > \delta\sqrt{B_k/L_2B_k}\}}\right) = \mathscr{L}((1-\zeta_j)Z_j),$$

$$\mathscr{L}(X_j) = \mathscr{L}\left(\eta_j Y_j + (1-\zeta_j)Z_j + (\zeta_j - \eta_j)Y_j\right).$$

Further $\{\{\zeta_j\}_{j \in I(k)}, \{\eta_j\}_{j \in I(k)}, \{Y_j\}_{j \in I(k)}, \{Z_j\}_{j \in I(k)}\}_{k \ge 1}$ is an independent system.

We now prove that

$$\sum_k \sup_{f \in B_1^*} P\left\{\left|\sum_{j \in I(k)} (\zeta_j - \eta_j) f(Y_j)\right| > \varepsilon\sqrt{2B_kL_2B_k}\right\} < +\infty. \tag{4.4}$$

First note that for each k,

$$\|Y_j\| \le \delta\sqrt{B_k/L_2B_k}\,, \qquad j \in I(k) \tag{4.5}$$

and

$$\begin{aligned}\sigma_k &\equiv \sup_{f \in B_1^*} \sum_{j \in I(k)} E\left((\zeta_j - \eta_j)^2 f^2(Y_j)\right)\\ &\le \sup_{f \in B_1^*} \sum_{j \in I(k)} 2P\left\{\|X_j\| > \delta\sqrt{B_k/L_2B_k}\right\} Ef^2(Z_j)\\ &\le 2 \max_{j \in I(k)} P\left\{\|X_j\| > \delta\sqrt{B_k/L_2B_k}\right\} \cdot B_k.\end{aligned}$$

By the Kolmogorov exponential estimate, there exists $\alpha > 0$ such that

$$\begin{aligned}&\sup_{f \in B_1^*} P\left\{\left|\sum_{j \in I(k)} (\zeta_j - \eta_j) f(Y_j)\right| > \varepsilon\sqrt{2B_kL_2B_k}\right\}\\ &\quad\le \exp\left\{-\alpha \cdot \frac{2B_kL_2B_k}{\sigma_k^2}\right\}\\ &\quad\le C_p\left(\frac{\sigma_k}{2B_kL_2B_k}\right)^p \le C_p(L_2B_k)^{-p} \max_{j \in I(k)} P\left\{\|X_j\| > \delta\sqrt{\frac{B_k}{L_2B_k}}\right\}.\\ &\quad\le C_p\delta^{-p}(B_kL_2B_k)^{-p/2} \sum_{j \in I(k)} E\left\{\|X_j\|^p I_{\{\|X_j\| > \delta\sqrt{B_k/L_2B_l}\}}\right\} \quad \text{for large } k,\end{aligned}$$

where $p \geq 2$ and C_p are constants. By (4.2) we have

$$B_k \sim (\rho - 1)^2 s_{n_k}^2. \tag{4.6}$$

It is easily seen by assumption (1.20) that we can find $p = p(\delta) \geq 2$ such that

$$\sum_k (B_k L_2 B_k)^{-p/2} \sum_{j \in I(k)} E\left\{\|X_j\|^p I_{\{\|X_j\| > \delta\sqrt{B_k/L_2 B_k}\}}\right\} < +\infty. \tag{4.7}$$

Therefore (4.4) follows from (4.7). Thus, we need only to prove

$$\sum_k \sup_{f \in B_1^*} P\left\{\left|\sum_{j \in I(k)} f(\eta_j Y_j + (1 - \zeta_j) Z_j)\right| > (1 - 2\varepsilon)\sqrt{2 B_k L_2 B_k}\right\} = +\infty. \tag{4.8}$$

Let

$$\tilde{Y}_j = \eta Y_j - E(\eta_j Y_j),$$
$$\tilde{Z}_j = (1 - \zeta_j) Z_j - E((1 - \zeta_j) Z_j), \qquad j = 1, 2 \dots .$$

By the independence of $\{Y_j, Z_j\}$, for each k and $f \in B_1^*$,

$$P\left\{\left|\sum_{j \in I(k)} f(\eta_j Y_j + (1 - \zeta_j) Z_j)\right| > (1 - 2\varepsilon)\sqrt{2 B_k L_2 B_k}\right\}$$
$$\geq P\left\{\left|\sum_{j \in I(k)} f(\tilde{Z}_j)\right| < \varepsilon\sqrt{2 B_k L_2 B_k}\right\} P\left\{\left|\sum_{j \in I(k)} f(\tilde{Y}_j)\right| > (1 - \varepsilon)\sqrt{2 B_k L_2 B_k}\right\}$$
$$\geq \left(1 - \frac{1}{\varepsilon^2} \cdot \frac{1}{2 L_2 B_k}\right) P\left\{\left|\sum_{j \in I(k)} f(\tilde{Y}_j)\right| > (1 - \varepsilon)\sqrt{2 B_k L_2 B_k}\right\}.$$

Therefore, we need only to prove

$$\sum_k \sup_{f \in B_1^*} P\left\{\left|\sum_{j \in I(k)} f(\tilde{Y}_j)\right| > (1 - \varepsilon)\sqrt{2 B_k L_2 B_k}\right\} = +\infty. \tag{4.9}$$

Let

$$\tilde{B}_k = \sup_{f \in B_1^*} \sum_{j \in I(k)} E f^2(\tilde{Y}_j), \qquad k = 1, 2, \dots .$$

We have

$$\tilde{B}_k \leq B_k, \qquad k = 1, 2, \dots . \tag{4.10}$$

Assume for the moment that we could sharpen (4.10) to

$$\tilde{B}_k \sim B_k. \tag{4.11}$$

The proof of (4.9) would be easy via the standard exponential estimate of Kolmogorov for the lower bound (the arguments will be seen in what follows).

Unfortunately (4.11) need not be true, but we shall show that this relation becomes true if $k \to \infty$ omitting a set of integers that is so sparse as not to affect what we prove. In fact, we can follow, by (4.7), the arguments in the proof of (3.8) and (3.9) to construct a subset $\mathbf{N}_1$ of $\mathbf{N}$ such that

$$\sum_{k \in \mathbf{N} \setminus \mathbf{N}_1} \frac{1}{(L_2 B_k)^p} < +\infty \tag{4.12}$$

and

$$\frac{1}{B_k} \sup_{f \in B_1^*} \sum_{j \in I(k)} E f^2(\tilde{Z}_j) \to 0, \qquad k \in \mathbf{N}_1,\ k \to \infty. \tag{4.13}$$

It is easily seen from (4.13) that (4.11) holds if $k \in \mathbf{N}_1$. So, for large $k \in \mathbf{N}_1$ and every $f \in B_1^*$, we have

$$P\left\{\left|\sum_{j \in I(k)} f(\tilde{Y}_j)\right| > (1-\varepsilon)\sqrt{2B_k L_2 B_k}\right\}$$
$$\geq P\left\{\left|\sum_{j \in I(k)} f(\tilde{Y}_j)\right| > \left(1 - \frac{\varepsilon}{2}\right)\sqrt{2\tilde{B}_k L_2 \tilde{B}_k}\right\}.$$

For each $k \in \mathbf{N}_1$, take $f_k \in B_1^*$ such that

$$\sum_{j \in I(k)} E f_k^2(\tilde{Y}_j) = \tilde{B}_k.$$

Note (4.5) and (4.10). By Wittmann [(1987), Lemma 2.1], we can make $\delta > 0$ so small that there exists $\alpha > 0$ such that

$$P\left\{\left|\sum_{j \in I(k)} f_k(\tilde{Y}_j)\right| > \left(1 - \frac{\varepsilon}{2}\right)\sqrt{2\tilde{B}_k L_2 \tilde{B}_k}\right\}$$
$$\geq \exp\{-(1-\alpha) L_2 \tilde{B}_k\} \geq \exp\{-(1-\alpha) L_2 B_k\}$$
$$\geq (\log B_k)^{-1+\alpha} \quad \text{for large } k \in \mathbf{N}_1.$$

Therefore,

$$\sup_{f \in B_1^*} P\left\{\left|\sum_{j \in I(k)} f(\tilde{Y}_j)\right| > (1-\varepsilon)\sqrt{2B_k L_2 B_k}\right\} \tag{4.14}$$
$$\geq (\log B_k)^{-1+\alpha} \sim \left(\log s_{n_k}^2\right)^{-1+\alpha} \quad \text{for large } k \in \mathbf{N}_1.$$

It is easily seen how (4.9) follows from (4.2), (4.12) and (4.14). □

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