# ON THE LAW OF THE ITERATED LOGARITHM FOR MAXIMA AND MINIMA 

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## 1. Introduction and summary

Let $w(t), 0 \leqq t \leqq \infty$, denote a standard Wiener process. The general law of the iterated logarithm (see [6], p. 21) says that if $g$ is a positive function such that $g(t) / \sqrt{t}$ is ultimately nondecreasing, then

$$
\begin{equation*}
P\{w(t) \geqq g(t) \text { i.o. } t \uparrow \infty\} \tag{1.1}
\end{equation*}
$$

equals 0 or 1 , according as

$$
\begin{equation*}
\int_{1}^{\infty} \frac{g(t)}{t^{3 / 2}} \exp \left\{-\frac{1}{2} \frac{g^{2}(t)}{t}\right\} d t<\infty \text { or }=\infty \tag{1.2}
\end{equation*}
$$

(The notation i.o. $t \uparrow \infty(t \downarrow 0)$ means for arbitrarily large (small) $t$.) In particular, for $k \geqq 3$ and

$$
\begin{equation*}
g(t)=\left[2 t\left(\log _{2} t+\frac{3}{2} \log _{3} t+\sum_{i=4}^{k} \log _{i} t+(1+\delta) \log _{k+1} t\right)\right]^{1 / 2} \tag{1.3}
\end{equation*}
$$

the probability (1.1) is 0 or 1 according as $\delta>0$ or $\delta \leqq 0$. (We write $\log _{2}=\log \log , e_{2}=e^{e}$, and so on.)

For applications in statistics it is of interest to compute as accurately as possible

$$
\begin{equation*}
P\{w(t) \geqq g(t) \text { for some } t \geqq \tau\} \tag{1.4}
\end{equation*}
$$

for functions $g$ for which this probability is $<1$; that is, functions for which (1.2) converges (see [3], [10], [12]). In [11], we gave a method for computing (1.4) exactly for a certain class of functions $g$. A sketch of this method follows. Since $\exp \left\{\theta w(t)-\frac{1}{2} \theta^{2} t\right\}, 0 \leqq t<\infty$, is a martingale for each $\theta$, Fubini's theorem shows that $\int_{0}^{\infty} \exp \left\{\theta w(t)-\frac{1}{2} \theta^{2} t\right\} d F(\theta), 0 \leqq t<\infty$, is also a martingale for any $\sigma$-finite measure $F$ on $(0, \infty)$. Let

$$
\begin{equation*}
f(x, t)=\int_{0}^{\infty} \exp \left\{\theta x-\frac{\theta^{2} t}{2}\right\} d F(\theta) \tag{1.5}
\end{equation*}
$$

and for each $t \geqq 0$ and $\varepsilon>0$ let $A(t, \varepsilon)$ be the solution of

$$
\begin{equation*}
f(x, t)=\varepsilon \tag{1.6}
\end{equation*}
$$

Then

$$
\begin{align*}
P\{w(t) & \geqq A(t, \varepsilon) \text { for some } t \geqq \tau\}  \tag{1.7}\\
& =P\{f(w(t), t) \geqq \varepsilon \text { for some } t \geqq \tau\}
\end{align*}
$$

and we use an elementary martingale equality to evaluate the right side of (1.7). The relation of $w(t)$ to the sequence of sums of i.i.d. random variables with mean 0 and variance 1 then permits the asymptotic evaluation of boundary crossing probabilities for partial sums.

In view of (1.3) a choice of $F$ of particular interest is, for $\delta>0$,

$$
d F(\theta)= \begin{cases}{\left[\theta\left(\log \frac{1}{\theta}\right) \cdots\left(\log _{k-1} \frac{1}{\theta}\right)\left(\log _{k} \frac{1}{\theta}\right)^{1+\delta}\right]^{-1} d \theta} & \text { for } \theta \leqq \frac{1}{e_{k}}  \tag{1.8}\\ 0 & \text { otherwise }\end{cases}
$$

for which it is shown in [11] that for any $\varepsilon>1 / \delta$,

$$
\begin{align*}
A(t, \varepsilon)=\left[2 t \left(\log _{2} t+\frac{3}{2} \log _{3} t\right.\right. & +\sum_{i=4}^{k} \log _{i} t  \tag{1.9}\\
& \left.\left.+(1+\delta) \log _{k+1} t+\log \frac{\varepsilon}{2 \sqrt{\pi}}+o(1)\right)\right]^{1 / 2}
\end{align*}
$$

as $\mathrm{t} \rightarrow \infty$ and

$$
\begin{equation*}
P\{w(t) \geqq A(t, \varepsilon) \text { for some } t \geqq 0\}=\frac{1}{\delta \varepsilon} \tag{1.10}
\end{equation*}
$$

The purpose of this paper is to obtain analogous results for maxima and minima of sequences $x_{1}, x_{2}, \cdots$ of i.i.d. random variables. We begin in Section 2 by establishing an analogue of the criterion (1.2) for a law of the iterated logarithm for sample minima. In Section 3, we give an application of this result to a conjecture of Darling and Erdös [2]. In Sections 4 and 5, we introduce a continuous time process $v_{t}, 0<t<\infty$, related to min ( $x_{1}, \cdots, x_{n}$ ) in much the same way that $w(t)$ is related to $x_{1}+\cdots+x_{n}$, and apply the methods of [11] to the study of this process. In spite of the dissimilarity in the behavior of $v_{t}$ and $w(t)$, the measure $F$ defined by (1.8) plays the same role for $v_{t}$ as for $w(t)$.

## 2. The law of the iterated logarithm for minima of uniform variables

Theorem 1. Let $u_{1}, u_{2}, \cdots$ be independent and uniform on (0,1), and let $V_{n}=\min \left(u_{1}, \cdots, u_{n}\right)$. Let ( $\left.c_{n}\right)$ be any sequence of positive numbers. Then:
(i) if $c_{n} / n \downarrow$ for all sufficiently large $n$, then $P\left\{n V_{n} \leqq c_{n} i .0.\right\}=0$ or 1 according as

$$
\begin{equation*}
\sum_{1}^{\infty} \frac{c_{n}}{n} \tag{2.1}
\end{equation*}
$$

converges or diverges;
(ii) if $c_{n} / n \downarrow$ and $c_{n} \uparrow$ for all sufficiently large $n$, then $P\left\{n V_{n} \geqq c_{n} i . o\right.$. $\}=0$ or 1 according as

$$
\begin{equation*}
\sum_{1}^{\infty} \frac{c_{n}}{n} e^{-c_{n}} \tag{2.2}
\end{equation*}
$$

converges or diverges.
Corollary 1. For $k \geqq 3$,

$$
\begin{equation*}
P\left\{n V_{n} \geqq \log _{2} n+2 \log _{3} n+\sum_{i=4}^{k} \log _{i} n+(1+\delta) \log _{k+1} n \text { i.o. }\right\} \tag{2.3}
\end{equation*}
$$

is equal to 0 or 1 according as $\delta>0$ or $\delta \leqq 0$.
Remark 2.1. The proof of (i) is an immediate consequence of the BorelCantelli lemma and the fact that if $c_{n} / n$ is ultimately decreasing, then $V_{n} \leqq c_{n} / n$ i.o. if and only if $u_{n} \leqq c_{n} / n$ i.o. The proof of (ii) is much harder and will be given below.

Remark 2.2. If $M_{n}=\max \left(u_{1}, \cdots, u_{n}\right)$, then Theorem 1 holds with $V_{n}$ replaced by l- $M_{n}$.

Remark 2.3. Under different regularity conditions on the sequence ( $c_{n}$ ), Ville [13] has shown that if (2.2) converges, then $P\left\{n V_{n} \geqq c_{n}\right.$ i.o. $\}=0$. His approach is similar to the one we take in Section 4. Pickands [8] has also obtained some results in the direction of Theorem 1.

Remark 2.4. The condition in (ii) that $c_{n}$ be ultimately increasing is bothersome in some applications (see Remark 2.5 below), but it cannot be dropped completely. For example, if $c_{n}=1 / n$, then both (2.1) and (2.2) converge. Hence by (i), $P\left\{n V_{n} \geqq c_{n}\right.$ for all sufficiently large $\left.n\right\}=1$, which is incompatible with the conclusion of (ii) applied to the same sequence $c_{n}$.

Remark 2.5. Let $x_{1}, x_{2}, \cdots$ be independent random variables with a common continuous distribution function $F$. Since $u_{n}=F\left(x_{n}\right)$ is uniform on $(0,1)$ and

$$
\begin{align*}
F\left[\min \left(x_{1}, \cdots, x_{n}\right)\right] & =\min \left[F\left(x_{1}\right), \cdots, F\left(x_{n}\right)\right]  \tag{2.4}\\
& =\min \left(u_{1}, \cdots, u_{n}\right)=V_{n}
\end{align*}
$$

Theorem 1 implies a law of the iterated logarithm for $\min \left(x_{1}, \cdots, x_{n}\right)$. In particular, (ii) implies that if $\left(a_{n}\right)$ is any sequence of numbers such that $a_{n}$ is ultimately decreasing and $n F\left(a_{n}\right)$ is ultimately increasing, then $P\left\{\min \left(x_{1}, \cdots\right.\right.$,
$\left.x_{m}\right) \geqq a_{n}$ i.o. $\}=0$ or 1 according as

$$
\begin{equation*}
\sum_{1}^{\infty} F\left(a_{n}\right) \exp \left\{-n F\left(a_{n}\right)\right\}<\infty \text { or }=\infty \tag{2.5}
\end{equation*}
$$

The condition that $n F\left(a_{n}\right)$ be ultimately increasing may be difficult to verify for a given $F$, and hence, it is worth observing (as will become apparent in the proof below) that the condition in (ii) that $c_{n}\left(=n F\left(a_{n}\right)\right)$ be ultimately increasing may be replaced by the growth condition

$$
\begin{equation*}
\liminf _{n \rightarrow \infty} \frac{c_{n}}{\log _{2} n} \geqq 1 \tag{2.6}
\end{equation*}
$$

Moreover, it follows a fortiori that if $c_{n} \leqq(\geqq) c_{n}^{\prime}$ and $P\left\{n V_{n} \geqq c_{n}^{\prime}\right.$ i.o. $\}=1(0)$, then $P\left\{n V_{n} \geqq c_{n}\right.$ i.o. $\}=1(0)$. Hence, (ii) may be applied indirectly to some sequences ( $c_{n}$ ) which satisfy neither the monotonicity conditions of (ii) nor the growth condition (2.6).

Remark 2.6. Let $x_{1}, x_{2}, \cdots$ be independent $N(0,1)$ random variables with distribution function $\Phi(x)=\int_{-\infty}^{x} \varphi(y) d y$, where $\varphi(y)=(2 \pi)^{-1 / 2} \exp \left\{-\frac{1}{2} y^{2}\right\}$. For $k \geqq 3$ and $\delta$ arbitrary, let

$$
\begin{align*}
a_{n}=-\left[2 \log \frac{n}{2 \sqrt{\pi}}-\log _{2} n-\right. & 2 \log \left(\log _{2} n+2 \log _{3} n\right.  \tag{2.7}\\
& \left.\left.+\sum_{i=4}^{k} \log _{i} n+(1+\delta) \log _{k+1} n\right)\right]^{1 / 2}
\end{align*}
$$

From the fact that

$$
\begin{equation*}
\Phi(x)=\frac{1}{|x|} \varphi(x)\left[1+O\left(\frac{1}{x^{2}}\right)\right], \quad \text { as } x \rightarrow-\infty \tag{2.8}
\end{equation*}
$$

it can be shown that $c_{n}=n \Phi\left(a_{n}\right)$ satisfies (2.6), and hence, by (ii) and the preceding remark, that $P\left\{\min \left(x_{1}, \cdots, x_{n}\right) \geqq a_{n}\right.$ i.o. $\}=0$ or 1 according as $\delta>0$ or $\delta \leqq 0$. Alternatively, it is possible using (2.8) to replace the criterion (2.5) by one involving the normal density $\varphi$; the argument of Lemma 8 below (together with (2.5), (2.8) and Remark 2.2) shows that if ( $a_{n}$ ) is any ultimately increasing sequence of positive numbers such that $n a_{n}^{-1} \varphi\left(a_{n}\right)$ is ultimately increasing, then $P\left\{\max \left(x_{1}, \cdots, x_{n}\right) \leqq a_{n}\right.$ i.o. $\}=0$ or 1 according as

$$
\begin{equation*}
\sum_{1}^{\infty} \frac{\varphi\left(a_{n}\right)}{a_{n}} \exp \left\{-n \frac{\varphi\left(a_{n}\right)}{a_{n}}\right\} \tag{2.9}
\end{equation*}
$$

converges or diverges.
The truth of (ii) follows from Theorem 2 and from Lemma 8 below which shows that the conditions of (ii) imply those of Theorem 2.

Theorem 2. Let $\alpha>0$ and $n_{k}=\exp \{\alpha k / \log k\}, k=2,3, \cdots$, and assume that $c_{n} / n$ is ultimately decreasing.
(i) If

$$
\begin{equation*}
\sum_{k} \exp \left\{-c_{\left[n_{k}\right]}\right\} \tag{2.10}
\end{equation*}
$$

converges for some $\alpha$, then $P\left\{n V_{n} \geqq c_{n}\right.$ i.o. $\}=0$.
(ii) If (2.6) holds and (2.10) diverges for some $\alpha$, then $P\left\{n V_{n} \geqq c_{n}\right.$ i.o. $\}=1$.

As usual, $[x]$ denotes the largest integer $\leqq x$. To avoid burdensome detail in the proof, we have ignored the difference between $n_{k}$ and [ $n_{k}$ ].

Proof. For (i), suppose that (2.10) converges for some $\alpha$. By replacing $c_{n}$ by $\min \left(c_{n}, 2 \log _{2} n\right)$, we may assume, without loss of generality, that

$$
\begin{equation*}
c_{n} \leqq 2 \log _{2} n \tag{2.11}
\end{equation*}
$$

By the Borel-Cantelli lemma, it suffices to show that

$$
\begin{equation*}
\sum_{k} P\left\{n V_{n} \geqq c_{n} \text { for some } n_{k}<n \leqq n_{k+1}\right\}<\infty \tag{2.12}
\end{equation*}
$$

and hence, by the monotonicity of $V_{n}$ and the ultimate monotonicity of $c_{n} / n$, to show that

$$
\begin{equation*}
\sum_{k} P\left\{V_{n_{k}} \geqq \frac{c_{n_{k+1}}}{n_{k+1}}\right\}<\infty \tag{2.13}
\end{equation*}
$$

But
(2.14) $\log P\left\{V_{n_{k}} \geqq \frac{c_{n_{k}+1}}{n_{k+1}}\right\}=\log \left(1-\frac{c_{n_{k+1}}}{n_{k+1}}\right)^{n_{k}} \leqq-\frac{n_{k}}{n_{k+1}} c_{n_{k+1}}$

$$
\begin{aligned}
& \leqq-c_{n_{k+1}} \exp \left\{\frac{\alpha k}{\log (k+1)}-\frac{\alpha(k+1)}{\log (k+1)}\right\} \\
& =-c_{n_{k+1}} \exp \left\{\frac{-\alpha}{\log (k+1)}\right\} \\
& \leqq-c_{n_{k+1}}\left(1-\frac{\alpha}{\log (k+1)}\right) \\
& \leqq-c_{n_{k+1}}+2 \alpha+o(1)
\end{aligned}
$$

where the last inequality follows from (2.11). Inequality 2.13 now follows immediately from the convergence of (2.10).

For (ii), assume that (2.6) holds and that the series (2.10) diverges for some $\alpha$. Let $c_{n}^{\prime}=\min \left(c_{n}, 2 \log _{2} n\right)$. Then $\Sigma_{k} \exp \left\{-c_{n_{k}}^{\prime}\right\} \geqq \Sigma_{k} \exp \left\{-c_{n_{k}}\right\}=\infty$, and since by the first part of the theorem

$$
\begin{equation*}
P\left\{n V_{n} \geqq 2 \log _{2} n \text { i.o. }\right\}=0, \tag{2.15}
\end{equation*}
$$

it follows that with no loss of generality we may again assume that (2.11) holds.

Let $A_{k}=\left\{n_{k} V_{n_{k}} \geqq c_{n_{k}}\right\}$. By Kolmogorov's 0-1 law, $P\left(\cap_{m=1}^{\infty} \cup_{k=m}^{\infty} A_{k}\right)=0$ or 1, and hence, to show that infinitely many of the events $A_{k}$ occur with probability l, it suffices to show that for all $k_{0}$

$$
\begin{equation*}
P\left(\bigcup_{k=k_{0}}^{\infty} A_{k}\right) \geqq \frac{1}{8} . \tag{2.16}
\end{equation*}
$$

Let $k_{1}>k_{0}$ and for $k_{0} \leqq k \leqq k_{1}$ let $B_{k}=A_{k} \cap A_{k+1}^{c} \cap \cdots \cap A_{k_{1}}^{c}$. Then

$$
\begin{equation*}
\bigcup_{k=k_{0}}^{\infty} A_{k} \supset \bigcup_{k=k_{0}}^{k_{1}} A_{k}=\bigcup_{k=k_{0}}^{k_{1}} B_{k} \tag{2.17}
\end{equation*}
$$

and the events $B_{k_{0}}, B_{k_{0}+1}, \cdots, B_{k_{1}}$ are disjoint. Hence, to prove (2.16), it suffices to show that there exists a $k_{1}>k_{0}, k_{1}$ depending on $k_{0}$, such that

$$
\begin{equation*}
\sum_{k=k_{0}}^{k_{1}} P\left(B_{k}\right) \geqq \frac{1}{8} \tag{2.18}
\end{equation*}
$$

But for each $k_{0} \leqq k \leqq k_{1}$,

$$
\begin{equation*}
B_{k}=A_{k} \cap\left\{V_{n_{r}}^{n_{k}}<\frac{c_{n_{r}}}{n_{r}} \text { for all } k<r \leqq k_{1}\right\} \tag{2.19}
\end{equation*}
$$

where we have set $V_{j}^{i}=\min _{i<n \leqq j} u_{n}$. Hence, by the independence of the $u_{n}$,

$$
\begin{align*}
P\left(B_{k}\right) & =P\left(A_{k}\right) P\left\{V_{n_{r}}^{n_{k}}<\frac{c_{n_{r}}}{n_{r}} \text { for all } k<r \leqq k_{1}\right\}  \tag{2.20}\\
& \geqq P\left(A_{k}\right)\left(1-\sum_{r=k+1}^{k_{1}} P\left\{V_{n_{r}}^{n_{k}} \geqq \frac{c_{n_{r}}}{n_{r}}\right\}\right) .
\end{align*}
$$

It is easy to see from (2.6) that as $k \rightarrow \infty, P\left(A_{k}\right) \rightarrow 0$, and from Lemma 1 below, $\Sigma_{k} P\left(A_{k}\right)=\infty$. Hence, there exists a number $K_{0}$ (to be further specified below) such that for any $k_{0} \geqq K_{0}$ and for some $k_{1}>k_{0}$,

$$
\begin{equation*}
\frac{1}{4} \leqq \sum_{k=k_{0}}^{k_{1}} P\left(A_{k}\right) \leqq \frac{5}{16} \tag{2.21}
\end{equation*}
$$

It follows from (2.20) and (2.21) that to prove (2.18) it suffices to show that

$$
\begin{equation*}
\sup _{k_{0} \leqq k \leqq k_{1}} \sum_{r=k+1}^{k_{1}} P\left\{V_{n_{r}}^{n_{k}} \geqq \frac{c_{n_{r}}}{n_{r}}\right\} \leqq \frac{1}{2} . \tag{2.22}
\end{equation*}
$$

It will be shown in Lemma 8 below that if (2.6) holds, then (2.10) converges or diverges simultaneously for all values of $\alpha$, and hence, it suffices to prove (2.22) for one value of $\alpha$. This will be done in Lemmas 2 through 7 below, completing the proof of the theorem.

Lemma 1. $\quad \Sigma_{k} P\left(A_{k}\right)=\infty$.

Proof. Since $\log (1-x) \geqq-x-x^{2}$ for all sufficiently small positive $x$, we have from (2.11) as $k \rightarrow \infty$,

$$
\begin{equation*}
\log P\left(A_{k}\right)=n_{k} \log \left(1-\frac{c_{n_{k}}}{n_{k}}\right) \geqq n_{k}\left(-\frac{c_{n_{k}}}{n_{k}}-\frac{c_{n_{k_{k}}}^{2}}{n_{k}^{2}}\right) \geqq-c_{n_{k}}+o(1) \tag{2.23}
\end{equation*}
$$

The lemma now follows from the divergence of (2.10).
In Lemmas 2 through 7 below, $\alpha>1$ and $0<\lambda<1$ will be fixed numbers satisfying

$$
\begin{equation*}
\lambda>\left(\frac{5}{6}\right)^{1 / 2} \tag{2.24}
\end{equation*}
$$

and

$$
\begin{equation*}
\exp \left\{\alpha \lambda^{3}\right\}>17 \tag{2.25}
\end{equation*}
$$

Lemma 2. There exists a number $K_{1}$ such that for all $k \geqq K_{1}$ and $r>k$,

$$
\begin{equation*}
\frac{n_{k}}{n_{r}} \leqq \exp \left\{\frac{-\lambda \alpha(r-k)}{\log r}\right\} \tag{2.26}
\end{equation*}
$$

Proof. Let $v=r-k$. Then since $\log (1+x) \leqq x$,

$$
\begin{align*}
\log \frac{n_{k}}{n_{r}} & =\frac{\alpha k}{\log k}-\frac{\alpha(k+v)}{\log (k+v)}=\frac{\alpha k \log (1+v / k)}{\log k \log (k+v)}-\frac{\alpha v}{\log (k+v)}  \tag{2.27}\\
& \leqq-\frac{\alpha v}{\log (k+v)}\left(1-\frac{1}{\log k}\right) \leqq-\lambda \alpha \frac{(r-k)}{\log r}
\end{align*}
$$

for $k \geqq K_{1}$, provided $\log K_{1} \geqq(1-\lambda)^{-1}$.
Lemma 3. For each $k$ let $r_{1}=r_{1}(k)$ be the largest integer $r$ such that $r-k<$ $(\log r)^{1 / 2}$. There exists a number $K_{2}$ such that for all $k \geqq K_{2}$ and $k<r \leqq r_{1}$,

$$
\begin{equation*}
P\left\{V_{n_{r}}^{n_{k}} \geqq \frac{c_{n_{r}}}{n_{r}}\right\} \leqq \exp \left\{-\alpha \lambda^{3}(r-k)\right\} \tag{2.28}
\end{equation*}
$$

Proof. From the inequality $1-x \leqq e^{-x}$ and Lemma 1 , for $r>k \geqq K_{1}$, we obtain

$$
\begin{align*}
P\left\{V_{n_{r}}^{n_{k}} \geqq \frac{c_{n_{r}}}{n_{r}}\right\} & =\left(1-\frac{c_{n_{r}}}{n_{r}}\right)^{n_{r}-n_{k}} \leqq \exp \left\{-\left(1-\frac{n_{k}}{n_{r}}\right) c_{n_{r}}\right\}  \tag{2.29}\\
& \leqq \exp \left\{-\left(1-\exp \left\{-\lambda \alpha\left(\frac{r-k}{\log r}\right)\right\}\right) c_{n_{r}}\right\}
\end{align*}
$$

For all sufficiently small positive $x, 1-e^{-x} \geqq \lambda x$. Hence, there exists $K_{3}$ so large that for all $k \geqq K_{3}$ and $k<r \leqq r_{1}$,

$$
\begin{equation*}
1-\exp \left\{-\lambda \alpha\left(\frac{r-k}{\log r}\right)\right\} \geqq \alpha \lambda^{2}\left(\frac{r-k}{\log r}\right) \tag{2.30}
\end{equation*}
$$

Finally, by (2.6), there exists $K_{4}$ so large that for $r>K_{4}$

$$
\begin{equation*}
c_{n_{r}} \geqq \lambda \log r . \tag{2.31}
\end{equation*}
$$

With $K_{2}=\max \left(K_{1}, K_{3}, K_{4}\right)$, the lemma follows from (2.29), (2.30), and (2.31).
Lemma 4. For each $k \geqq K_{2}$ and $r>r_{1}$,

$$
\begin{equation*}
P\left\{V_{n_{r}}^{n_{k}} \geqq \frac{c_{n_{r}}}{n_{r}}\right\} \leqq \exp \left\{-\lambda^{3}(\log r)^{1 / 2}\right\} \tag{2.32}
\end{equation*}
$$

Proof. From (2.29) and (2.31), we have

$$
\begin{align*}
P\left\{V_{n_{r}}^{n_{k}} \geqq \frac{c_{n_{r}}}{n_{r}}\right\} & \leqq \exp \left\{-\left(1-\exp \left\{-\lambda \frac{r-k}{\log r}\right\}\right) c_{n_{r}}\right\}  \tag{2.33}\\
& \leqq \exp \left\{-\left(1-\exp \left\{-\lambda(\log r)^{-1 / 2}\right\}\right) \lambda \log r\right\} \\
& \leqq \exp \left\{-\lambda^{3}(\log r)^{1 / 2}\right\} .
\end{align*}
$$

Lemma 5. For each $k$, let $r_{2}=r_{2}(k)$ be the least integer $r>k$ such that $r \geqq k+(\log r)^{2}$. Then for all $k>K_{1}$ and $r \geqq r_{2}, n_{k} / n_{r} \leqq 1 / r$.

Proof. By (2.27), for $k \geqq K_{1}$ and $r \geqq r_{2}$,

$$
\begin{equation*}
\log \frac{n_{k}}{n_{r}} \leqq-\lambda \alpha\left(\frac{r-k}{\log r}\right) \leqq-\lambda \alpha \log r<-\log r \tag{2.34}
\end{equation*}
$$

Lemma 6. There exists a number $K_{5}$ such that for all $k \geqq K_{5}$ and $r>r_{2}$ 。

$$
\begin{equation*}
P\left\{V_{n_{r}}^{n_{k}} \geqq \frac{c_{n_{r}}}{n_{r}}\right\} \leqq \frac{1}{\lambda^{2}} P\left(A_{r}\right) . \tag{2.35}
\end{equation*}
$$

Proof. From (2.23), we have

$$
\begin{equation*}
P\left(A_{r}\right) \geqq \lambda \exp \left\{-c_{n_{r}}\right\} \tag{2.36}
\end{equation*}
$$

for all $r \geqq$ some $K_{6}$. Hence, by Lemma 5, (2.11), (2.29), and (2.36), we have, for all $k \geqq K_{5} \geqq \max \left(K_{1}, K_{6}\right)$ and $r \geqq r_{2}$,

$$
\begin{align*}
P\left\{V_{n_{r}}^{n_{k}} \geqq \frac{c_{n_{r}}}{n_{r}}\right\} & \leqq \exp \left\{-\left(1-\frac{n_{k}}{n_{r}}\right) c_{n_{r}}\right\}  \tag{2.37}\\
& \leqq \frac{1}{\lambda} P\left(A_{r}\right) \exp \left\{\frac{2 \log _{2} n_{r}}{r}\right\} \leqq \frac{1}{\lambda^{2}} P\left(A_{r}\right)
\end{align*}
$$

Note that $r_{2}(k) \sim k$ as $k \rightarrow \infty$. Let $K_{7}$ be so large that for all $k \geqq K_{7}$,

$$
\begin{equation*}
r_{2}(k) \leqq 2 k \tag{2.38}
\end{equation*}
$$

and

$$
\begin{equation*}
8(\log k)^{2} \exp \left\{-\lambda^{3}(\log k)^{1 / 2}\right\}<\frac{1}{16} . \tag{2.39}
\end{equation*}
$$

Lemma 7. For $K_{0}=\max \left(K_{1}, \cdots, K_{7}\right)$ and all $k_{0} \geqq K_{0}$, (2.22) holds.

Proof. For all $k_{0} \leqq k \leqq k_{1}$,

$$
\begin{equation*}
\sum_{r=k+1}^{k_{1}} P\left\{V_{n_{r}}^{n_{k}} \geqq \frac{c_{n_{r}}}{n_{r}}\right\} \leqq \sum_{r=k+1}^{r_{1}}+\sum_{r=r_{1}+1}^{r_{2}}+\sum_{r=r_{2}+1}^{k_{1}}, \tag{2.40}
\end{equation*}
$$

which by Lemmas 3, 4, and 6, equations (2.21), (2.24), (2.25), (2.38), and (2.39), does not exceed

$$
\begin{align*}
& \sum_{r=k+1}^{\infty} \exp \left\{-\alpha \lambda^{3}(r-k)\right\}+2\left(\log r_{2}\right)^{2} \exp \left\{-\lambda^{3}(\log k)^{1 / 2}\right\}+\frac{1}{\lambda^{2}} \sum_{r=r_{2}}^{k_{1}} P\left(A_{r}\right)  \tag{2.41}\\
& \quad \leqq \frac{\exp \left\{-\alpha \lambda^{3}\right\}}{1-\exp \left\{-\alpha \lambda^{3}\right\}}+2(2 \log k)^{2} \exp \left\{-\lambda^{3}(\log k)^{1 / 2}\right\}+\frac{1}{\lambda^{2}}\left(\frac{5}{16}\right) \\
& \quad \leqq \frac{1}{16}+\frac{1}{16}+\frac{6}{16}=\frac{1}{2} .
\end{align*}
$$

The following lemma shows that the conditions of Theorem 1 (ii) imply those of Theorem 2. Note that the condition that $\left(c_{n}\right)$ be ultimately increasing is used only to show that ( $c_{n}$ ) may without loss of generality be assumed to satisfy (2.6). This substantiates Remark (2.5) above.

Lemma 8. Let $\left(c_{n}\right)$ be any sequence of positive numbers such that $c_{n} / n$ is ultimately decreasing and either $\left(c_{n}\right)$ is ultimately increasing or (2.6) holds. Then (2.2) converges if and only if (2.10) converges for all $\alpha>0$.

Proof. First observe that without loss of generality we may assume that (2.11) holds. In fact, if $c_{n}$ is ultimately increasing, so is $c_{n}^{\prime}=\min \left(c_{n}, 2 \log _{2} n\right)$, while if (2.6) holds then it also holds for $c_{n}^{\prime}$, and it is easy to see that replacing $c_{n}$ by $c_{n}^{\prime}$ does not alter the convergence or divergence of either (2.2) or (2.10).

We next show that with no loss of generality it may be assumed that (2.6) holds. Suppose that $c_{n} \leqq c_{n+1}$ for all $n \geqq n_{0}$. If $\lim _{n \rightarrow \infty} c_{n}<\infty$, then (2.2) and (2.10) both diverge and continue to do so if $c_{n}$ is replaced by $c_{n}^{\prime}=$ $\max \left(c_{n}, \log _{2} n\right)$. Suppose on the other hand that $c_{n} \rightarrow \infty$. Since $x e^{-x}$ is decreasing for large $x$, we have

$$
\begin{equation*}
\sum_{n_{0}}^{n} \frac{c_{k}}{k} e^{-c_{k}} \geqq c_{n} e^{-c_{n}} \sum_{n_{0}}^{n} k^{-1} \geqq c_{n} e^{-c_{n}} \log n-O(1), \tag{2.42}
\end{equation*}
$$

which $\rightarrow \infty$ along any subsequence $n^{\prime}$ for which $c_{n^{\prime}} \leqq \log _{2} n^{\prime}$. Hence, if (2.2) converges, (2.6) holds. If (2.2) diverges, we see from (2.42) that we may replace $c_{n}$ by $c_{n}^{\prime}=\max \left(c_{n}, \log _{2} n\right)$ and maintain divergence, so in this case as well we may assume that (2.6) holds.

It remains to prove the lemma under the assumption

$$
\begin{equation*}
\frac{1}{2} \log _{2} n \leqq c_{n} \leqq 2 \log _{2} n \tag{2.43}
\end{equation*}
$$

Now

$$
\begin{equation*}
\left(1-\frac{n_{k}}{n_{k+1}}\right) \sim 1-\exp \left\{-\frac{\alpha}{\log k}\right\} \sim \frac{\alpha}{\log k} \sim\left(\frac{n_{k+1}}{n_{k}}-1\right) \text { as } k \rightarrow \infty \tag{2.44}
\end{equation*}
$$

and hence, from (2.43),

$$
\begin{equation*}
c_{n_{k+1}}\left(1-\frac{n_{k}}{n_{k+1}}\right), \quad c_{n_{k}}\left(\frac{n_{k+1}}{n_{k}}-1\right) \tag{2.45}
\end{equation*}
$$

are bounded away from 0 and $\infty$. Since $c_{n} / n$ is decreasing for large $n$, if (2.2) diverges, we have

$$
\begin{align*}
\infty & =\sum_{k} \sum_{n_{k}<n \leqq n_{k+1}} \frac{c_{n}}{n} \exp \left\{\frac{-c_{n}}{n} n\right\} \leqq \sum_{k}\left(\frac{c_{n_{k}}}{n_{k}} \exp \left\{-\frac{c_{n_{k+1}}}{n_{k+1}} n_{k}\right\}\right)\left(n_{k+1}-n_{k}\right)  \tag{2.46}\\
& \leqq \sum_{k} c_{n_{k}}\left(\frac{n_{k+1}}{n_{k}}-1\right) \exp \left\{-c_{n_{k+1}}+c_{n_{k+1}}\left(1-\frac{n_{k}}{n_{k+1}}\right)\right\} \\
& \leqq \text { const. } \sum_{k} \exp \left\{-c_{n_{k+1}}\right\} .
\end{align*}
$$

The case in which (2.2) converges is treated similarly.

## 3. A conjecture of Darling and Erdös

In [2] Darling and Erdös obtained the limiting distribution of

$$
\begin{equation*}
y_{t}=\max _{0 \leqq \tau \leqq t} \frac{w(\tau)}{(\tau+1)^{1 / 2}} \quad \text { as } t \rightarrow \infty \tag{3.1}
\end{equation*}
$$

(This question was suggested by an inequality in [9] concerning the statistical consequences of "optional stopping.") They also conjectured an iterated logarithm law for the process $y_{t}$, namely:
(a) there exists a constant $c_{1}>0$ such that

$$
\begin{align*}
P\left\{y_{t} \geqq\left(2 \log _{2} t\right)^{1 / 2}+\right. & \left.\frac{\log _{3} t}{2\left(2 \log _{2} t\right)^{1 / 2}}+\frac{\left(c_{1}+\delta\right) \log _{4} t}{\left(2 \log _{2} t\right)^{1 / 2}} \text { i.o. } t \uparrow \infty\right\}  \tag{3.2}\\
& =0 \text { or } 1 \quad \text { according as } \delta>0 \text { or } \delta<0
\end{align*}
$$

and
(b) there exists a constant $c_{2}>0$ such that

$$
\begin{align*}
P\left\{y_{t} \leqq\left(2 \log _{2} t\right)^{1 / 2}+\frac{\log _{3} t}{2\left(2 \log _{2} t\right)^{1 / 2}}-\frac{\left(c_{2}+\delta\right) \log _{4} t}{\left(2 \log _{2} t\right)^{1 / 2}} \text { i.o. } t \uparrow \infty\right\}  \tag{3.3}\\
=0 \text { or } 1 \quad \text { according as } \delta>0 \text { or } \delta<0 .
\end{align*}
$$

Since $y_{t}$ is increasing in $t$, it follows that for any ultimately increasing function $\psi(t), y_{t} \geqq \psi(t)$ for arbitrarily large $t$ if and only if $w(t) \geqq(t+1)^{1 / 2} \psi(t)$ for arbitrarily large $t$ (see Remark 2.1). Hence, from (1.3) we see that the probability (3.2) is 1 for all $c_{1}$ and $\delta$; that is, conjecture (a) is false. A correct version of $(a)$ is

$$
\begin{gather*}
P\left\{y_{t} \geqq\left(2 \log _{2} t\right)^{1 / 2}+\frac{3 \log _{3} t}{2\left(2 \log _{2} t\right)^{1 / 2}}+\frac{(1+\delta) \log _{4} t}{\left(2 \log _{2} t\right)^{1 / 2}} \text { i.o. } t \uparrow \infty\right\}  \tag{3.4}\\
=0 \text { or } 1 \quad \text { according as } \delta>0 \text { or } \delta<0
\end{gather*}
$$

Conjecture (b) is correct, but more difficult to prove. In this section, we shall use Theorem 1 (ii) to verify (b) and identify the constant $c_{2}$ as 1 . We shall only sketch the proof, which relies greatly on the method of Motoo [6]. (Motoo's method illuminates the entire paper [2] as well as the relation of Theorem l(i) to (a) and Theorem 1 (ii) to (b).)

Let

$$
\begin{equation*}
U(t)=e^{-t} w\left(e^{2 t}-1\right) \tag{3.5}
\end{equation*}
$$

It is easy to verify that

$$
\begin{equation*}
P\{U(t+s) \varepsilon d x \mid U(s)=u\}=\varphi\left(\frac{x-u e^{-t}}{\left(1-e^{-2 t}\right)^{1 / 2}}\right) \frac{d x}{\left(1-e^{-2 t}\right)^{1 / 2}} \tag{3.6}
\end{equation*}
$$

and hence, that $U(t), 0 \leqq t<\infty$, is a Markov process with stationary transition probabilities and infinitesimal generator

$$
\begin{equation*}
D f(x)=f^{\prime \prime}(x)-x f^{\prime}(x) \tag{3.7}
\end{equation*}
$$

(This $U(t)$ is the Ornstein-Uhlenbeck process with $U(0)=0$.) To prove (b) with $c_{2}=1$, it suffices to show that

$$
\begin{align*}
P\left\{\max _{0 \leqq r \leqq t} U(\tau) \leqq(2 \log t\right. & \left.\left.+\log _{2} t-(2+\delta) \log _{3} t\right)^{1 / 2} \text { i.o. } t \uparrow \infty\right\}  \tag{3.8}\\
& =0 \text { or } 1 \quad \text { according as } \delta>0 \text { or } \delta<0 .
\end{align*}
$$

Define $T_{0}=0$ and for each $n=1,2, \cdots$,

$$
\begin{align*}
T_{2 n-1} & =\inf \left\{t: t>T_{2 n-2}, U(t)=1\right\} \\
T_{2 n} & =\inf \left\{t: t>T_{2 n-1}, U(t)=0\right\} . \tag{3.9}
\end{align*}
$$

It may be shown that $\gamma=E T_{2}<\infty$, and since $T_{2}-T_{0}, T_{4}-T_{2}, \cdots$ are independent and identically distributed, it follows from the strong law of large numbers that

$$
\begin{equation*}
P\left\{\frac{T_{2 n}}{n} \rightarrow \gamma\right\}=1 \tag{3.10}
\end{equation*}
$$

Let $x_{n}=\max _{T_{2 n-2 \leqq t}<T_{2 n}} U(t), n=1,2, \cdots$. Then $x_{1}, x_{2}, \cdots$ are i.i.d., and for $a>1, P\left\{x_{n}>a\right\}$ is the probability that the process $U(t)$ starting from 1 reaches the level $a$ before it reaches 0 . From (3.7) and standard diffusion theory it follows that $P\left\{x_{n}>a\right\}=g(1)$, where $g(x)$ satisfies $g^{\prime \prime}(x)-x g(x)=0$, $0<x<a$, subject to the boundary conditions $g(0)=0, g(a)=1$. Hence,

$$
\begin{equation*}
P\left\{x_{n}>a\right\}=\frac{\int_{0}^{1} \exp \left\{\frac{1}{2} y^{2}\right\} d y}{\int_{0}^{a} \exp \left\{\frac{1}{2} y^{2}\right\} d y}=\eta a \exp \left\{-\frac{1}{2} a^{2}\right\}\left(1+O\left(\frac{1}{a^{2}}\right)\right) \tag{3.11}
\end{equation*}
$$

as $a \rightarrow \infty$, where we have put $\eta=\int_{0}^{1} \exp \left\{\frac{1}{2} y^{2}\right\} d y$. Let $\delta>0$ and

$$
\begin{equation*}
\psi(t)=\left[2 \log t+\log _{2} t-(2+\delta) \log _{3} t\right]^{1 / 2} \tag{3.12}
\end{equation*}
$$

Since $\psi$ is ultimately increasing, we have by (3.10) for any $\varepsilon>0$,

$$
\begin{align*}
P\left\{\max _{0 \leqq r \leqq t} U(t) \leqq \psi(t) \text { i.o. } t \uparrow \infty\right\} & \leqq P\left\{\max _{0 \leqq t \leqq T_{2 n}} U(t) \leqq \psi\left(T_{2 n+2}\right) \text { i.o. }\right\}  \tag{3.13}\\
& \leqq P\left\{\max _{1 \leqq k \leqq n} x_{k} \leqq \psi(n(\gamma+\varepsilon)) \text { i.o. }\right\} .
\end{align*}
$$

It follows from Remarks 2.2 and 2.5 and some straightforward calculation using (3.11) that

$$
\begin{equation*}
P\left\{\max _{0 \leqq r \leqq t} U(t) \leqq \psi(t) \text { i.o. } t \uparrow \infty\right\}, \tag{3.14}
\end{equation*}
$$

that is, the probability (3.8), is 0 for $\delta>0$. A similar argument shows that (3.8) is 1 for $\delta>0$.

Motoo's method of proof of the criterion (1.2) for the Wiener process [6] is essentially a combination of the preceding argument with Theorem 1 (i) instead of Theorem 1 (ii). It is interesting to note that neither Motoo's nor our argument requires knowledge of the exact value of $\gamma=E T_{2}$ or of the constant $\eta$ appearing in (3.11). A more careful analysis shows that under certain regularity conditions on the function $\psi$,

$$
\begin{align*}
& P\left\{\max _{0 \leqq r \leqq t} U(\tau) \leqq \psi(t) \text { i.o. } t \uparrow \infty\right\},  \tag{3.15}\\
& =0 \text { or 1, according as } \int_{1}^{\infty} f(\psi(t)) \exp \left\{-\frac{t}{\sqrt{2 \pi}} f(\psi(t))\right\} d t
\end{align*}
$$

is convergent or divergent, where we have set $f(x)=x \exp \left\{-\frac{1}{2} x^{2}\right\}$. However, establishing this deeper criterion requires knowledge of the constants $\gamma$ and $\eta$ and in particular that $\gamma / \eta=(2 \pi)^{1 / 2}$. (It is interesting to observe that if we had defined the stopping times $T_{n}$ in our proof in terms of 0 and an arbitrary number $b>0$, then $\gamma$ and $\eta$ would depend on $b$, but the ratio $\gamma / \eta$ would not.) It is also necessary to sharpen (3.10) to, say,

$$
\begin{equation*}
P\left\{\frac{T_{2 n}-n \gamma}{n^{1 / 2} \log n} \rightarrow 0\right\}=1 \tag{3.16}
\end{equation*}
$$

which is a consequence of the fact that $E T_{2}^{2}<\infty$ and the usual proof of the strong law of large numbers using Kolmogorov's inequality and Kronecker's lemma (see [7]). We omit the details.

## 4. A continuous time extremal process

In this section, we introduce a continuous time process $v_{t}$ which bears more or less the same relation to the process $V_{n}$ as the Wiener process does to the sequence of partial sums of i.i.d. mean 0 , variance 1 , random variables, and give boundary crossing probabilities for this process analogous to those of [11] for the Wiener process.

Consider the sequence of processes $n V_{[n t]}, 0 \leqq t<\infty$. For any $0=$ $t_{0}<t_{1}<t_{2}<\cdots<t_{r}$ and $a_{1} \geqq a_{2} \geqq \cdots \geqq a_{r}>0$, we have as $n \rightarrow \infty$

$$
\begin{align*}
P\left\{n V_{\left[n t_{1}\right]} \geqq a_{1}\right. & \left., \cdots, n V_{\left[n t_{r}\right]} \geqq a_{r}\right\}  \tag{4.1}\\
& =\prod_{i=1}^{r}\left(1-\frac{a_{i}}{n}\right)^{\left[n t_{i}\right]-\left[n t_{i-1}\right]} \rightarrow \exp \left\{-\sum_{i=1}^{n} a_{i}\left(t_{i}-t_{i-1}\right)\right\} .
\end{align*}
$$

This suggests defining $v_{t}, 0<t<\infty$, by the following consistent family of joint distributions: for $0=t_{0}<t_{1}<\cdots<t_{r}$ and $-\infty<a_{i}<\infty, i=$ $1,2, \cdots, r$,

$$
\begin{equation*}
P\left\{v_{t_{1}} \geqq a_{1}, \cdots, v_{t_{r}} \geqq a_{r}\right\}=\exp \left\{-\sum_{i=1}^{r} \max \left(a_{i}, \cdots, a_{r}\right)^{+}\left(t_{i}-t_{i-1}\right)\right\} \tag{4.2}
\end{equation*}
$$

By Kolmogorov's consistency theorem, there exists a process, say $\tilde{v}_{t}$, having the finite dimensional joint distributions given by (4.2). Defining $v_{t}=\lim _{s \downarrow t} \tilde{v}_{s}$, where $s$ runs through rationals greater than $t$, we obtain a process having the same finite dimensional joint distributions as $\tilde{v}_{t}$ and in addition right continuous, decreasing sample paths. We shall call any such process a standard extremal process.

It is easy to see from (4.2) that the process $v_{t}, 0<t<\infty$, is Markovian with stationary transition probability

$$
P\left\{v_{t} \geqq a_{2} \mid v_{\tau}=a_{1}\right\}= \begin{cases}\exp \left\{-a_{2}(t-\tau)\right\} & \text { for } a_{1} \geqq a_{2} \geqq 0  \tag{4.3}\\ 0 & \text { for } 0 \leqq a_{1}<a_{2}\end{cases}
$$

For each $\tau>0$, let $h=h(\tau)=\inf \left\{t: t>\tau, v_{t}<v_{\tau}\right\}$. Then $\{h>t\}=\left\{v_{t}=\right.$ $\left.v_{\tau}\right\}$, and hence, by (4.3),

$$
\begin{equation*}
P\left\{h>t \mid v_{\tau}=a\right\}=\exp \{-a(t-\tau)\} . \tag{4.4}
\end{equation*}
$$

Also for $a_{2}<a_{1}$,

$$
\begin{align*}
P\left\{v_{h} \leqq\right. & \left.a_{2}, t<h<t+\delta \mid v_{\tau}=a_{1}\right\}  \tag{4.5}\\
& =P\left\{v_{t+\delta} \leqq a_{2}, t<h<t+\delta \mid v_{\tau}=a_{1}\right\}+o(\delta) \\
& =P\left\{v_{t+\delta} \leqq a_{2}, v_{t} \geqq a_{1} \mid v_{\tau}=a_{1}\right\}+o(\delta) \\
& =\left(1-\exp \left\{-\delta a_{2}\right\}\right) \exp \left\{-(t-\tau) a_{1}\right\}+o(\delta),
\end{align*}
$$

so

$$
\begin{equation*}
P\left\{v_{h} \leqq a_{2} \mid v_{\tau}=a_{1}\right\}=\int_{\tau}^{\infty} P\left\{v_{h} \leqq a_{2}, h \in d t \mid v_{\tau}=a_{1}\right\}=\frac{a_{2}}{a_{1}} \tag{4.6}
\end{equation*}
$$

It follows that the sample paths of the process $v_{t}, 0<t<\infty$, may be described as follows: for any $\tau>0$, if the process is in the state $a$ at time $\tau$, it remains there for a random length of time having a negative exponential distribution with parameter $a$ and then jumps to a point uniformly distributed on $(0, a) . \mathrm{By}(4.2), P\left\{v_{0_{+}}=+\infty\right\}=1$, and with probability 1 there are infinitely
many jumps in each neighborhood of $t=0$. Except for the behavior of $v_{t}$ near $t=0$, this description is analogous to that of the discrete time process $V_{n}$, which holds in each state $a$ a random length of time which is geometrically distributed with parameter $a$ and then moves to a point uniformly distributed on ( $0, a$ ). If $x_{1}, x_{2}, \cdots$ are i.i.d. with $P\left\{x_{i} \geqq x\right\}=e^{-x}$, and $v_{n}^{*}=\min \left(x_{1}\right.$, $\cdots, x_{n}$ ), then the process $v_{t}$ interpolates the process $v_{n}^{*}$ in the sense that the two sequences $v_{n}$ and $v_{n}^{*}, n=1,2, \cdots$, have the same joint distributions.

Trivial modifications of the proof of Theorem 1 (ii) prove:
Theorem 3. If $c(t) \geqq 0$ is ultimately increasing and $c(t) / t$ is ultimately decreasing, then $P\left\{v_{t} \geqq c(t) / t\right.$, i.o. $\left.t \uparrow \infty\right\}=0$ or 1 , according as $\int_{1}^{\infty}(c(t) / t) e^{-c(t)} d t$ converges or diverges.

Since $P\left\{v_{0_{+}}=+\infty\right\}=1$, it is of interest to obtain a description of the rate of growth of $v_{t}$ as $t \downarrow 0$. A law of the iterated logarithm for $v_{t}$ as $t \downarrow 0$ follows from Theorem 3 and the following inversion theorem.

Theorem 4. For each $v>0$, let $T(v)=\sup \left\{t: v_{t} \geqq v\right\}$. The process $T(v)$, $0<v<\infty$, is a standard extremal process.

Proof. The fact that the sample paths of $T(v), 0<v<\infty$ are decreasing, right continuous step functions follows at once from the corresponding properties of $v_{t}, 0<t<\infty$. Hence, it suffices to show that $T(v)$ and $v_{t}$ have the same finite dimensional joint distributions. For $0<u<v$ and $\tau>t>0$, except for a set of probability 0 ,

$$
\begin{equation*}
\{T(u) \geqq \tau, T(v) \geqq t\}=\left\{v_{t} \geqq v, v_{\tau} \geqq u\right\} \tag{4.7}
\end{equation*}
$$

and hence, by (4.2),

$$
\begin{align*}
P\{T(u) \geqq \tau, T(v) & \geqq t\}  \tag{4.8}\\
& =\exp \{-t v-(\tau-t) u\}=\exp \{-u \tau-(v-u) t\}
\end{align*}
$$

The general case of an arbitrary finite number of time points $u, v, \cdots, z$ follows by the same argument.

For any strictly decreasing function $\psi$ defined on $(0, \infty), v_{t} \geqq \psi(t)$ i.o. $t \downarrow 0$ if and only if $T(v)>\psi^{-1}(v)$ i.o. $v \uparrow \infty$. Hence, by Theorem 4,

$$
\begin{equation*}
P\left\{v_{t} \geqq \psi(t) \text { i.o. } t \downarrow 0\right\}=P\left\{v_{t}>\psi^{-1}(t) \text { i.o. } t \uparrow \infty\right\} . \tag{4.9}
\end{equation*}
$$

For example, by Theorem 3 and (4.9), we have

$$
\begin{equation*}
P\left\{\limsup _{t \rightarrow 0} \frac{t v_{t}}{\log _{2} \frac{1}{t}}=1\right\}=1 \tag{4.10}
\end{equation*}
$$

We now show that the method of [11] yields boundary crossing probabilities for the process $v_{t}, 0<t<\infty$, analogous to those obtained there for the Wiener process.

Let $F$ denote a measure on $(0, \infty)$ assigning finite measure to bounded intervals, and define for $x>0, t \geqq 0, \varepsilon>0$,

$$
\begin{align*}
f(x, t) & =\int_{\{0<y \leqq x\}} e^{y t} d F(y) \\
A(t, \varepsilon) & =\inf \{x: f(x, t) \geqq \varepsilon\}(=\infty \text { if } f(x, t)<\varepsilon \text { for all } x) \tag{4.11}
\end{align*}
$$

It is easily seen that for all $t \geqq 0, x \geqq A(t, \varepsilon)$ if and only if $f(x, t) \geqq \varepsilon$, and that if $\tau_{0}=\inf \{\tau: A(\tau, \varepsilon)<\infty\}$, then $A(\cdot, \varepsilon)$ is continuous and decreasing on $\left(\tau_{0}, \infty\right)$ and $A\left(\tau_{0}, \varepsilon\right)=\lim _{t \downarrow \tau_{0}} A(t, \varepsilon)$. Moreover, if $F\{A(t, \varepsilon)\}=0$, then $f(A(t, \varepsilon), t)=\varepsilon$.

Let $\mathscr{F}_{t}=\mathscr{B}\left(v_{\tau}, \tau \leqq t\right)$. Since $\left\{I\left[v_{t} \geqq y\right] e^{y t}, \mathscr{F}_{t}, 0 \leqq t<\infty\right\}$ is a martingale for each $y>0$, as may be verified by direct computation using (4.3), it follows from Fubini's theorem that $\left\{f\left(v_{t}, t\right), \mathscr{F}_{t}, 0<t<\infty\right\}$ is also a martingale.

Theorem 5. For any $\varepsilon>F\{(0, \infty)\}$

$$
\begin{equation*}
P\left\{v_{t} \geqq A(t, \varepsilon) \text { for some } t>0\right\}=\frac{F\{(0, \infty)\}}{\varepsilon} \tag{4.12}
\end{equation*}
$$

For each $\tau>0$,

$$
\begin{align*}
P\left\{v_{t} \geqq A(t, \varepsilon) \text { for some } t \geqq \tau\right\} & =\exp \{-\tau A(\tau, \varepsilon)\}  \tag{4.13}\\
+\varepsilon^{-1}[F\{(0, A(\tau, \varepsilon))\} & \left.-\exp \{-\tau A(\tau, \varepsilon)\} \int_{\{0<y<A(\tau, \varepsilon)\}} e^{\tau y} d F(y)\right] \\
& \left(=\varepsilon^{-1} F\{(0, A(\tau, \varepsilon))\} \text { if } F\{A(\tau, \varepsilon)\}=0\right)
\end{align*}
$$

Proof. The parenthetical part of (4.13) follows at once from the preceding line and the observation that if $F\{A(\tau, \varepsilon)\}=0$, then

$$
\begin{equation*}
\int_{\{0<y<A(\tau, \varepsilon)\}} e^{\tau y} d F(y)=f(A(\tau, \varepsilon), \tau)=\varepsilon ; \tag{4.14}
\end{equation*}
$$

equation (4.12) follows from the parenthetical part of (4.13), by letting $\tau \downarrow \tau_{0}=$ $\inf \{t: A(t, \varepsilon)<\infty\}$ through any sequence of values such that $\mathscr{F}\{A(\tau, \varepsilon)\}=0$. The proof of (4.13) follows from Lemma 1 of [11], Remark (d) at the end of Section 3 of [11], and Lemma 9 below (see the proof of Theorem 1 of [11]).

Lemma 9. The function $f\left(v_{t}, t\right)$ tends to 0 in probability.
Proof. Let $c>0$. By the weak convergence of the family $F_{t}\{\cdot\}=$ $F\{(\cdot) \cap(0, c] / t\}$ to the 0 measure,

$$
\begin{equation*}
\int_{\{0<y \leqq c / t\}} e^{y t} d F(y)=\int_{\{0<y \leqq c\}} e^{y} d F\left(\frac{y}{t}\right) \rightarrow 0 \tag{4.15}
\end{equation*}
$$

as $t \rightarrow \infty$. Hence, for any $\varepsilon>0$, for all $t$ sufficiently large,

$$
\begin{equation*}
P\left\{f\left(v_{t}, t\right) \geqq \varepsilon\right\} \leqq P\left\{v_{t} \geqq \frac{c}{t}\right\} \stackrel{\doteq}{=} e^{-c} \tag{4.16}
\end{equation*}
$$

which can be made arbitrarily small by taking $c$ sufficiently large.

## 5. Asymptotic expansions for $\boldsymbol{A}(\boldsymbol{t}, \boldsymbol{\varepsilon})$

If the measure $F$ of the preceding section is taken to be Lebesgue measure on $(0, \infty)$, it is easily seen from (4.11) that

$$
\begin{equation*}
A(t, \varepsilon)=\frac{1}{t} \log (1+\varepsilon t)=\frac{1}{t}\left[\log t+\log \varepsilon+O\left(\frac{1}{t}\right)\right] \tag{5.1}
\end{equation*}
$$

as $t \rightarrow \infty$. By Theorem 3 there exist functions $g(t) \sim \log _{2} t / t$ as $t \rightarrow \infty$ such that $P\left\{v_{t} \geqq g(t)\right.$ for some $\left.t>0\right\}<1$, and it is natural to ask whether we can find boundaries with this rate of growth to which Theorem 5 applies.

Theorem 6. If $F$ is defined by (1.8), then for $k=2$

$$
\begin{equation*}
A(t, \varepsilon)=\frac{1}{t}\left[\log _{2} t+(2+\delta) \log _{3} t+\log \varepsilon+o(1)\right] \tag{5.2}
\end{equation*}
$$

as $t \rightarrow \infty$, while for $k \geqq 3$,

$$
\begin{align*}
A(t, \varepsilon)=\frac{1}{t}\left[\log _{2} t+2 \log _{3} t+\right. & \sum_{i=4}^{k} \log _{i} t  \tag{5.3}\\
& \left.+(1+\delta) \log _{k+1} t+\log \varepsilon+o(1)\right]
\end{align*}
$$

as $t \rightarrow \infty$.
(See equation (10) of [11] which describes the cofresponding result for the Wiener process.)

To prove (5.2), let $F$ be given by (1.8) with $k=2$ and let $F^{\prime}(y)=d F / d y$. (The proof of (5.3) when $F$ is given by (1.8) with $k \geqq 3$ is similar and will be omitted.) It follows easily from (4.11) that

$$
\begin{equation*}
A=A(t, \varepsilon) \rightarrow 0 \tag{5.4}
\end{equation*}
$$

as $t \rightarrow \infty$. Similarly,

$$
\begin{equation*}
t A \rightarrow \infty \tag{5.5}
\end{equation*}
$$

In fact, if there exists a number $C$ such that $t A<C$ along a sequence of $t$ values, then

$$
\begin{equation*}
\varepsilon \leqq \int_{0}^{c} e^{z} d F\left(\frac{z}{t}\right) \tag{5.6}
\end{equation*}
$$

But $F(z / t) \rightarrow 0$ as $t \rightarrow \infty$ for all $z>0$, and hence, $\int_{0}^{C} e^{z} d F(z / t) \rightarrow 0$ as $t \rightarrow \infty$, contradicting our supposition. Since $F^{\prime}$ is decreasing in a neighborhood of the origin, we have by (5.4) for all $t$ sufficiently large,

$$
\begin{equation*}
\varepsilon=f(A, t)=\int_{0}^{A} e^{y t} F^{\prime}(y) d y \geqq \frac{F^{\prime}(A)\left(e^{t A}-1\right)}{t}, \tag{5.7}
\end{equation*}
$$

and hence, by (1.8) and (5.5),

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} \frac{e^{t A}}{t A \log \frac{1}{A}\left(\log _{2} \frac{1}{A}\right)^{1+\delta}} \leqq \varepsilon \tag{5.8}
\end{equation*}
$$

Rewriting (5.8) as

$$
\begin{equation*}
A \leqq \frac{1}{t}\left[\log \varepsilon t+\log A+\log _{2} \frac{1}{A}+(1+\delta) \log _{3} \frac{1}{A}+o(1)\right] \tag{5.9}
\end{equation*}
$$

we see by (5.4) and (5.5) that $A \leqq(\log t / t)[1+o(1)]$, and hence $\log A \leqq$ $\log _{2} t-\log t+o(1)$. Since (5.5) implies a fortiori that for all sufficiently large $t$ we have $1 / A \leqq t$, and hence

$$
\begin{equation*}
\log _{k} 1 / A \leqq \log _{k} t \tag{5.10}
\end{equation*}
$$

we have from (5.9),

$$
\begin{equation*}
A \leqq \frac{1}{t}\left[\log \varepsilon+2 \log _{2} t+(1+\delta) \log _{3} t+o(1)\right] \tag{5.11}
\end{equation*}
$$

Thus $\log A \leqq-\log t+\log _{3} t+O(1)$, which by substituting once more in (5.9) yields $A \leqq \log _{2} t / t(1+o(1))$ and hence

$$
\begin{equation*}
\log A \leqq-\log t+\log _{3} t+o(1) \tag{5.12}
\end{equation*}
$$

Finally, substituting (5.12) and (5.10) in (5.9), yields one half of (5.2), to wit

$$
\begin{equation*}
A \leqq \frac{1}{t}\left[\log _{2} t+(2+\delta) \log _{3} t+\log \varepsilon+o(1)\right] \tag{5.13}
\end{equation*}
$$

In particular,

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} \frac{t A}{\log _{2} t} \leqq 1 \tag{5.14}
\end{equation*}
$$

To prove (5.13) with the inequality reversed let $0<b<c<1$. From (4.11), we have for all $t$ sufficiently large

$$
\begin{align*}
\varepsilon & =\left(\int_{0}^{b A}+\int_{b A}^{c A}+\int_{c A}^{A}\right) e^{y t} F^{\prime}(y) d y  \tag{5.15}\\
& \leqq e^{b t A} F(b A)+\frac{e^{c t A}}{t} F^{\prime}(b A)+\frac{e^{t A}}{t} F^{\prime}(c A) \\
& \leqq \frac{e^{b t A}}{\delta\left(\log \frac{1}{2 A}\right)^{\delta}}+\frac{e^{c t A}}{b t A \log \frac{1}{A}\left(\log \frac{1}{2 A}\right)^{1+\delta}}+\frac{e^{t A}}{c t A \log \frac{1}{A}\left(\log \frac{1}{2 A}\right)^{1+\delta}}
\end{align*}
$$

Let $b=1 / \log _{2} t$. Then from (5.4), (5.5), (5.14), and (5.15), we obtain for large $t$

$$
\begin{equation*}
\varepsilon \leqq o(1)+\frac{e^{t A}}{\log 1 / A} \leqq o(1)+\frac{e^{t A}}{\frac{1}{2} \log t}, \tag{5.16}
\end{equation*}
$$

and hence,

$$
\begin{equation*}
\frac{\log _{2} t}{t}=O(A) \tag{5.17}
\end{equation*}
$$

From (5.14) and (5.17), it follows that for all $\eta$ sufficiently small,

$$
\begin{equation*}
\eta \leqq \frac{t A}{\log _{2} t} \leqq 1+\eta \tag{5.18}
\end{equation*}
$$

for all sufficiently large $t$. Using (5.18), we see that the second term on the right side of (5.15) is majorized by $\exp \left\{c(1+\eta) \log _{2} t\right\} /(\eta / 2 \log t)$ for large $t$, which converges to 0 as $t \rightarrow \infty$ for $\eta$ so small that $c(1+\eta)<1$. Hence, letting $t \rightarrow \infty$, then $c \rightarrow 1$ in (5.15), we have

$$
\begin{equation*}
\liminf _{t \rightarrow \infty} \frac{e^{t A}}{t A \log \frac{1}{A}\left(\log _{2} \frac{1}{A}\right)^{1+\delta}} \geqq \varepsilon \tag{5.19}
\end{equation*}
$$

The reverse of inequality (5.13) now follows from (5.19) by an argument similar to that which led from (5.8) to (5.13). This completes the proof of (5.2).

## 6. Remarks

Remark 6.1. Extremal processes in continuous time have been studied by Dwass [4] and Lamperti [5]. Lamperti proved an invariance theorem which is helpful in the proof of (6.2) below.

Remark 6.2. Theorem 2(i) of [11] states that if $g$ is a positive continuous function such that $g(t) / t^{1 / 2}$ is ultimately increasing and (1.2) converges, then for each $\tau>0$,

$$
\begin{align*}
P\{w(t) \geqq g(t) \text { for some } t & \geqq \tau\}  \tag{6.1}\\
& =\lim _{m \rightarrow \infty} P\left\{S_{n} \geqq m^{1 / 2} g(n / m) \text { for some } n \geqq \tau m\right\}
\end{align*}
$$

where $S_{n}=x_{1}+\cdots+x_{n}$, and $x_{1}, x_{2}, \cdots$ is any sequence of i.i.d. random variables having $E x_{1}=0, E x_{1}^{2}=1$. An analogous limit theorem for minima of uniform random variables is that if $g$ is continuous and decreasing on some interval $\left(\tau_{0}, \infty\right)\left(\left[\tau_{0}, \infty\right)\right)$ and $\equiv \infty$ on $\left[0, \tau_{0}\right]\left(\left[0, \tau_{0}\right)\right)$, and if $(2.10)$ converges with $c_{n}=n g(n)$, then for each $\tau>0$,

$$
\begin{align*}
P\left\{v_{t} \geqq g(t) \text { for some } t\right. & \geqq \tau\}  \tag{6.2}\\
& =\lim _{m \rightarrow \infty} P\left\{V_{n} \geqq \frac{1}{m} g\left(\frac{n}{m}\right) \text { for some } n \geqq \tau m\right\} .
\end{align*}
$$

(As in Theorem 2(ii) of [11] a similar result holds if in (6.2) we replace $t \geqq \tau$ by $t>0$ and $n \geqq \tau m$ by $n \geqq 1$.) The proof is similar in spirit to the proof of Theorem 2 of [11], but the details are much simpler.

Remark 6.3. Using the probability integral transform, one can immediately obtain analogous results for random variables having arbitrary continuous distributions. (See Remark 2.5.) For example, if $x_{1}, x_{2}, \cdots$ are i.i.d. with $P\left\{x_{i} \geqq x\right\}=e^{-x}(x>0)$ and if $g=e^{-h}$ satisfies the conditions of Remark 6.2 above, then the left side of (6.2) equals

$$
\begin{equation*}
\lim _{m \rightarrow \infty} P\left\{\max _{1 \leqq k \leqq n} x_{k} \leqq \log m+h\left(\frac{n}{m}\right) \text { for some } n \geqq \tau m\right\} . \tag{6.3}
\end{equation*}
$$

If the $x$ are standard normal random variables, the left side of (6.2) equals

$$
\begin{align*}
& \lim _{m \rightarrow \infty} P\left\{\max _{1 \leqq k \leqq n} x_{k} \leqq 2^{1 / 2}\left[\log m+h\left(\frac{n}{m}\right)-\frac{1}{2} \log \left(\log m+h\left(\frac{n}{m}\right)\right)\right.\right.  \tag{6.4}\\
&\left.-\log 2 \sqrt{\pi}]^{1 / 2} \quad \text { for some } n \geqq \tau m\right\}
\end{align*}
$$

Remark 6.4. It is possible to give a proof of Theorem 3 which is in the spirit of Motoo's proof [6] of the law of the iterated logarithm for the Wiener process. In fact, it may be shown that $\left\{x_{t} \equiv e^{t} v_{e^{t}}, 0 \leqq t<\infty\right\}$ is a positive recurrent Markov process, and since the sample paths of this process are continuous and increasing except for jumps in the negative direction, Motoo's method (as sketched in Section 3) applies with minor changes. To complete the argument it is necessary to compute (at least asymptotically as $a \rightarrow \infty$ )

$$
\begin{equation*}
P\{x(T) \geqq a \mid x(0)=1\} \tag{6.5}
\end{equation*}
$$

where $T=\inf \{t: x(t) \notin[1, a)\}$. Since the generator $A$ of the process $x(t)$ is given by

$$
\begin{equation*}
A f(x)=x f^{\prime}(x)+\int_{0}^{x}(f(u)-f(x)) d u \tag{6.6}
\end{equation*}
$$

and since $p_{a}(x) \equiv P\{x(T) \geqq a \mid x(0)=x\}$ satisfies $A p_{a}(x)=0,1<x<a$, subject to $p_{a}(a)=1$ and $p_{a}(x)=0,0<x \lessdot 1$, it may be shown that

$$
\begin{equation*}
p_{a}(x)=\frac{e+\int_{1}^{x} \frac{e^{u}}{u} d u}{e+\int_{1}^{a} \frac{e^{u}}{u} d u}, \quad 1 \leqq x \leqq a \tag{6.7}
\end{equation*}
$$

and hence $p_{a}(1) \sim a e^{-a}$ as $a \rightarrow \infty$.
Extensions of this approach are being investigated by Mr. J. Frankel.
Remark 6.5. Minor changes in the method of proof of Theorem 1 applied to $\max _{0 \leqq s \leq t}|w(s)|$ yield Chung's law of the iterated logarithm [1]; to wit: if ultimately $c(t) \uparrow$ and $c(t) / t \downarrow$, then

$$
\begin{equation*}
P\left\{\max _{0 \leqq s \leqq t}|w(s)| \leqq\left(\frac{t}{c(t)}\right)^{1 / 2} \text { for arbitrarily large } t\right\}=0 \text { or } 1, \tag{6.8}
\end{equation*}
$$

according as

$$
\begin{equation*}
\int^{\infty} \frac{c(t)}{t} \exp \left\{-\frac{\pi^{2}}{8} c(t)\right\} d t<\infty \text { or }=\infty \tag{6.9}
\end{equation*}
$$

The required computations are virtually identical with those of Lemmas 2 through 7 in light of the observation that for $0<\tau<t, 0<y<x$,

$$
\begin{equation*}
P\left\{\max _{0 \leqq s \leqq t}|w(s)| \leqq x\left|\max _{0 \leqq s \leqq \tau}\right| w(s) \mid \leqq y\right\} \leqq P\left\{\max _{0 \leqq s \leqq t-\tau}|w(s)| \leqq x\right\} . \tag{6.10}
\end{equation*}
$$

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