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ON THE LAW OF THE ITERATED LOGARITHM FOR LACUNARY TRIGONOMETRIC SERIES II

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1. Introduction. In this note we set

$$S_N(t) = \sum_1^N a_m \cos 2\pi n_m(t + \alpha_m) \text{ and } A_N = \left(2^{-1} \sum_1^N a_m^2\right)^{1/2},$$

where $a_m \geqq 0$ and $\{n_m\}$ is a sequence of positive integers satisfying the gap condition

$$n_{m+1}/n_m \geqq 1 + cm^{-\alpha}, \text{ for some } c > 0 \text{ and } 0 \leqq \alpha \leqq 1/2 . \tag{1.1}$$

For $\alpha = 0$, M. Weiss [5] proved that if

$$A_N \longrightarrow +\infty \text{ and } a_N = o(A_N(\log\log A_N)^{-1/2}), \text{ as } N \longrightarrow +\infty ,$$

then for any sequence of $\{\alpha_m\}$

$$\overline{\lim_N}\, (2A_N^2 \log\log A_N)^{-1/2} S_N(t) = 1 , \quad \text{a.e.} .$$

For $\alpha > 0$, we proved the following

THEOREM A [4]. *If*

$$A_N \longrightarrow +\infty \textit{ and } a_N = O(A_N N^{-\alpha}(\log A_N)^{-(1+\varepsilon)/2}), \textit{ as } N \longrightarrow +\infty ,$$

where ε is a positive number, then we have

$$\overline{\lim_N}\, (2A_N^2 \log\log A_N)^{-1/2} S_N(t) \leqq 1 , \quad \textit{a.e.} .$$

The purpose of the present note is to prove the

THEOREM B. *Suppose*

$$A_N \longrightarrow +\infty \textit{ and } a_N = O(A_N N^{-\alpha}\omega_N^{-1}), \textit{ as } N \longrightarrow +\infty , \tag{1.2}$$

where $\omega_N = (\log N)^\beta (\log A_N)^4 + (\log A_N)^8$ and $\beta > 1/2$, then we have

$$\overline{\lim_N}\, (2A_N^2 \log\log A_N)^{-1/2} S_N(t) \geqq 1 , \quad \textit{a.e.} .$$

If $\alpha < 1/2$ and $\{a_m\}$ is non-increasing, then by Theorem A and B we obtain

$$\overline{\lim_N}\,(2A_N^2 \log\log A_N)^{-1/2} S_N(t) = 1\ , \quad \text{a.e.}\ .$$

In §§ 2–5 we prove Theorem B. The method of the proof is to approximate $S_N(t)$ by the sums of a "almost strongly multiplicative" system and apply the method of P. Révész [2].

2. **Preliminaries.** Let us put, for $k = 0, 1, 2\cdots$,

$$p(k) = \max\{m;\ n_m \leqq 2^k\}\ ,$$
$$\Delta_k(t) = S_{p(k+1)}(t) - S_{p(k)}(t) \quad \text{and} \quad B_k = A_{p(k+1)}\ .$$

If $p(k) + 1 < p(k+1)$, (1.1) implies that

$$2 > n_{p(k+1)}/n_{p(k)+1} > \prod_{m=p(k)+1}^{p(k+1)-1} (1 + cm^{-\alpha})$$
$$> 1 + c\{p(k+1) - p(k) - 1\} p^{-\alpha}(k+1)\ ,$$

and hence

$$\text{(2.1)} \qquad \begin{cases} p(k+1) - p(k) = O(p^{\alpha}(k))\ , \\ p(k+1)/p(k) \to 1\ , \qquad \text{as} \quad k \to +\infty\ . \end{cases}$$

Therefore, we have, by (1.2) and (2.1),

$$\text{(2.2)} \qquad \begin{cases} b_k = \max\{|a_m|,\ p(k) < m \leqq p(k+1)\} = O(B_k \omega_{p(k)}^{-1} p^{-\alpha}(k)) \\ \sum_{p(k)+1}^{p(k+1)} |a_m| \leqq b_k\{p(k+1) - p(k)\} = O(B_k \omega_{p(k)}^{-1})\ , \quad \text{as} \quad k \to +\infty\ . \end{cases}$$

LEMMA 1. *For any given k, j, q and h satisfying $p(j) + 1 < h \leqq p(j+1) < p(k) + 1 < q \leqq p(k+1)$, the number of solutions (n_r, n_i) of the equations*

$$n_q - n_r = n_h \pm n_i{}^{*)}$$

where $p(j) < i < h$ and $p(k) < r < q$, is at most $C2^{j-k}p^{\alpha}(k)$ where C is a positive constant independent of k, j, q and h.

PROOF. If $k < j + 3$, the lemma is evident by (2.1). We assume that $k \geqq j + 3$. If we denote m the smallest number r of the solutions (n_r, n_i), then the number of solutions is not greater than $q - m$. Since $(n_h \pm n_i) \leqq 2^{j+2}$, we have

$$n_m \geqq n_q - 2^{j+2} > n_q(1 - 2^{j+2-k}) \geqq n_q(1 + 2^{j-k}\cdot 5)^{-1}\ .$$

Therefore, we have, by (1.1)

$$1 + 2^{j-k}\cdot 5 > n_q/n_m > \prod_{s=m}^{q-1} (1 + cs^{-\alpha}) \geqq 1 + c(q - m) p^{-\alpha}(k+1)\ .$$

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*) Clearly, $n_q + n_r = n_h \pm n_i$ has no solutions.

Thus, by (2.1) we can prove the lemma.

In the same way we can prove the following

LEMMA 1'. *For any given k, j, q and h such that $j \leqq k-2$, $p(j+1) < h \leqq p(j+2)$ and $p(k+1) < q \leqq p(k+2)$, the number of solutions (n_r, n_i) of the equations*

$$n_q - n_r = n_h \pm n_i$$

where $p(j) < i \leqq p(j+1)$ and $p(k) < r \leqq p(k+1)$, is at most $C2^{j-k}p^{\alpha}(k)$, where C is a positive constant independent of k, j, q and h.

LEMMA 2. *We have, for any M and N $(M < N)$,*

$$\text{(i)} \qquad \left\| B_N^{-2} \sum_M^N (\Delta_m^2 - \|\Delta_m\|^2) \right\| = O((\log B_N)^{-8}) ,$$

$$\text{(ii)} \qquad \left\| B_N^{-2} \sum_M^N \Delta_m \Delta_{m-1} \right\| = O((\log B_N)^{-8}) ,^{*)} \quad as \quad N \to +\infty .$$

PROOF. (i) Let us put, for $k = 1, 2 \cdots$

$$U_k(t) = \Delta_k^2(t) - \|\Delta_k\|^2 - 2^{-1} \sum_{p(k)+1}^{p(k+1)} a_m^2 \cos 4\pi n_m (t + \alpha_m) .$$

Then we have, by (2.2),

$$\| U_k \|_\infty = O(B_N^2 (\log B_N)^{-16}) ,$$

$$\left\| B_N^{-2} \sum_M^N (\Delta_m^2 - \|\Delta_m\|^2) \right\|^2 = 2B_N^{-4} \sum_{k=M+1}^N \sum_{j=M}^{k-1} \int_0^1 U_k(t) U_j(t) dt + O((\log B_N)^{-16}) ,$$

$$\text{as} \quad N \to +\infty .$$

Further, by Lemma 1 and (2.2), we have, for $k > j$

$$\left| \int_0^1 U_k U_j dt \right| \leqq C 2^{j-k} p^{\alpha}(k) \sum_{q=p(k)+1}^{p(k+1)} |a_q| b_k \sum_{h=p(j)+1}^{p(j+1)} |a_h| b_j$$

$$= O(2^{j-k} \|\Delta_k\| \|\Delta_j\| p^{\alpha/2}(k) p^{-\alpha/2}(j) B_N^2 (\log B_N)^{-16}) ,$$

$$\text{as} \quad N \to +\infty .$$

Since $p(j+1)/p(j) \to 1$, as $j \to +\infty$, we have, for every k,

$$\sum_{j=1}^{k-1} 2^{j-k} p^{-\alpha}(j) \leqq C' p^{-\alpha}(k) , \quad \text{for some } C' > 0 .$$

Hence, we have

*) $\|f\|$ denotes L^2-norm unless otherwise stated.

$$\sum_{k=M+1}^{N}\sum_{j=M}^{k-1} 2^{j-k} \| \Delta_k \| \| \Delta_j \| p^{\alpha/2}(k) p^{-\alpha/2}(j)$$
$$\leqq C' \sum_{k=M+1}^{N} \| \Delta_k \| \Big(\sum_{j=1}^{k-1} 2^{j-k} \| \Delta_j \|^2\Big)^{1/2}$$
$$\leqq C' \Big(\sum_{k=M+1}^{N} \| \Delta_k \|^2\Big)^{1/2} \Big(\sum_{k=M+1}^{N}\sum_{j=1}^{k-1} 2^{j-k} \| \Delta_j \|^2\Big)^{1/2} = O(B_N^2) ,$$
$$\text{as} \quad N \to +\infty .$$

Therefore, by the above relations we can prove (i).

(ii) Using Lemma 1′ we can prove (ii) in the same way.

LEMMA 3. *If* $M < N$ *and* $\lambda_N = o((\log A_N)^{3-1/2\beta})$, *as* $N \to +\infty$, *then*

(i) $$\int_0^1 \exp\Big\{\frac{\lambda_N^2}{B_N^2} \sum_M^N (\Delta_m^2 - \| \Delta_m \|^2)\Big\} dt = 1 + o(1) ,$$

(ii) $$\int_0^1 \exp\Big\{\frac{\lambda_N^2}{B_N^2} \sum_M^N \Delta_m \Delta_{m+1}\Big\} dt = 1 + o(1) , \quad \textit{as} \quad N \to +\infty .$$

PROOF. (i) From (1.1), the frequencies of terms of $\Delta_m^2 - \| \Delta_m \|^2$ are in the interval $[2^m c p^{-\alpha}(m+1), 2^{m+2}]$. Since $p(j+1)/p(j) \to 1$, as $j \to +\infty$, we may assume that

(2.3) $$2^m c p^{-\alpha}(m+1) \uparrow +\infty , \quad \text{as} \quad m \uparrow +\infty .$$

We set $m(0) = M$ and if $m(j)$ is defined, then we put

(2.4) $$m(j+1) = \min \{m + m(j); c 2^{m(j)+m} p^{-\alpha}(m(j) + m + 1) > 2^{m(j)+2}\} .$$

By (2.1) we can define $m(j)$ for every j and if $m(j') \leqq N < m(j'+1)$, then we put

$$T_j(t) = \begin{cases} \displaystyle\sum_{m(j)}^{m(j+1)-1} \{\Delta_m^2(t) - \| \Delta_m \|^2\} , & \text{if} \quad 0 \leqq j < j' , \\ \displaystyle\sum_{m(j')}^{N} \{\Delta_m^2(t) - \| \Delta_m \|^2\} , & \text{if} \quad j = j' . \end{cases}$$

From (2.2) it is seen that

$$\| T_j \|_\infty \leqq 2 \max_m \| \Delta_m \|_\infty^{(2\beta-1)/2\beta} \sum_{m(j)}^{m(j+1)-1} \| \Delta_m \|_\infty^{(2\beta+1)/2\beta}$$
$$= O(B_N^2 (\log B_N)^{(4-24\beta)/2\beta} \sum_{m(j)}^{m(j+1)-1} (\log p(m+1))^{-(2\beta+1)/2} ,$$
$$\text{as} \quad N \to +\infty .$$

If $1 \leqq m < m(j+1) - m(j)$, we have, by (2.4)

$$p^\alpha(m(j) + m + 1) \geqq C' 2^{m+1} , \quad \text{for some} \quad C' > 0 .$$

Hence we have, for some constants A and A',

$$\sum_{m(j)}^{m(j+1)-1}(\log p(m+1))^{-(2\beta+1)/2} \leqq A\sum_{1}^{\infty} m^{-(2\beta+1)/2} = A' ,$$

and we obtain

(2.5) $$\varepsilon_N = \max(\| T_j \|_\infty ; 0 \leqq j \leqq j') = O(B_N^2(\log B_N)^{(4-24\beta)/2\beta}) , \quad \text{as} \quad N \to +\infty .$$

Therefore,

$$T_j^2 \leqq \varepsilon_N | T_j | < \varepsilon_N \sum_{m(j)}^{m(j+1)-1}(\Delta_m^2 - \| \Delta_m \|^2) + 2\varepsilon_N \sum_{m(j)}^{m(j+1)} \| \Delta_m \|^2 .$$

Using the inequality $e^x \leqq (1+x)e^{x^2}$ for $|x| \leqq 1/2$, we have, by (2.5)

$$\begin{aligned}\exp\left\{\frac{\lambda_N^2}{B_N^2}\sum_0^{j'} T_j\right\} &\leqq \left\{\prod_0^{j'}\left(1+\frac{2\lambda_N^2}{B_N^2}T_j\right)\right\}^{1/2}\exp\left\{\frac{2\lambda_N^4}{B_N^4}\sum_0^{j'}T_j^2\right\} \\ &= \left\{\prod_0^{j'}\left(1+\frac{2\lambda_N^2}{B_N^2}T_j\right)\right\}^{1/2}\exp\left\{\frac{2\varepsilon_N\lambda_N^4}{B_N^4}\sum_0^{j'}T_j + o(1)\right\} , \quad \text{as} \quad N \to +\infty .\end{aligned}$$

This shows that,

$$\begin{aligned}&\int_0^1 \exp\left\{\frac{\lambda_N^2}{B_N^2}\left(1-\frac{2\varepsilon_N\lambda_N^2}{B_N^2}\right)\sum_0^{j'}T_j\right\}dt \\ &= \left\{\int_0^1 \prod_0^{j'}\left(1+\frac{2\lambda_N^2}{B_N^2}T_j\right)^{1/2}dt\right\}e^{o(1)} \\ &\leqq \left\{\int_0^1 \prod\nolimits_1\left(1+\frac{2\lambda_N^2 T_{2j}}{B_N^2}\right)dt\int_0^1 \prod\nolimits_2\left(1+\frac{2\lambda_N^2 T_{2j+1}}{B_N^2}\right)dt\right\}^{1/2}e^{o(1)} , \quad \text{as} \quad N \to +\infty ,\end{aligned}$$

where $\prod_1$ (or $\prod_2$) denotes the product over all j satisfying $0 \leqq 2j \leqq j'$ (or $0 \leqq 2j+1 \leqq j'$). From the definitions of $\{T_j\}$ and (2.3), the frequencies of $T_{2j}(t)$ are not less than $c2^{m(2j)}p^{-\alpha}(m(2j)+1)$ and

$$\left\{\text{frequencies of terms of } \prod_0^{j-1}\left(1+\frac{2\lambda_N^2}{B_N^2}T_{2k}\right)\right\} \leqq 2^{m(2j-1)+2} ,$$

therefore we have, by (2.4)

$$\int_0^1 \prod\nolimits_1\left(1+\frac{2\lambda_N^2}{B_N^2}T_{2j}\right)dt = 1 \text{ and } \int_0^1 \prod\nolimits_2\left(1+\frac{2\lambda_N^2}{B_N^2}T_{2j+1}\right)dt = 1 .$$

Hence, we have

$$\int_0^1 \exp\left\{\frac{\lambda_N^2}{B_N^2}\left(1-\frac{2\varepsilon_N\lambda_N^2}{B_N^2}\right)\sum_0^{j'}T_j\right\}dt = 1 + o(1) ,^{*)} \quad \text{as} \quad N \to +\infty .$$

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*) By Jenssen's inequality we have $\int_0^1 \exp\{\lambda_N^2 B_N^{-2}\sum_0^{j'}T_j\}dt \geqq 1$.

Since $\varepsilon_N \lambda_N^2 B_N^{-2} = o(1)$, as $N \to +\infty$, the above relation proves (i). Using the same method and (i) we can prove (ii).

3. Almost Multiplicatively Orthogonal Summands. Putting $\phi(k) = \sum_{m=1}^{k} (\log\log B_m + 1)$, we take a sequence $\{q(k)\}$ of integers satisfying

$$q(0) = 0 \text{ and } \| \Delta_{q(k)-1} \| = \min\{\| \Delta_m \|; \phi(2k-1) < m \leqq \phi(2k)\} .$$

Set

$$Q_k(t) = \sum_{q(k-1)}^{q(k)-2} \Delta_m(t) \quad \text{and} \quad D_k = \left\| \sum_1^k Q_m \right\|$$

then

$$\left\| \sum_1^N \Delta_{q(k)-1} \right\| = o(D_N) , \quad D_N \sim B_{q(N)-2} , \quad \text{as } N \to +\infty \tag{3.1}$$

and

$$\begin{aligned} \sum_{q(k-1)}^{q(k)-2} \| \Delta_m \|_\infty &= O(D_k (\log D_k)^{-8} \log\log B_{2k}) \\ &= O(D_k (\log D_k)^{-8} \log\log D_k) , \quad \text{as} \quad k \to +\infty , \end{aligned} \tag{3.2}$$

since $q(k) - 2 \geqq \phi(2k-1) > 2k - 1$ and $B_k / B_{k+1} \to 1$, as $k \to +\infty$.

LEMMA 4. *If $M < N$, then*

$$\left\| D_N^{-2} \sum_M^N (Q_k^2 - \| Q_k \|^2) \right\| = O((\log D_N)^{-7}) , \quad \textit{as} \quad N \to +\infty .$$

PROOF. Let us put

$$\Delta'_m(t) = \begin{cases} \Delta_m(t) , & \text{if } q(k-1) \leqq m \leqq q(k) - 2 , \quad k = 1, 2, \cdots , \\ 0 , & \text{if otherwise} , \end{cases} \tag{3.3}$$

and

$$T'_m(t) = \begin{cases} \sum_{j=q(k-1)}^{m-2} \Delta'_j , & \text{if} \quad q(k-1) + 2 \leqq m < q(k) - 2 , \quad k = 1, 2, \cdots , \\ 0 , & \text{if otherwise} . \end{cases} \tag{3.4}$$

Then we have

$$\sum_M^N (Q_k^2 - \| Q_k \|^2) = 2 \sum_{q(M-1)}^{q(N)-2} \Delta'_m \Delta'_{m-1} + 2 \sum_{q(M-1)}^{q(N)-2} \Delta'_m T'_m + \sum_{q(M-1)}^{q(N)-2} (\Delta'^2_m - \| \Delta'_m \|^2) .$$

By Lemma 2, it is sufficient to show that

$$\left\| D_N^{-2} \sum_{q(M-1)}^{q(N)-2} \Delta'_m T'_m \right\| = O((\log D_N)^{-7}) , \quad \text{as} \quad N \to +\infty .$$

Since $\int_0^1 \Delta'_m T'_m \Delta'_n T'_n dt = 0$ if $| m - n | \geqq 2$, we have, by (3.2)

$$\left\|\sum_{q(M-1)}^{q(N)-2} \Delta'_m T'_m\right\|^2 \leqq 2 \sum_{q(M-1)}^{q(N)-2} \int_0^1 \Delta'^2_m T'^2_m dt$$
$$= O(D_N^2(\log D_N)^{-16}(\log\log D_N)^2 D_N^2) = o(D_N^4(\log D_N)^{-14})) ,$$
$$\text{as} \quad N \to +\infty .$$

LEMMA 5. *If* $\lambda_N = o((\log D_N)^{3-(1/2\beta)})$, *as* $N \to +\infty$, *then we have*

$$\int_0^1 \exp\left\{\frac{\lambda_N^2}{D_N^2} \sum_M^N (Q_m^2 - \| Q_m \|^2)\right\} dt = 1 + o(1) , \quad as \quad N \to +\infty .$$

PROOF. We use the same notation as in the proof of Lemma 4. Therefore, by Lemma 3 and Jenssen's inequality it is sufficient to show that

$$\text{(3.5)} \qquad \int_0^1 \exp\left\{\frac{\lambda_N^2}{D_N^2} \sum_{q(M-1)}^{q(N)-2} \Delta'_m T'_m\right\} dt = 1 + o(1) , \quad \text{as} \quad N \to +\infty .$$

By (3.2) and (3.4), we have

$$\exp\left\{\frac{\lambda_N^2}{D_N^2} \sum \Delta'_m T'_m\right\} \leqq \left\{\prod \left(1 + \frac{2\lambda_N^2}{D_N^2} \Delta'_m T'_m\right)\right\}^{1/2} \exp\left\{\frac{2\lambda_N^4}{D_N^4} \sum \Delta'^2_m T'^2_m\right\}$$
$$= \left\{\prod \left(1 + \frac{2\lambda_N^2 \Delta'_m T'_m}{D_N^2}\right)\right\}^{1/2} \exp\{o(D_N^{-2} \sum \Delta'^2_m)\} , \quad \text{as} \quad N \to +\infty .$$

Hence, for the proof of (3.5) it is enough to show that

$$\text{(3.6)} \qquad \int_0^1 \prod\nolimits_1 \left(1 + \frac{2\lambda_N^2 \Delta'_{2m} T'_{2m}}{D_N^2}\right) dt \int_0^1 \prod\nolimits_2 \left(1 + \frac{2\lambda_N^2 \Delta'_{2m+1} T'_{2m+1}}{D_N^2}\right) dt = 1 .$$

Further, both of the sequences $\{\Delta'_{2m} T'_{2m}\}$ and $\{\Delta'_{2m+1} T'_{2m+1}\}$ are multiplicatively orthogonal, we can prove (3.6).

We take a constant $\theta > 1$ which will be determined more precisely in § 5 and put

$$N(0) = 1 , \quad N(k) = \min\{m; D_m^2 > \theta^{2k}\} , \quad X_k(t) = \sum_{N(k)+1}^{N(k+1)} Q_m(t) ,$$
$$V_k = \| X_k \| \text{ and } \eta_k = \max(\| Q_m \|_\infty V_k^{-1}, N(k) < m \leqq N(k+1)) .$$

Then by (3.1) and (3.2), we have

$$\text{(3.7)} \qquad \begin{cases} D_{N(k)}^2 \sim \theta^{2k} , \quad V_k^2 \sim \theta^{2k+2} - \theta^{2k} \\ \eta_k = O(k^{-8} \log k) , \qquad \text{as} \quad k \to +\infty . \end{cases}$$

LEMMA 6. *We have*

(i) $\displaystyle\overline{\lim_k}\, (2D_{N(k)}^2 \log\log D_{N(k)})^{-1/2} \sum_{m=1}^{N(k)} Q_m(t) \leqq 1$, *a.e.* ,

(ii) $\displaystyle\overline{\lim_k}\, (2D_{N(k)}^2 \log\log D_{N(k)})^{-1/2} \sum_{m=1}^{N(k)} \Delta_{q(m)-1}(t) = 0$, *a.e.* .

PROOF. Cf. [4] p. 326 (i) and (ii).

Hence for the proof of our theorem it is sufficient to show that

(3.8) $$(2\theta^{2k+2}\log k)^{-1/2}\sum_{1}^{k} X_m(t) \geqq 1\ , \quad \text{a.e.}\ .$$

4. Characteristic Functions. In the following let $f_{k,l}(u, v)$ denote the characteristic function of the random vector $(X_k V_k^{-1}, X_l V_l^{-1})$, that is,

$$f_{k,l}(u, v) = \int_0^1 \exp\{iuX_k(t)V_k^{-1} + ivX_l(t)V_l^{-1}\}dt\ .$$

LEMMA 7. *Let ε be a positive number satisfying*

(4.1) $$\varepsilon < 1/7 \quad \textit{and} \quad 2\varepsilon + \frac{1}{2\beta} < 1\ .$$

Then for any (k, l) and (u, v) such that

(4.2) $$k^{1/(1+\varepsilon)} \leqq l \leqq k \quad \textit{and} \quad \max(|u|, |v|) \leqq k^2\ ,$$

if $k > k_0$, then we have

$$\begin{aligned}|f_{k,l}(u, v) - \exp\{-(u^2 + v^2)/2\}|\\ \leqq C(k^{-8}|u|^3\log k + l^{-8}|v|^3\log k + k^{-7}|u|^2 + l^{-7}|v|^2)\ ,\end{aligned}$$

where C is a positive constant.

PROOF. We have

$$\begin{aligned}&\left|\exp\left\{\frac{iuX_k}{V_k} + \frac{ivX_l}{V_l}\right\} - P_k(u, t)P_l(v, t)\exp\left\{\frac{-u^2P_k'(t) - v^2P_l'(t)}{2}\right\}\right|\\ &\leqq \left|\exp\left(\frac{iuX_k}{V_k}\right) - P_k\exp\left(\frac{-u^2P_k'}{2}\right)\right|\\ &\quad + \left|\exp\frac{ivX_l}{V_l}\right) - P_l\exp\left(-\frac{v^2P_l'}{2}\right)\right|\ ,\end{aligned}$$

where $P_k(u, t) = \prod_{m=N(k)+1}^{N(k+1)}\{1+iuQ_m(t)/V_k\}$ and $P_k'(t) = V_k^{-2}\sum_{N(k)+1}^{N(k+1)} Q_m^2(t)$. Since (3.7) and (4.2) imply that $u\eta_k = o(1)$ and $v\eta_l = o(1)$, as $k \to +\infty$, we have, for $k > k_0$,

$$\exp(iuX_k V_k^{-1}) = P_k(u, t)\exp\{-u^2 2^{-1}P_k'(t) + R_k(u, t)\}$$

where

$$|R_k(u, t)| \leqq |u|^3\sum_{N(k)+1}^{N(k+1)}|Q_m V_k^{-1}|^3 \leqq \eta_k|u|^3 P_k'(t)\ .$$

By Lemma 4 and 5, we have

$$\begin{aligned}
&\int_0^1 |\exp(iuX_k V_k^{-1}) - P_k(u, t)\exp\{-u^2 2^{-1} P_k'(t)\}|\,dt \\
&\quad \leqq \int_0^1 |\exp\{R_k(u, t)\} - 1|\,dt \leqq \int_0^1 |R_k(u, t)| \exp\{|R_k(u, t)|\}dt \\
&\quad \leqq \eta_k |u|^3 \int_0^1 P_k'(t)\exp\{\eta_k |u|^3 P_k'(t)\}dt \\
&\quad < \eta_k |u|^3 \|P_k'\| \exp\{\eta_k |u|^3\}(1 + o(1)) \\
&\quad < C\eta_k |u|^3 , \qquad \text{for some constant } C > 0 ,
\end{aligned}$$

and the same inequality holds for l.

On the other hand since $\{Q_m(t)\}$ is multiplicatively orthogonal, it is seen that

$$\int_0^1 P_k(u, t)P_l(v, t)dt = 1 ,$$

and we have, by Lemma 4 and 5,

$$\begin{aligned}
&\left| \int_0^1 P_k(u, t)P_l(v, t)\exp\left\{\frac{-u^2 P_k'(t) - v^2 P_l'(t)}{2}\right\}dt - e^{-(u^2+v^2)/2} \right| \\
&\quad = \left| \int_0^1 P_k(u, t)P_l(v, t)\left[\exp\left\{\frac{-u^2 P_k'(t) - v^2 P_l'(t)}{2}\right\} - e^{-(u^2+v^2)/2}\right]dt \right| \\
&\quad \leqq \int_0^1 |1 - \exp\{2^{-1}u^2(P_k' - 1) + 2^{-1}v^2(P_l' - 1)\}|\,dt \\
&\quad \leqq \int_0^1 |u^2(P_k' - 1) + v^2(P_l' - 1)| \\
&\qquad \times [\exp\{2^{-1}u^2(P_k' - 1) + 2^{-1}v^2(P_l' - 1)\} + 1]dt \\
&\quad \leqq \{u^2 \|P_k' - 1\| + v^2 \|P_l' - 1\|\} \\
&\qquad \times \{\|\exp\{2^{-1}u^2(P_k' - 1) + 2^{-1}v^2(P_l' - 1)\}\| + 1\} \\
&\quad \leqq C(u^2 k^{-7} + v^2 l^{-7}) \\
&\qquad \times \{\|\exp(2^{-1}u^2(P_k' - 1)\|_4 \|\exp(2^{-1}v^2(P_l' - 1)\|_4 + 1\} \\
&\quad \leqq C(u^2 k^{-7} + v^2 l^{-7}) , \qquad \text{for some } C > 0 .
\end{aligned}$$

LEMMA 8. [3] *Let $F(x, y)$ and $G(x, y)$ be two dimensional distribution functions. Denote the corresponding characteristic functions by $f(u, v)$ and $g(u, v)$. Suppose that $G(x, y)$ has a bounded density function. Further set*

$$\hat{f}(u, v) = f(u, v) - f(u, 0)f(0, v)$$

and

$$\hat{g}(u, v) = g(u, v) - g(u, 0)g(0, v) .$$

Then

$$\sup_{x,y} |F(x, y) - G(x, y)| \leqq C\Big(\int_{-T}^{T}\int_{-T}^{T}\Big|\frac{\hat{f}(u, v) - \hat{g}(u, v)}{uv}\Big| dudv + \int_{-T}^{T}\Big|\frac{f(u, 0) - g(u, 0)}{u}\Big| du + \int_{-T}^{T}\Big|\frac{f(0, v) - g(0, v)}{v}\Big| dv + \frac{1}{T}\Big)$$

for any $T > 0$, *where* C *is a positive constant.*

Making use of Lemmas 7 and 8 we can prove the

LEMMA 9. *Let* $F_{k,l}(x, y)$ *denote the distribution function of the vector* $(X_k(t)V_k^{-1}, X_l(t)V_l^{-1})$. *Then we have*

$$\sup_{x,y}\Big|F(x, y) - (2\pi)^{-1}\int_{-\infty}^{x}\int_{-\infty}^{y} \exp\{-(z^2 + z'^2)/2\} dzdz'\Big| \leqq C(\log k)^2 k^6 l^{-8}$$

for $k^{1/(1+\varepsilon)} \leqq l \leqq k$, *where* ε *satisfies* (4.1) *and* C *is a constant.*

PROOF. Set $f(u, v) = f_{k,l}(u, v)$ and $g(u, v) = e^{-(u^2+v^2)/2}$. Then $\hat{g}(u, v) = 0$ and by Lemma 4,

$$\hat{f}(u, v) = \int_0^1\Big[\exp\Big\{\frac{iuX_k(t)}{V_k}\Big\} - f(u, 0)\Big]\Big[\exp\Big\{\frac{ivX_l(t)}{V_l}\Big\} - f(0, v)\Big]dt$$
$$\leqq |uv|\, V_k^{-1}V_l^{-1}\int_0^1\Big[\int_0^1 |X_k(t) - X_k(t')|\, dt'\Big]\Big[\int_0^1 |X_l(t) - X_l(t')|\, dt'\Big]dt$$
$$\leqq 4|uv| .$$

In Lemma 8 we put $T = k^2$. Then we have

$$\int_{-T}^{T}\int_{-T}^{T}\Big|\frac{\hat{f}(u, v) - \hat{g}(u, v)}{uv}\Big| dudv = \iint_{A(k)}\Big|\frac{\hat{f}(u, v)}{uv}\Big| dudv + \iint_{B(k)}\Big|\frac{\hat{f}(u, v)}{uv}\Big| dudv ,$$

where $A(k) = \{(u, v); k^{-4} < |u| \leqq k^2, k^{-4} < |v| \leqq k^2\}$ and $B(k) = \{(u, v); |u| \leqq k^2, |v| \leqq k^2\} - A(k)$. By Lemma 7, we have

$$\begin{cases} \displaystyle\iint_{A(k)}\Big|\frac{\hat{f}(u, v)}{uv}\Big| dudv \leqq Ck^6(\log k)^2 l^{-8} , \\ \displaystyle\iint_{B(k)}\Big|\frac{\hat{f}(u, v)}{uv}\Big| dudv \leqq 8k^{-2} . \end{cases}$$

In the same way we can obtain

$$\int_{-T}^{T} \left| \frac{f(u, 0) - g(u, 0)}{u} \right| du \leqq Ck^{-2} \log k$$

and

$$\int_{-T}^{T} \left| \frac{f(0, v) - g(0, v)}{v} \right| dv < Ck^{6} l^{-8} \log k \, .$$

Thus, we can complete the proof.

5. **Proof of** (3.8). The following lemma is an extension of the Borel-Cantelli lemma.

LEMMA 10. [1] *If $\{E_k\}$ is a sequence of arbitrary events, fulfilling the conditions*

$$\sum P(E_k) = +\infty \text{ and } \lim_{n} \sum_{k=1}^{n} \sum_{l=1}^{n} P(E_k E_l) \Big/ \left\{ \sum_{1}^{n} P(E_k) \right\}^2 = 1 \, ,$$

then we have $P\{E_k \text{ i.o.}\} = 1$.

LEMMA 11. *Let ε be a positive number satisfying the condition* (4.1). *Then we have*

$$|\{t; X_k(t) \geqq \{(2 - \varepsilon) \log k\}^{1/2} V_k \text{ i.o.}\}| = 1 \, .$$

PROOF. Let us put $C_r = [t; X_r(t) \geqq \{(2-\varepsilon)\log r\}^{1/2} V_r]$ and

(5.1) $$\gamma = \varepsilon/7 \, , \quad u_r = \sqrt{(2-\varepsilon')\log r} \, , \quad y_r = u_r/2 \, ,$$

where ε' is a positive number satisfying

(5.2) $$\varepsilon < \varepsilon' < 2\varepsilon\{1 + (1+\gamma)^{-1}\}^{-1} \, .$$

Further, let $\sum_1$, $\sum_2$ and $\sum_3$ denote the summation over the (k, l)-sets $\{1 \leqq k \leqq n, \ k^{1/(1+\gamma)} \leqq l < k\}$, $\{1 \leqq k \leqq n, \ 1 \leqq l \leqq n^{\varepsilon/4}\}$ and $\{n^{\varepsilon/4} \leqq k \leqq n, \ n^{\varepsilon/4} < l < k^{1/(1+\gamma)}\}$ respectively. On the other hand by Lemma 9, we have

(5.3) $$\begin{aligned} P(C_k) &= (2\pi)^{-1/2} \int_{\sqrt{(2-\varepsilon)\log k}} e^{-z^2/2} dz + O(k^{-2}(\log k)^2) \\ &\sim (2\pi)^{-1/2} k^{-1+\varepsilon/2} ((2-\varepsilon)\log k)^{-1/2} \, ,^{*)} \quad \text{as } k \to +\infty \, . \end{aligned}$$

Therefore, we have

(5.4) $$\sum\nolimits_2 P(C_k C_l) \leqq n^{\varepsilon/4} \sum_{k=1}^{n} P(C_k) = o\left\{ \left(\sum_{k=1}^{n} P(C_k) \right)^2 \right\} \, , \quad \text{as } n \to +\infty \, .$$

By Lemma 9 we have, for $k^{1/(1+\varepsilon)} < k^{1/(1+\gamma)} \leqq l < k$,

$$|P(C_k C_l) - P(C_k)P(C_l)| = o(P(C_k)P(C_l)) \, , \quad \text{as } k \to +\infty$$

and by (5.3), it is seen that

*) P denotes the Lebesgue measure on [0, 1].

(5.5) $$\left\{\sum_{k=1}^{n} P(C_k)\right\}^2 \sim 2\sum_1 P(C_k)P(C_l) \sim 2\sum_1 P(C_kC_l)\,, \quad \text{as } n \to +\infty\,.$$

Using the inequality $e^x \leqq (1+x)\exp\{2^{-1}(x^2+|x|^3)\}$ for $|x|<1/3$ and the multiplicative orthogonality of $\{Q_m(t)\}$, we have

$$\int_0^1 \exp\left\{\frac{u_kX_k}{V_k} - \frac{u_k^2P_k'(t)}{2}(1+u_k\eta_k) + \frac{u_lX_l}{V_l} - \frac{u_l^2P_l'(t)}{2}(1+u_l\eta_l)\right\}dt$$
$$\leqq \int_0^1 \prod_{m=N(k)+1}^{N(k+1)} (1+u_kQ_mV_k^{-1}) \prod_{s=N(l)+1}^{N(l+1)} (1+u_lQ_sV_l^{-1})dt = 1\,.$$

By Tschebyschev's inequality, it is seen that

$$P\{X_rV_r^{-1} - 2^{-1}P_r'(t)(1+u_r\eta_r)u_r \geqq y_r,\ r=k,\,l\}$$
$$\leqq \exp(-y_ku_k - y_lu_l)\,.$$

Putting $\lambda_N = r^2$ in Lemma 5, it is seen that

$$P\{P_r'(t) > 1 + r^{-1}\} \leqq Ce^{-r}\,, \quad \text{for some constant } C>0\,.$$

Since (5.1), (5.2) and (3.7) imply that $C_r \subset \{X_rV_r^{-1} > y_r + 2^{-1}(1+r^{-1})(1+u_r\eta_r)u_r\}$ for $r>r_0$, we have, for $n>n_0$ and $k>l\geqq n^{\varepsilon/4}$

$$P(C_kC_l) \leqq P[C_kC_l \text{ and } \bigcup_{r=k,l}\{P_r'(t)>1+r^{-1}\}]$$
$$+ P\{X_rV_r^{-1} > y_r + 2^{-1}P_r'(t)(1+u_r\eta_r)u_r,\ P_r'(t)\leqq 1+r^{-1},\ r=k,\,l\}$$
$$\leqq 2C\exp(-n^{\varepsilon/4}) + \exp\{-(1-\varepsilon'/2)\log k - (1-\varepsilon'/2)\log l\}$$
$$\leqq C'k^{-1+\varepsilon'/2}l^{-1+\varepsilon'/2}\,, \quad \text{for some } C'>0\,.$$

Therefore, by (5.2) and (5.3)

(5.6) $$\sum\nolimits_2 P(C_kC_l) = O(n^{\varepsilon'\{1+(1+\gamma)^{-1}\}/2})$$
$$= o\left\{\left(\sum_{k=1}^{n} P(C_k)\right)^2\right\}\,, \quad \text{as } n\to+\infty\,.$$

By (5.4), (5.5) and (5.6) we can prove the lemma.

Since ε in Lemma 11 is small as we please, we have

(5.1) $$\overline{\lim_k}\,(2V_k^2\log k)^{-1/2}X_k(t) \geqq 1 \quad \text{a.e.}\,.$$

Let δ, $0<\delta<1/2$, be an arbitrary number. Then by (3.7) we can take the constant θ which is used to define $\{N(k)\}$ in § 4 so large that

$$D_{N(k)}^2 \leqq \delta^2 D_{N(k+1)}^2\,,$$

then

$$V_k^2 = D_{N(k+1)}^2 - D_{N(k)}^2 \geqq (1-\delta^2)D_{N(k+1)}^2 \geqq (1-\delta)^2\theta^{2(k+1)}\,.$$

By Lemma 5 and (5.1), we have

$$\begin{aligned}
&\overline{\lim_k}\,(2D^2_{N(k+1)}\log\log D_{N(k+1)})^{-1/2}\sum_1^k X_m(t)\\
&\geqq \overline{\lim_k}\,(2\theta^{2(k+1)}\log k)^{-1/2}\sum_1^k X_m(t)\\
&\geqq \overline{\lim_k}\,(2\theta^{2(k+1)}\log k)^{-1/2}X_k(t) - \overline{\lim_k}\,(2\theta^{2(k+1)}\log k)^{-1/2}\sum_1^{k-1} X_m(t)\\
&\geqq (1-\delta)-\delta = 1-2\delta\,. \quad \text{a.e.}\,.
\end{aligned}$$

Since δ is arbitrary we can prove (3.8).

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