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ON THE LAW OF THE SUPREMUM OF LÉVY PROCESSES[1]

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We show that the law of the overall supremum $\overline{X}_t = \sup_{s \le t} X_s$ of a Lévy process X, before the deterministic time t is equivalent to the average occupation measure $\mu_t^+(dx) = \int_0^t \mathbb{P}(X_s \in dx)\,ds$, whenever 0 is regular for both open halflines $(-\infty, 0)$ and $(0, \infty)$. In this case, $\mathbb{P}(\overline{X}_t \in dx)$ is absolutely continuous for some (and hence for all) $t > 0$ if and only if the resolvent measure of X is absolutely continuous. We also study the cases where 0 is not regular for both halflines. Then we give absolute continuity criterions for the laws of $(g_t, \overline{X}_t)$ and $(g_t, \overline{X}_t, X_t)$, where g_t is the time at which the supremum occurs before t. The proofs of these results use an expression of the joint law $\mathbb{P}(g_t \in ds, X_t \in dx, \overline{X}_t \in dy)$ in terms of the entrance law of the excursion measure of the reflected process at the supremum and that of the reflected process at the infimum. As an application, this law is made (partly) explicit in some particular instances.

1. Introduction. The law of the past supremum $\overline{X}_t = \sup_{s \le t} X_s$ of Lévy processes before a deterministic time $t > 0$ presents some major interest in stochastic modeling, such as queuing and risk theories, as it is related to the law of the first passage time T_x above any positive level x, through the relation $\mathbb{P}(\overline{X}_t \ge x) = \mathbb{P}(T_x \le t)$. The importance of knowing features of this law, for some domains of application, mainly explains the abundance of the literature on this topic. From the works of Lévy on Brownian motion [15] to the recent developments of Kuznetsov [13] for a very large class of stable Lévy processes, an important number of papers have appeared. Most of them concern explicit computations for stable processes and basic features, such as tail behavior of this law, are still unknown in the general case.

The present work is mainly concerned with the study of the nature of the law of the overall supremum $\overline{X}_t$ and, more specifically, with the existence of a density for this distribution. In a recent paper, Bouleau and Denis [5] proved that the law of $\overline{X}_t$ is absolutely continuous whenever the Lévy measure of X is itself absolutely continuous and satisfies some additional conditions; see Proposition 3

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in [5]. This result has raised our interest on the subject, and we propose to determine “exploitable” necessary and sufficient conditions, under which the law of $\overline{X}_t$ is absolutely continuous. Doing so, we also obtained conditions for the absolute continuity of the random vectors $(g_t, \overline{X}_t)$ and $(g_t, \overline{X}_t, X_t)$, where g_t is the time at which the maximum of X occurs on $[0,t]$. The proofs are based on two main ingredients. The first one is the equivalence between the law of X_t in $\mathbb{R}_+$ and the entrance law of the excursions of the reflected process at its minimum; see Lemma 1. The second argument is an expression of the law of $(g_t, \overline{X}_t, X_t)$ in terms of the entrance laws q_t and q_t^* of the excursions of both reflected processes, at the maximum and at the minimum, respectively: if 0 is regular for both half lines $(-\infty, 0)$ and $(0,\infty)$, then

$$\mathbb{P}(g_t \in ds, \overline{X}_t \in dx, \overline{X}_t - X_t \in dy) = q_s^*(dx)q_{t-s}(dy)\mathbb{1}_{[0,t]}(s)\,ds.$$

This expression is extended to the nonregular cases in Theorem 6. As another application, we may recover the law of the triplet $(g_t, \overline{X}_t, X_t)$ for Brownian motion with drift and derive an explicit form of this law, for the symmetric Cauchy process. The law of $(g_t, \overline{X}_t)$, may also be computed in some instances of spectrally negative Lévy processes.

The remainder of this paper is organized as follows. In Section 2, we give some definitions and we recall some basic elements of excursion theory and fluctuation theory for Lévy processes, which are necessary for the proofs. The main results of the paper are stated in Sections 3 and 4. In Section 3, we state continuity properties of the triple $(g_t, \overline{X}_t, X_t)$, whereas Section 4 is devoted to some representations and explicit expressions for the law of this triple. Then the proofs of the results are postponed to Section 5.

2. Preliminaries. We denote by $\mathcal{D}$ the space of càdlàg paths $\omega : [0,\infty) \to \mathbb{R} \cup \{\infty\}$ with lifetime $\zeta(\omega) = \inf\{t \ge 0 : \omega_t = \omega_s, \forall s \ge t\}$, with the usual convention that $\inf\{\varnothing\} = +\infty$. The space $\mathcal{D}$ is equipped with the Skorokhod topology, its Borel σ-algebra $\mathcal{F}$ and the usual completed filtration $(\mathcal{F}_s, s \ge 0)$, generated by the coordinate process $X = (X_t, t \ge 0)$ on the space $\mathcal{D}$. We write $\overline{X}$ and $\underline{X}$ for the supremum and infimum processes,

$$\overline{X}_t = \sup\{X_s : 0 \le s \le t\} \quad \text{and} \quad \underline{X}_t = \inf\{X_s : 0 \le s \le t\}.$$

For $t > 0$, the last passage times by X at its supremum and at its infimum before t are, respectively, defined by

$$g_t = \sup\{s \le t : X_s = \overline{X}_t \text{ or } X_{s-} = \overline{X}_t\} \quad \text{and}$$
$$g_t^* = \sup\{s \le t : X_s = \underline{X}_t \text{ or } X_{s-} = \underline{X}_t\}.$$

We also define the first passage time by X in the open halfline $(0,\infty)$ by

$$\tau_0^+ = \inf\{t \ge 0 : X_t > 0\}.$$

We denote by $\mathbb{P}$ the law on $\mathcal{D}$ of a Lévy process starting from 0. When $(X, \mathbb{P})$ or $(-X, \mathbb{P})$ is a subordinator, the past supremum at time t corresponds to the value X_t or 0, respectively. So these cases will be excluded in the sequel. Besides, the technics which are used in this paper are not quite adapted to the case of compound Poisson processes which will be treated apart, in Theorem 4. So unless explicitly mentioned, in the sequel, we assume that X is not a compound Poisson process.

Note that under our assumptions, 0 is always regular for $(-\infty, 0)$ or/and $(0, \infty)$. It is well known that the reflected processes $\overline{X} - X$ and $X - \underline{X}$ are strong Markov processes. Under $\mathbb{P}$, the state 0 is regular for $(0, \infty)$ [resp., for $(-\infty, 0)$] if and only if it is regular for $\{0\}$, for the reflected process $\overline{X} - X$ (resp., for $X - \underline{X}$). If 0 is regular for $(0, \infty)$, then the local time at 0 of the reflected process $\overline{X} - X$ is the unique continuous, increasing, additive functional L with $L_0 = 0$, a.s., such that the support of the measure dL_t is the set $\overline{\{t : \overline{X}_t = X_t\}}$ and which is normalized by

$$\mathbb{E}\left(\int_0^\infty e^{-t}\, dL_t\right) = 1. \tag{2.1}$$

Let G be the set of the left endpoints of the excursions away from 0 of $\overline{X} - X$, and for each $s \in G$, call ε^s the excursion which starts at s. Denote by E the set of excursions, that is, $E = \{\omega \in \mathcal{D} : \omega_t > 0, \text{for all } 0 < t < \zeta(\omega)\}$, and let $\mathcal{E}$ be the Borel σ-algebra which is the trace of $\mathcal{F}$ on the subset E of $\mathcal{D}$. The Itô measure n of the excursions away from 0 of the process $\overline{X} - X$ is characterized by the so-called *compensation formula*,

$$\mathbb{E}\left(\sum_{s\in G} F(s, \omega, \varepsilon^s)\right) = \mathbb{E}\left(\int_0^\infty dL_s\left(\int F(s, \omega, \varepsilon) n(d\varepsilon)\right)\right), \tag{2.2}$$

which is valid whenever F is a positive and predictable process, that is, $\mathcal{P}(\mathcal{F}_s) \otimes \mathcal{E}$-measurable, where $\mathcal{P}(\mathcal{F}_s)$ is the predictable σ-algebra associated to the filtration $(\mathcal{F}_s)$. We refer to [3], Chapter IV, [14], Chapter 6 and [8] for more detailed definitions and some constructions of L and n.

If 0 is not regular for $(0, \infty)$, then the set $\{t : (\overline{X} - X)_t = 0\}$ is discrete, and following [3] and [14], we define the local time L of $\overline{X} - X$ at 0 by

$$L_t = \sum_{k=0}^{R_t} \mathbf{e}^{(k)}, \tag{2.3}$$

where $R_t = \text{Card}\{s \in (0, t] : \overline{X}_s = X_s\}$, and $\mathbf{e}^{(k)}$, $k = 0, 1, \dots$ is a sequence of independent and exponentially distributed random variables with parameter

$$\gamma = \left(1 - \mathbb{E}(e^{-\tau_0^+})\right)^{-1}. \tag{2.4}$$

In this case, the measure n of the excursions away from 0 is proportional to the distribution of the process X under the law $\mathbb{P}$, returned at its first passage time in the positive halfline. More formally, let us define $\varepsilon^{\tau_0^+} = (-X_s, 0 \le s < \tau_0^+)$, then for any bounded Borel functional K on $\mathcal{E}$,

$$\int_{\mathcal{E}} K(\varepsilon) n(d\varepsilon) = \gamma \mathbb{E}[K(\varepsilon^{\tau_0^+})]. \tag{2.5}$$

Define G and ε^s as in the regular case. Then from definitions (2.3), (2.5) and an application of the strong Markov property, we may check that the normalization (2.1) and the compensation formula (2.2) are still valid in this case.

The local time at 0 of the reflected process at its infimum $X - \underline{X}$, and the measure of its excursions away from 0 are defined in the same way as for $\overline{X} - X$. They are respectively denoted by L^* and n^*. Then the ladder time processes τ and τ^*, and the ladder height processes H and H^* are the following (possibly killed) subordinators:

$$\tau_t = \inf\{s : L_s > t\}, \qquad \tau_t^* = \inf\{s : L_s^* > t\},$$
$$H_t = X_{\tau_t}, \qquad H_t^* = -X_{\tau_t^*}, \qquad t \ge 0,$$

where $\tau_t = H_t = +\infty$, for $t \ge \zeta(\tau) = \zeta(H)$ and $\tau_t^* = H_t^* = +\infty$, for $t \ge \zeta(\tau^*) = \zeta(H^*)$. The characteristic exponent κ of the ladder process (τ, H) may be defined by

$$\mathbb{E}\left(\int_0^\infty dL_t e^{-qt} \exp(-\alpha t - \beta \overline{X}_t)\right) = \frac{1}{\kappa(q+\alpha, \beta)}, \qquad q > 0, \alpha, \beta \ge 0.$$

From (2.1), we derive that $\kappa(1,0) = \kappa^*(1,0) = 1$, so that the Wiener–Hopf factorization in time (which is stated in [3], page 166 and in [14], page 166) is normalized as follows:

$$\kappa(\alpha, 0)\kappa^*(\alpha, 0) = \alpha, \qquad \text{for all } \alpha \ge 0. \tag{2.6}$$

Recall also that the drifts $\mathtt{d}$ and $\mathtt{d}^*$ of the subordinators τ and τ^* satisfy $\mathtt{d} = 0$ (resp., $\mathtt{d}^* = 0$) if and only if 0 is regular for $(-\infty, 0)$ [resp., for $(0, \infty)$], and that

$$\int_0^t \mathbb{1}_{\{X_s = \overline{X}_s\}} ds = \mathtt{d} L_t \quad \text{and} \quad \int_0^t \mathbb{1}_{\{X_s = \underline{X}_s\}} ds = \mathtt{d}^* L_t^*. \tag{2.7}$$

Suppose that 0 is not regular for $(0, \infty)$, and let $\mathbf{e}$ be an independent exponential time with mean 1, then from (2.1) and (2.7), $\mathbb{P}((X - \underline{X})_{\mathbf{e}} = 0) = \mathtt{d}^*$. From the time reversal property of Lévy processes, $\mathbb{P}((X - \underline{X})_{\mathbf{e}} = 0) = \mathbb{P}(\overline{X}_{\mathbf{e}} = 0) = \mathbb{P}(\tau_0^+ \ge \mathbf{e}) = \gamma^{-1}$, so that $\mathtt{d}^* = \gamma^{-1}$.

We will denote by q_t^* and q_t the entrance laws of the reflected excursions at the maximum and at the minimum, that is, for $t > 0$,

$$q_t(dx) = n(X_t \in dx, t < \zeta) \quad \text{and} \quad q_t^*(dx) = n^*(X_t \in dx, t < \zeta).$$

They will be considered as measures on $\mathbb{R}_+ = [0, \infty)$. Recall that the law of the lifetime of the reflected excursions is related to the Lévy measure of the ladder time processes, through the equalities

$$
\begin{aligned}
q_t(\mathbb{R}_+) &= n(t < \zeta) = \overline{\pi}(t) + a \quad \text{and} \\
q_t^*(\mathbb{R}_+) &= n^*(t < \zeta) = \overline{\pi}^*(t) + a^*,
\end{aligned}
\tag{2.8}
$$

where $\overline{\pi}(t) = \pi(t, \infty)$ and $\overline{\pi}^*(t) = \pi^*(t, \infty)$ and a, a^* are the killing rates of the subordinators τ and τ^*.

In this paper, we will sometimes write $\mu \ll \nu$, when μ is absolutely continuous with respect to ν. We will say that μ and ν are *equivalent* if $\mu \ll \nu$ and $\nu \ll \mu$. We will denote by λ the Lebesgue measure on $\mathbb{R}$. A measure which is absolutely continuous with respect to the Lebesgue measure will sometimes be called *absolutely continuous*. A measure which has no atoms will be called *continuous*.

3. Continuity properties of the law of $(g_t, \overline{X}_t, X_t)$. In this section, X is any Lévy process such that $|X|$ is not subordinator, and except in Theorem 4, we assume that X is not a compound Poisson process.

For $t > 0$ and $q > 0$, we will denote, respectively by $p_t(dx)$ and $U_q(dx)$, the semigroup and the resolvent measure of X, that is, for any positive Borel function f,

$$
\mathbb{E}(f(X_t)) = \int_0^\infty f(x) p_t(dx) \quad \text{and}
$$
$$
\int_0^\infty f(x) U_q(dx) = \mathbb{E}\left(\int_0^\infty e^{-qt} f(X_t)\, dt\right).
$$

Since $U_q(A) = 0$ if and only if $\mathbb{P}(X_t \in A) = 0$, for λ almost every t, it follows that for all q and q', the resolvent measures $U_q(dx)$ and $U_{q'}(dx)$ are equivalent. For the same reason, each measure U_q is equivalent to the potential measure $U_0(dx) = \int_0^\infty \mathbb{P}(X_t \in dx)\, dt$. In what follows, when comparing the law of $\overline{X}_t$ to the measures U_q, $q \ge 0$, we will take $U(dx) \stackrel{\text{def}}{=} U_1(dx)$ as a reference measure. We will say that a Lévy process X is of:

- type 1 if 0 is regular for both $(-\infty, 0)$ and $(0, \infty)$;
- type 2 if 0 is not regular for $(-\infty, 0)$;
- type 3 if 0 is not regular for $(0, \infty)$.

We emphasize that since X is not a compound Poisson process, types 1, 2 and 3 define three exhaustive cases. Recall that $\mathbb{R}_+ = [0, \infty)$, and let $\mathcal{B}_{\mathbb{R}_+}$ be the Borel σ-field on $\mathbb{R}_+$. For $t > 0$, let μ_t^+ be the restriction to $(\mathbb{R}_+, \mathcal{B}_{\mathbb{R}_+})$ of the average occupation measure of X, on the time interval $[0, t)$, that is,

$$
\int_{[0,\infty)} f(x) \mu_t^+(dx) = \mathbb{E}\left(\int_0^t f(X_s)\, ds\right),
$$

for every nonnegative Borel function f on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$, such that $f \equiv 0$ on $(-\infty, 0)$. Moreover, we will denote by $p_t^+(dx)$ the restriction of the semigroup $p_t(dx)$ to $(\mathbb{R}_+, \mathcal{B}_{\mathbb{R}_+})$. In particular, we have $\mu_t^+ = \int_0^t p_s^+ \, ds$. The law of $\overline{X}_t$ will be considered as a measure on $(\mathbb{R}_+, \mathcal{B}_{\mathbb{R}_+})$. In all the remainder of this article, we assume that the time t is deterministic and finite.

THEOREM 1. *For $t > 0$, the law of the past supremum $\overline{X}_t$ can be compared to the occupation measure μ_t^+ as follows*:

(1) *If X is of type* 1, *then for all $t > 0$, the law of $\overline{X}_t$ is equivalent to μ_t^+.*
(2) *If X is of type* 2, *then for all $t > 0$, the law of $\overline{X}_t$ is equivalent to $p_t^+(dx) + \mu_t^+(dx)$.*
(3) *If X is of type* 3, *then for all $t > 0$, the law of $\overline{X}_t$ has an atom at* 0 *and its restriction to the open halfline* $(0, \infty)$ *is equivalent to the restriction of the measure $\mu_t^+(dx)$ to* $(0, \infty)$.

It appears clearly from this theorem that the law of $\overline{X}_t$ is absolutely continuous for all $t > 0$, whenever 0 is regular for $(0, \infty)$ and p_t is absolutely continuous, for all $t > 0$. We will see in Theorem 3 that a stronger result actually holds. Let $U^+(dx)$ be the restriction to $(\mathbb{R}_+, \mathcal{B}_{\mathbb{R}_+})$ of the resolvent measure $U(dx)$. Since μ_t^+ is absolutely continuous with respect to U^+ for all $t > 0$, the law of the past supremum before t can be compared to U^+ as follows.

COROLLARY 1. *Under the same assumptions as in Theorem* 1:

(1) *If X is of type* 1, *then for any $t > 0$, the law of $\overline{X}_t$ is absolutely continuous with respect to the resolvent measure $U^+(dx)$.*
(2) *If X is of type* 2, *then for any $t > 0$, the law of $\overline{X}_t$ is absolutely continuous with respect to the measure $p_t^+(dx) + U^+(dx)$.*
(3) *If X is of type* 3, *then the same conclusions as in* 1. *hold for the measures restricted to* $(0, \infty)$.

Whenever X is not a compound Poisson process, the resolvent measure $U^+(dx)$ is continuous; see Proposition I.15 in [3]. Moreover, the measure $p_t^+(dx)$ is also continuous for all $t > 0$; see Theorem 27.4 in Sato [18]. Hence from Corollary 1, for all $t > 0$, when X is of type 1 or 2, the law of $\overline{X}_t$ is continuous, and when it is of type 3, this law has only one atom at 0. This fact has already been observed in [17], Lemma 1.

It is known that for a Lévy process X, the law of X_t may be absolutely continuous for all $t > t_0$, whereas it is continuous singular for $t \in (0, t_0)$; see Theorem 27.23 and Remark 27.24 in [18]. The following theorem shows that when X is of type 1, this phenomenon cannot happen for the law of the supremum, that is, either absolute continuity of the law of $\overline{X}_t$ holds at any time t or it never holds.

We denote by $V(dt, dx)$ the potential measure of the ladder process (τ, H) and by $V(dx)$ the potential measure of the ladder height process H, that is,

$$V(dt, dx) = \int_0^\infty \mathbb{P}(\tau_s \in dt, H_s \in dx)\, ds \quad \text{and} \quad V(dx) = \int_0^\infty \mathbb{P}(H_s \in dx)\, ds.$$

Then let λ^+ be the Lebesgue measure on $\mathbb{R}_+$.

THEOREM 2. *Suppose that X is of type* 1. *The following assertions are equivalent*:

(1) *The law of $\overline{X}_t$ is absolutely continuous with respect to λ^+, for all $t > 0$.*
(2) *The law of $\overline{X}_t$ is absolutely continuous with respect to λ^+, for some $t > 0$.*
(3) *The resolvent measure $U^+(dx)$ is absolutely continuous with respect to λ^+.*
(4) *The resolvent measure $U(dx)$ is absolutely continuous with respect to λ.*
(5) *The potential measure $V(dx)$ is absolutely continuous with respect to λ^+.*

Moreover assertions 1–5 *are equivalent to the same assertions formulated for the dual process $-X$. In particular,* 1–5 *hold if and only if the law of $-\underline{X}_t$ is absolutely continuous with respect to λ^+, for all $t > 0$.*

Condition 4 of the above theorem is satisfied whenever the drift coefficient of the subordinator H is positive; see Theorem II.16 and Corollary II.20 in [3]. Let us also mention that necessary and sufficient conditions for $U(dx)$ to be absolutely continuous may be found in Theorem 41.15 of [18], and in Proposition 10, Chapter I of [3]. Formally, $U \ll \lambda$ if and only if for some $q > 0$ and for all bounded Borel function f, the function $x \mapsto \mathbb{E}_x(\int_0^\infty f(X_t)e^{-qt}\, dt)$ is continuous. However, we do not know any necessary and sufficient conditions bearing directly on the characteristic exponent ψ of X. Let us simply recall the following sufficient condition. From Theorem II.16 in [3], if

$$\int_{-\infty}^{\infty} \Re\left(\frac{1}{1+\psi(x)}\right) dx < \infty, \tag{3.1}$$

then $U(dx) \ll \lambda$, with a bounded density. Therefore, if X is of type 1, then from Theorem 2, condition (3.1) implies that both the laws of $\underline{X}_t$ and $\overline{X}_t$ are absolutely continuous for all $t > 0$.

A famous result from [11] asserts that when X is a symmetric process, condition $U \ll \lambda$ implies that $p_t \ll \lambda$, for all $t > 0$. Then it follows from Theorem 2 that in this particular case, absolute continuity of the law of $\overline{X}_t$, for some $t > 0$ (hence for all $t > 0$) is equivalent to the absolute continuity of the semigroup p_t, for all $t > 0$.

THEOREM 3. *If* 0 *is regular for* $(0, \infty)$, *then the following assertions are equivalent*:

(1) *The measures p_t^+ are absolutely continuous with respect to λ^+, for all $t > 0$.*

(2) *The measures* p_t *are absolutely continuous with respect to* λ, *for all* $t > 0$.
(3) *The potential measure* $V(dt, dx)$ *is absolutely continuous with respect to the Lebesgue measure on* $\mathbb{R}_+^2$.

If moreover X *is of type* 1, *then each of the following assertions is equivalent to* 1–3:

(4) *The law of* $(g_t, \overline{X}_t)$ *is absolutely continuous with respect to the Lebesgue measure on* $[0, t] \times \mathbb{R}_+$, *for all* $t > 0$.
(5) *The law of* $(g_t, \overline{X}_t, X_t)$ *is absolutely continuous with respect to the Lebesgue measure on* $[0, t] \times \mathbb{R}_+ \times \mathbb{R}$, *for all* $t > 0$.

We may wonder if the equivalence between assertions (1) and (2) of Theorem 3 still holds when t is fixed, that is, when 0 is regular for $(0, \infty)$, does the condition $p_t^+ \ll \lambda^+$, imply that $p_t \ll \lambda$? A counterexample in the case where 0 is not regular for $(0, \infty)$ may easily be found. Take for instance, $X_t = Y_t - S_t$, where Y is a compound Poisson process with absolutely continuous Lévy measure, and S is a subordinator independent of Y, whose law at time $t > 0$ is continuous singular. Then clearly $p_t^+ \ll \lambda^+$, and there exists a Borel set $A \subset (-\infty, 0)$, such that $\lambda(A) = 0$ and $\mathbb{P}(-S_t \in A) > 0$, so that $p_t(A) > \mathbb{P}(Y_t = 0)\mathbb{P}(S_t \in A) > 0$.

Let Y be a càdlàg stochastic process such that $Y_0 = 0$, a.s. We say that Y is an *elementary process* if there is an increasing sequence (T_n) of nonnegative random variables, such that $T_0 = 0$ and $\lim_{n\to+\infty} T_n = +\infty$, a.s. and two sequences of finite real-valued random variables $(a_n, n \geq 0)$ and $(b_n, n \geq 0)$ such that $b_0 = 0$ and

$$Y_t = a_n t + b_n \qquad \text{if } t \in [T_n, T_{n+1}). \tag{3.2}$$

We say that Y is a *step process* if it is an elementary process with $a_n = 0$, for all n in the above definition.

PROPOSITION 1. *Suppose that* 0 *is regular for* $(0, \infty)$.

(1) *If* 0 *is regular for* $(-\infty, 0)$, *and if the law of* $\overline{X}_t$ *is absolutely continuous for some* $t > 0$, *then for any step process* Y *which is independent of* X, *the law of* $\sup_{s\leq t}(X + Y)_s$ *is absolutely continuous for all* $t > 0$.
(2) *If* $p_t^+ \ll \lambda^+$, *for all* $t > 0$, *or if* X *has unbounded variation*, *and if at least one of the ladder height processes* H *and* H^* *has a positive drift*, *then for any elementary stochastic process* Y *which is independent of* X, *the law of* $\sup_{s\leq t}(X + Y)_s$ *is absolutely continuous for all* $t > 0$.

Sufficient conditions for the absolute continuity of the semigroup may be found in Chapter 5 of [18] and in Section 5 of [12]. In particular if $\Pi(\mathbb{R}) = \infty$ and $\Pi \ll \lambda$, then $p_t \ll \lambda$ for all $t > 0$. Proposition 20 in Bouleau and Denis [5] asserts that under a slight reinforcement of this condition, for any independent càdlàg process Y, the law of $\sup_{s\le t}(X+Y)_s$ is absolutely continuous, provided it has no atom at 0. In the particular case where Y is an elementary process, this result is a consequence of part 2 of Proposition 1.

In view of Theorems 2 and 3, it is natural to look for instances of Lévy processes of type 1 such that the law of $\overline{X}_t$ is absolutely continuous, whereas $p_t(dx)$ is not, as well as instances of Lévy processes of type 1 such that the law of $\overline{X}_t$ is not absolutely continuous. The following corollary is inspired from Orey's example [16]; see also [18], Exercise 29.12 and Example 41.23.

COROLLARY 2. *Let X be a Lévy process whose characteristic exponent ψ, that is, $\mathbb{E}(e^{i\lambda X_t}) = e^{-t\psi(\lambda)}$ is given by*

$$\psi(\lambda) = \int_{\mathbb{R}} \big(1 - e^{i\lambda x} + i\lambda x \mathbb{1}_{\{|x|<1\}}\big)\Pi(dx).$$

Let $\alpha \in (1,2)$ and c be an integer such that $c > 2/(2-\alpha)$, and set $a_n = 2^{-c^n}$.

(1) *If $\Pi(dx) = \sum_{n=1}^{\infty} a_n^{-\alpha}\delta_{-a_n}(dx)$, then X is of type 1, and for all $t>0$, the law of $\overline{X}_t$ is absolutely continuous, whereas $p_t(dx)$ is continuous singular.*
(2) *If $\Pi(dx) = \sum_{n=1}^{\infty} a_n^{-\alpha}(\delta_{-a_n}(dx) + \delta_{a_n}(dx))$, then X is of type 1, and for all $t>0$, the law $\overline{X}_t$ is not absolutely continuous.*

We end this section with the case of compound Poisson processes. Recall that any such process can be expressed as:

$$X_t = S_{N_t}, \qquad t \ge 0,$$

where $S_0 = 0$, $S_n = \sum_{k=1}^{n} X_k$, $n \ge 1$, $(X_k)_{k\ge1}$ are i.i.d. random variables, and $(N_t)_{t\ge0}$ is a Poisson process with any intensity, which is independent from the sequence $(X_k)_{k\ge1}$. We keep the same notation for the measures p_t^+, μ_t^+ and U^+, which are defined with respect to X, as before. We denote by υ^+ the restriction to $(\mathbb{R}_+, \mathcal{B}_{\mathbb{R}_+})$ of the potential measure of the random walk $(S_n)_{n\ge0}$, that is,

$$\upsilon^+(A) = \sum_{n=0}^{\infty} \mathbb{P}(S_n \in A), \qquad A \in \mathcal{B}_{\mathbb{R}_+}.$$

THEOREM 4. *Let X be a compound Poisson process. Then for all $t>0$, the measures*

$$\mathbb{P}(\overline{X}_t \in dx), \qquad p_t^+(dx), \qquad \mu_t^+(dx), \qquad U^+(dx) \quad \textit{and} \quad \upsilon^+(dx)$$

are equivalent.

As a consequence, when X is a compound Poisson process, for any $t>0$ and $t'>0$, the laws of $\overline{X}_t$ and $\overline{X}_{t'}$ are equivalent. This question is still open in the general case.

4. An expression for the joint law of $(g_t, \overline{X}_t, X_t)$. In this section, we assume that $|X|$ is not subordinator and that X is not a compound Poisson process.

The following theorem presents a path decomposition of the Lévy process X, over the interval $[0, t]$, at time g_t. More specifically, it states that conditionally on $g_t = s$, the returned pre-g_t part and the post-g_t part are distributed according to the laws $n^*(\cdot|s < \zeta)$ and $n(\cdot|t - s < \zeta)$, respectively. Actually, we will essentially focus on its corollaries which provide some representations of the joint law of g_t, $\overline{X}_t$ and X_t, at a fixed time t, in terms of the entrance laws (q_s) and (q_s^*). Besides they will be applied in Section 5 for the proofs of the results of Section 3.

For $\omega \in \mathcal{D}$ and $s \ge 0$, we set $\Delta_s^{\pm}(\omega) = (\omega_s - \omega_{s-})^{\pm}$, where $\omega_{0-} = \omega_0$. Then we define the (special) *shift* operator by

$$\theta_s(\omega) = \big(\omega_{s-} - \omega_{s+u} + \Delta_s^+(\omega), u \ge 0\big).$$

The *killing* operator and the *return* operator are respectively defined as follows:

$$k_s(\omega) = \begin{cases} \omega_u, & 0 \le u < s, \\ \omega_s, & u \ge s, \end{cases}$$

$$r_s(\omega) = \begin{cases} \omega_s - \omega_{(s-u)-} - \Delta_s^-(\omega), & 0 \le u < s, \\ \omega_s - \omega_0 - \Delta_s^-(\omega), & u \ge s. \end{cases}$$

We also denote by ω^0 the path which is identically equal to 0.

THEOREM 5. *Fix $t > 0$, let f be any bounded Borel function and let F and K be any bounded Borel functionals which are defined on the space $\mathcal{D}$.*

(1) *If X is of type* 1, *then*

$$\begin{aligned} &\mathbb{E}\big(f(g_t) \cdot F \circ r_{g_t} \cdot K \circ k_{t-g_t} \circ \theta_{g_t}\big) \\ &\quad = \int_0^t f(s) n^*(F \circ k_s, s < \zeta) n(K \circ k_{t-s}, t - s < \zeta)\, ds. \end{aligned} \tag{4.1}$$

(2) *If X is of type* 2, *then*

$$\begin{aligned} &\mathbb{E}\big(f(g_t) \cdot F \circ r_{g_t} \cdot K \circ k_{t-g_t} \circ \theta_{g_t}\big) \\ &\quad = \int_0^t f(s) n^*(F \circ k_s, s < \zeta) n(K \circ k_{t-s}, t - s < \zeta)\, ds \\ &\qquad + \mathtt{d} f(t) n^*(F \circ k_t, t < \zeta) K(\omega^0). \end{aligned} \tag{4.2}$$

(3) *If X is of type* 3, *then*

$$\begin{aligned} &\mathbb{E}\big(f(g_t) \cdot F \circ r_{g_t} \cdot K \circ k_{t-g_t} \circ \theta_{g_t}\big) \\ &\quad = \int_0^t f(s) n^*(F \circ k_s, s < \zeta) n(K \circ k_{t-s}, t - s < \zeta)\, ds \\ &\qquad + \mathtt{d}^* f(0) F(\omega^0) n(K \circ k_t, t < \zeta). \end{aligned} \tag{4.3}$$

Simultaneously to our work, a similar path decomposition has been obtained in [19], when X is of type 1. In the later work, the post-g_t part of $(X_s, 0 \le s \le t)$ is expressed in terms of the law of the meander, that is, $\mathbb{M}^{(t)} = n(\cdot | t < \zeta)$; see Theorem 5.1.

By applying Theorem 5 to the joint law of g_t, together with the terminal values of the pre-g_t and the post-g_t parts of $(X_s, 0 \le s \le t)$, we obtain the following representation for the law of the triple: $(g_t, \overline{X}_t, X_t)$. Moreover, when $\lim_{t\to\infty} X_t = -\infty$, a.s., we define $\overline{X}_\infty = \sup_t X_t$, the overall supremum of X and $g_\infty = \sup\{t : X_t = \overline{X}_\infty \text{ or } X_{t-} = \overline{X}_\infty\}$, the location of this supremum. Then we obtain the same kind of representation for $(g_\infty, \overline{X}_\infty)$. We emphasize that in the next result, as well as in Corollaries 3 and 4, at least one of the drift coefficients $\mathtt{d}$ and $\mathtt{d}^*$ is zero.

THEOREM 6. *The law of* $(g_t, \overline{X}_t, X_t)$ *fulfills the following representation*:

$$\begin{aligned}
&\mathbb{P}(g_t \in ds, \overline{X}_t \in dx, \overline{X}_t - X_t \in dy) \\
&\quad = q_s^*(dx) q_{t-s}(dy) \mathbb{1}_{[0,t]}(s)\, ds + \mathtt{d}\delta_{\{t\}}(ds) q_t^*(dx) \delta_{\{0\}}(dy) \\
&\qquad + \mathtt{d}^* \delta_{\{0\}}(ds) \delta_{\{0\}}(dx) q_t(dy).
\end{aligned} \tag{4.4}$$

If moreover $\lim_{t\to\infty} X_t = -\infty$, *a.s., then*

$$\mathbb{P}(g_\infty \in ds, \overline{X}_\infty \in dx) = a q_s^*(dx)\, ds + \mathtt{d}^* a \delta_{\{(0,0)\}}(ds, dx), \tag{4.5}$$

where a *is the killing rate of the ladder time process* τ.

We derive from Theorem 6 that when X is of type 1, the law of the time g_t is equivalent to the Lebesgue measure, with density $s \mapsto n^*(s < \zeta) n(t - s < \zeta) \mathbb{1}_{[0,t]}(s)$. This theorem illustrates the importance of the entrance laws q_t and q_t^* for the computation of some distributions involved in fluctuation theory. We give below a couple of examples where some explicit forms can be obtained for q_t, q_t^* and the law of $(g_t, \overline{X}_t, X_t)$. When $q_t(dx) \ll \lambda^+$ [resp., $q_t^*(dx) \ll \lambda^+$], we will denote by $q_t(x)$ [resp., $q_t^*(x)$] the density of $q_t(dx)$ [resp., $q_t^*(dx)$].

EXAMPLE 1. Suppose that X is a Brownian motion with drift, that is, $X_t = B_t + ct$, where B is the standard Brownian motion and $c \in \mathbb{R}$. We derive for instance from Lemma 1 in Section 5 that

$$q_t(dx) = \frac{x}{\sqrt{\pi t^3}} e^{-(x-c)^2/2t}\, dx \quad \text{and} \quad q_t^*(dx) = \frac{x}{\sqrt{\pi t^3}} e^{-(x+c)^2/2t}\, dx.$$

Then expression (4.4) in Theorem 6 allows us to compute the law of the triple $(g_t, \overline{X}_t, X_t)$.

EXAMPLE 2. Recently, the density of the measure $q_t(dx)$ for the symmetric Cauchy process has been computed in [6].

$$\begin{aligned} q_t(x) &= q_t^*(x) \\ &= \sqrt{2}\frac{\sin(\pi/8 + 3/2\arctan(x/t))}{(t^2+x^2)^{3/4}} \\ &\quad - \frac{1}{2\pi}\int_0^\infty \frac{y}{(1+y^2)(xy+t)^{3/2}}\exp\left(-\frac{1}{\pi}\int_0^\infty \frac{\log(y+s)}{1+s^2}\,ds\right)dy. \end{aligned}$$

As far as we know, this example, together with the case of Brownian motion with drift (Example 1), are the only instances of Lévy processes where the measures $q_t(dx)$, $q_t^*(dx)$ and the law of the triplet $(g_t, \overline{X}_t, X_t)$ can be computed explicitly.

EXAMPLE 3. Recall from (2.8) that $q_t(\mathbb{R}_+) = n(t < \zeta)$ and $q_t^*(\mathbb{R}_+) = n^*(t < \zeta)$, so that we can derive from Theorem 6, all possible marginal laws in the triplet $(g_t, \overline{X}_t, X_t)$. In particular, when X is stable, the ladder time process τ also satisfies the scaling property with index $\rho = \mathbb{P}(X_1 \geq 0)$, so we derive from the normalization $\kappa(1, 0) = 1$ and (2.8) that $n(t < \zeta) = t^{-\rho}/\Gamma(1-\rho)$. Moreover q_t^* and q_t are absolutely continuous in this case (it can be derived, e.g., from part 4 of Lemma 1 in the next section). Then a consequence of (4.4) is the following form of the joint law of $(g_t, \overline{X}_t)$:

$$\mathbb{P}(g_t \in ds, \overline{X}_t \in dx) = \frac{(t-s)^{-\rho}}{\Gamma(1-\rho)}\mathbb{1}_{[0,t]}(s)q_s^*(x)\,ds\,dx. \tag{4.6}$$

Note that this computation is implicit in [1]; see Corollary 3 and Theorem 5. A more explicit form is given in (4.13), after Proposition 2, in the case where the process has no positive jumps. Note also that when X is stable, the densities q_t and q_t^* satisfy the scaling properties

$$q_t(y) = t^{-\rho-1/\alpha}q_1(t^{-1/\alpha}y) \quad \text{and} \quad q_t^*(x) = t^{\rho-1-1/\alpha}q_1^*(t^{-1/\alpha}x).$$

These properties together with Theorem 6 imply that the three r.v.s g_t, $\overline{X}_t/g_t^{1/\alpha}$ and $(\overline{X}_t - X_t)/(t-g_t)^{1/\alpha}$ are independent and have densities

$$\frac{\sin(\pi\rho)}{\pi}s^{\rho-1}(t-s)^{-\rho}\mathbb{1}_{[0,t]}(s), \qquad \Gamma(\rho)q_1^*(x) \quad \text{and} \quad \Gamma(1-\rho)q_1(y),$$

respectively. The independence between g_t, $\overline{X}_t/g_t^{1/\alpha}$ and $(\overline{X}_t - X_t)/(t-g_t)^{1/\alpha}$ has recently been proved in Proposition 2.39 of [7].

It is clear that an expression for the law of $\overline{X}_t$ follows directly from Theorem 6 by integrating (4.4) over s and y. However, for convenience in the proofs of Section 5, we write it here in a proper statement. An equivalent version of Corollary 3 may also be found in [10], Lemma 6.

COROLLARY 3. *The law of $\overline{X}_t$ fulfills the following representation*:

$$\mathbb{P}(\overline{X}_t \in dx) = \int_0^t n(t-s<\zeta) q_s^*(dx)\,ds + \mathrm{d}q_t^*(dx) + \mathrm{d}^* n(t<\zeta)\delta_{\{0\}}(dx). \tag{4.7}$$

Another remarkable, and later useful, direct consequence of Theorem 6 is the following representation of the semigroup of X in terms of the entrance laws (q_s) and (q_s^*).

COROLLARY 4. *Let us denote the measure $q_t(-dx)$ by $\overline{q}_t(dx)$. We extend the measures $\overline{q}_t(dx)$ and $q_t^*(dx)$ to $\mathbb{R}$ by setting $\overline{q}_t(A) = \overline{q}_t(A \cap \mathbb{R}_-)$ and $q_t^*(A) = q_t^*(A \cap \mathbb{R}_+)$, for any Borel set $A \subset \mathbb{R}$. Then we have the following identity between measures on $\mathbb{R}$*:

$$p_t = \int_0^t \overline{q}_s * q_{t-s}^*\,ds + \mathrm{d}q_t^* + \mathrm{d}^*\overline{q}_t. \tag{4.8}$$

Now we turn to the particular case where X has no positive jumps. Then, 0 is always regular for $(0,\infty)$. When moreover 0 is regular for $(-\infty,0)$, since $H_t \equiv t$, it follows from Theorem 2 and the remark thereafter that the law of $\overline{X}_t$ is absolutely continuous. In the next result, we present an explicit form of its density. We set $c = \Phi(1)$, where Φ is the Laplace exponent of the first passage process $T_x = \inf\{t : X_t > x\}$, which in this case, is related to the ladder time process by $T_x = \tau_{cx}$.

PROPOSITION 2. *Suppose that the Lévy process X has no positive jumps.*

(1) *If* 0 *is regular for* $(-\infty,0)$, *then for* $t>0$, *the couple* $(g_t, \overline{X}_t)$ *has law*

$$\mathbb{P}(g_t \in ds, \overline{X}_t \in dx) = cxp_s^+(dx) n(t-s<\zeta) s^{-1} \mathbb{1}_{(0,t]}(s)\,ds \tag{4.9}$$

$$= cn(t-s<\zeta)\mathbb{1}_{(0,t]}(s)\mathbb{P}(\tau_{cx} \in ds)\,dx. \tag{4.10}$$

In particular, the density of the law of $\overline{X}_t$ is given by the function

$$x \mapsto \int_0^t cn(t-s<\zeta)\mathbb{P}(\tau_{cx} \in ds).$$

(2) *If* 0 *is not regular for* $(-\infty,0)$, *then for all* $t>0$,

$$\begin{aligned}\mathbb{P}(g_t \in ds, \overline{X}_t \in dx) &= cxn(t-s<\zeta)s^{-1}\mathbb{1}_{(0,t]}(s) p_s^+(dx)\,ds \\ &\quad + \mathrm{d}cxt^{-1} p_t^+(dx)\delta_{\{t\}}(ds).\end{aligned} \tag{4.11}$$

Moreover, we have the following identity between measures on $[0,\infty)^3$:

$$\begin{aligned}\mathbb{P}(g_t \in ds, \overline{X}_t \in dx)\,dt &= cn(t-s<\zeta)\mathbb{1}_{(0,t]}(s)\mathbb{P}(\tau_{cx} \in ds)\,dx\,dt \\ &\quad + \mathrm{d}c\mathbb{P}(\tau_{cx} \in dt)\delta_{\{t\}}(ds)\,dx.\end{aligned} \tag{4.12}$$

EXAMPLE 4. Using the series development (14.30), page 88 in [18] for $p_s^+(dx)$, we derive from (4.9) in Proposition 2, the following reinforcement of expression (4.6). When X is stable and spectrally negative, the density of $(g_t, \overline{X}_t)$ is given by

$$\frac{c}{\pi\Gamma((\alpha-1)/\alpha)(t-s)^{1/\alpha}}\sum_{n=1}^{\infty}\frac{\Gamma(1+n/\alpha)}{n!}\sin\left(\frac{\pi n}{\alpha}\right)s^{-(n+\alpha)/\alpha}x^n, \tag{4.13}$$
$$s\in[0,t], x\ge 0,$$

which completes Proposition 1, page 282 in [4].

We end this section with a remark on the existence of a density with respect to the Lebesgue measure, for the law of the local time of general Markov processes. From (4.12), we derive that $\mathbb{P}(\tau_x \ge t)\,dt = \int_0^x \int_{(0,t]} n^*(t-s<\zeta)\mathbb{P}(\tau_y \in ds)\,dy\,dt + \mathrm{d}\mathbb{P}(\tau_x \in dt)$. Actually, this identity may be generalized to any subordinator S with drift b, killing rate k and Lévy measure ν. Set $\bar{\nu}(t) = \nu(t,\infty)+k$, then the characteristic exponent Φ of S is given by

$$\Phi(\alpha) = \alpha b + \alpha\int_0^\infty e^{-\alpha t}\bar{\nu}(t)\,dt,$$

from which and Fubini theorem, we derive that for all $x \ge 0$ and $\alpha > 0$,

$$\frac{1}{\alpha}\mathbb{E}(1-e^{-\alpha S_x}) = \left(b+\int_0^\infty e^{-\alpha t}\bar{\nu}(t)\,dt\right)\frac{\mathbb{E}(1-e^{-\alpha S_x})}{\Phi(\alpha)},$$
$$\int_0^\infty e^{-\alpha t}\mathbb{P}(S_x > t)\,dt = \left(b+\int_0^\infty e^{-\alpha t}\bar{\nu}(t)\,dt\right)\int_0^\infty e^{-\alpha t}\int_0^x \mathbb{P}(S_y \in dt)\,dy.$$

Inverting the Laplace transforms on both sides of this identity gives for all $x \ge 0$, the following identity between measures:

$$\mathbb{P}(S_x > t)\,dt = \int_0^x\int_{(0,t]}\bar{\nu}(t-s)\mathbb{P}(S_y \in ds)\,dy\,dt + b\int_0^x \mathbb{P}(S_y \in dt)\,dy.$$

In particular, if S has no drift coefficient, then the law of $L_t \stackrel{\text{def}}{=} \inf\{u : S_u > t\}$ has density

$$\frac{\mathbb{P}(L_t \in dx)}{dx} = \int_{(0,t]}\bar{\nu}(t-s)\mathbb{P}(S_x \in ds).$$

This computation shows that if $a \in \mathbb{R}$ is a regular state of any real Markov process M such that $\int_0^t \mathbb{1}_{\{M_s=a\}}\,ds = 0$, a.s. for all t, then the law of the local time of M, at level a, is absolutely continuous, for any time $t > 0$. This last result is actually a particular case of [9], where it is proved that for any non creeping Lévy process, the law of the first passage time over $x > 0$ is always absolutely continuous.

5. Proofs and further results. We first prove Theorems 5 and 6, since they will be used in the proofs of the results of Section 3.

PROOF OF THEOREM 5. Let $\mathbf{e}$ be an exponential time with parameter $q > 0$ which is independent of $(X, \mathbb{P})$. Recall the notations of Section 2, and for $\omega \in \mathcal{D}$, define $d_s(\omega) = \inf\{u > s : \omega_u = 0\}$, so that $d_s(\overline{X} - X)$ corresponds to the right extremity of the excursion of $\overline{X} - X$, which straddles the time s. From the independence of $\mathbf{e}$ and Fubini theorem, we have for all bounded function f on $\mathbb{R}_+$ and for all bounded Borel functionals F and K on $\mathcal{D}$,

$$\begin{aligned}
&\mathbb{E}\big(f(g_{\mathbf{e}}) F \circ r_{g_{\mathbf{e}}} K \circ k_{\mathbf{e}-g_{\mathbf{e}}} \circ \theta_{g_{\mathbf{e}}}\big)\\
&\quad = \mathbb{E}\left(\int_0^\infty q e^{-qt} f(g_t) F \circ r_{g_t} K \circ k_{t-g_t} \circ \theta_{g_t}\, dt\right)\\
&\quad = \mathbb{E}\left(\sum_{s \in G} q e^{-qs} f(s) F \circ r_s \int_s^{d_s} e^{-q(u-s)} K \circ k_{u-s} \circ \theta_s\, du\right)\\
&\qquad + \mathbb{E}\left(\int_0^\infty q e^{-qt} f(t) F \circ r_t \mathbb{1}_{\{g_t = t\}}\, dt\right) K(\omega^0).
\end{aligned}$$

Recall from Section 2 that ε^s denotes the excursion starting at s. Then

$$\begin{aligned}
&\mathbb{E}\big(f(g_{\mathbf{e}}) F \circ r_{g_{\mathbf{e}}} K \circ k_{\mathbf{e}-g_{\mathbf{e}}} \circ \theta_{g_{\mathbf{e}}}\big)\\
&\quad = \mathbb{E}\left(\sum_{s \in G} q e^{-qs} f(s) F \circ r_s \int_0^{d_s - s} e^{-qu} K(\varepsilon^s \circ k_u)\, du\right)\\
&\qquad + \mathbb{E}\left(\int_0^\infty q e^{-qt} f(t) F \circ r_t \mathbb{1}_{\{X_t = \overline{X}_t\}}\, dt\right) K(\omega^0).
\end{aligned}\tag{5.1}$$

The process

$$(s, \omega, \varepsilon) \mapsto e^{-qs} f(s) F \circ r_s(\omega) \int_0^{\zeta(\varepsilon)} e^{-qu} K \circ k_u(\varepsilon)\, du$$

is $\mathcal{P}(\mathcal{F}_s) \otimes \mathcal{E}$-measurable, so that by applying (2.2) and (2.7) to equality (5.1), we obtain

$$\begin{aligned}
&\frac{1}{q}\mathbb{E}\big(f(g_{\mathbf{e}}) F \circ r_{g_{\mathbf{e}}} K \circ k_{\mathbf{e}-g_{\mathbf{e}}} \circ \theta_{g_{\mathbf{e}}}\big)\\
&\quad = \mathbb{E}\left(\int_0^\infty dL_s e^{-qs} f(s) F \circ r_s\right) n\left(\int_0^\zeta e^{-qu} K \circ k_u\, du\right)\\
&\qquad + \mathrm{d}\mathbb{E}\left(\int_0^\infty dL_s e^{-qs} f(s) F \circ r_s\right) K(\omega^0).
\end{aligned}\tag{5.2}$$

From the time reversal property of Lévy processes (see Lemma 2, page 45 in [3]) under $\mathbb{P}$, we have $X \circ k_e \stackrel{(d)}{=} X \circ r_e$, so that

$$\mathbb{E}\big(f(g_{\mathbf{e}}) F \circ r_{g_{\mathbf{e}}} K \circ k_{\mathbf{e}-g_{\mathbf{e}}} \circ \theta_{g_{\mathbf{e}}}\big) = \mathbb{E}\big(f(\mathbf{e} - g^*_{\mathbf{e}}) K \circ r_{g^*_{\mathbf{e}}} F \circ k_{\mathbf{e}-g^*_{\mathbf{e}}} \circ \theta_{g^*_{\mathbf{e}}}\big).\tag{5.3}$$

Doing the same calculation as in (5.2) for the reflected process at its minimum $X - \underline{X}$, we get

$$\begin{aligned}
&\frac{1}{q}\mathbb{E}(f(\mathbf{e}-g^*_{\mathbf{e}})K\circ r_{g^*_{\mathbf{e}}}F\circ k_{\mathbf{e}-g^*_{\mathbf{e}}}\circ\theta_{g^*_{\mathbf{e}}})\\
&\quad=\mathbb{E}\biggl(\int_0^\infty dL^*_s e^{-qs}K\circ r_s\biggr)n^*\biggl(\int_0^\zeta e^{-qu}f(u)F\circ k_u\,du\biggr)\\
&\qquad+\mathtt{d}^*\mathbb{E}\biggl(\int_0^\infty dL^*_s e^{-qs}K\circ r_s\biggr)f(0)F(\omega^0).
\end{aligned}\tag{5.4}$$

Then we derive from (5.2), (5.3) and (5.4), the following equality:

$$\begin{aligned}
&\mathbb{E}\biggl(\int_0^\infty dL_s e^{-qs}f(s)F\circ r_s\biggr)n\biggl(\int_0^\zeta e^{-qu}K\circ k_u\,du\biggr)\\
&\quad+\mathtt{d}\mathbb{E}\biggl(\int_0^\infty dL_s e^{-qs}f(s)F\circ r_s\biggr)K(\omega^0)\\
&\quad=\mathbb{E}\biggl(\int_0^\infty dL^*_s e^{-qs}K\circ r_s\biggr)n^*\biggl(\int_0^\zeta e^{-qu}f(u)F\circ k_u\,du\biggr)\\
&\qquad+\mathtt{d}^*\mathbb{E}\biggl(\int_0^\infty dL^*_s e^{-qs}K\circ r_s\biggr)f(0)F(\omega^0).
\end{aligned}\tag{5.5}$$

Then by taking $f \equiv 1$, $F \equiv 1$ and $K \equiv 1$, we derive from (5.2) that

$$\kappa(q,0)=n(1-e^{-q\zeta})+q\mathtt{d}.\tag{5.6}$$

Now suppose that X is of type 1 or 2, so that $\mathtt{d}^*=0$, from what has been recalled in Section 2. Hence with $K\equiv 1$ in (5.5) and using (5.6), we have

$$\begin{aligned}
&\mathbb{E}\biggl(\int_0^\infty dL_s e^{-qs}f(s)F\circ r_s\biggr)\kappa(q,0)\kappa^*(q,0)\\
&\quad=qn^*\biggl(\int_0^\zeta e^{-qu}f(u)F\circ k_u\,du\biggr).
\end{aligned}\tag{5.7}$$

But using (2.6) and plugging (5.7) into (5.2) gives

$$\begin{aligned}
&\mathbb{E}\biggl(\int_0^\infty e^{-qt}f(g_t)F\circ r_{g_t}K\circ k_{t-g_t}\circ\theta_{g_t}\,dt\biggr)\\
&\quad=n^*\biggl(\int_0^\zeta e^{-qu}f(u)F\circ k_u\,du\biggr)n\biggl(\int_0^\zeta e^{-qu}K\circ k_u\,du\biggr)\\
&\qquad+\mathtt{d}n^*\biggl(\int_0^\zeta e^{-qu}f(u)F\circ k_u\,du\biggr)K(\omega^0),
\end{aligned}$$

so that identities (4.1) and (4.2) follow for λ-almost every $t>0$, by inverting the Laplace transforms in this equality. Then we easily check that for all $t>0$, the functionals g_t, r_{g_t} and $k_{t-g_t} \circ \theta_{g_t}$ are a.s. continuous, at any time t. Hence for any bounded and continuous functions f, F and K, from Lebesgue's theorem of dominated convergence, the left-hand sides of (4.1) and (4.2) are continuous in t. From the same arguments, the functions $t \mapsto n^*(F \circ k_t, t < \zeta)$ and $t \mapsto n(K \circ k_k, t < \zeta)$ are continuous. Hence, from general properties of the convolution product, the right-hand sides of (4.1) and (4.2) are continuous, so that these identities are valid for all $t > 0$. Then we extend this result to any bounded Borel functions f, F and K through a classical density argument. Finally, (4.3) is obtained in the same way as parts 1 and 2. □

PROOF OF THEOREM 6. Let g and h be two bounded Borel functions on $\mathbb{R}_+$, and define the functionals K and F on $\mathcal{D}$ by $F(\omega) = g(\omega_\zeta)$ and $K(\omega) = h(\omega_\zeta)$. Then we may check that for $\varepsilon \in \mathcal{E}$ and $t < \zeta(\varepsilon)$, $F \circ k_t(\varepsilon) = g(\varepsilon_t)$ and $K \circ k_t(\varepsilon) = h(\varepsilon_t)$. Moreover since the lifetime of the path $k_{t-g_t} \circ \theta_{g_t}(\omega)$ is $t - g_t(\omega)$ and $\Delta^+_{g_t}(\omega) = (\omega_{g_t} - \omega_{g_t-})^+ = (\overline{X}_t - \overline{X}_{t-})(\omega)$, we have $F \circ r_{g_t} \circ X = g(\overline{X}_t)$ and $K \circ k_{t-g_t} \circ \theta_{g_t} \circ X = h(\overline{X}_t - X_t)$, so that by applying Theorem 5 to the functionals F and K, we obtain (4.4).

To prove (4.5), we first note that $\lim_{t\to\infty}(g_t, \overline{X}_t) = (g_\infty, \overline{X}_\infty)$, a.s. Then let f be a bounded and continuous function which is defined on $\mathbb{R}^2_+$. We have from (4.4),

$$\mathbb{E}(f(g_t, \overline{X}_t)) = \int_0^t f(s,x) n(t-s<\zeta) q^*_s(dx)\, ds + \mathrm{d} \int_0^\infty f(t,x) q^*_t(dx) + \mathrm{d}^* n(t<\zeta) f(0,0).$$

On the one hand, we see from (2.8) that $\lim_{t\to\infty} n(t < \zeta) = n(\zeta = \infty) = a > 0$. On the other hand, $\lim_{t\to\infty} n^*(t<\zeta) = 0$, and since the term $\mathrm{d} \int_0^\infty f(t,x) q^*_t(dx)$ is bounded by $Cn^*(t<\zeta)$, where $|f(s,x)| \le C$, for all s, x, it converges to 0 as t tends to ∞. This allows us to conclude. □

Recall that the definition of the ladder height process (H_t) has been given in Section 2. Then define $(\ell_x, x \ge 0)$ as the right continuous inverse of H, that is,

$$\ell_x = \inf\{t : H_t > x\}.$$

Note that for types 1 and 2, since H is a strictly increasing subordinator, the process $(\ell_x, x \ge 0)$ is continuous, whereas in type 3, since H is a compound Poisson process, then ℓ is a càdlàg jump process. Parts 1 and 2 of the following lemma are reinforcements of Theorems 3 and 5 in [1]. Part 1 is also stated in Proposition 9 of [8]. Recall that $V(dt, dx)$ denotes the potential measure of the ladder process (τ, H).

LEMMA 1. *Let X be a Lévy process which is not a compound Poisson process and such that $|X|$ is not a subordinator.*

(1) *The following identity between measures holds on* $\mathbb{R}_+^3$:

$$u\mathbb{P}(X_t \in dx, \ell_x \in du)\,dt = t\mathbb{P}(\tau_u \in dt, H_u \in dx)\,du. \tag{5.8}$$

(2) *The following identity between measures holds on* $\mathbb{R}_+^2$:

$$\mathrm{d}^*\delta_{\{(0,0)\}}(dt, dx) + q_t^*(dx)\,dt = V(dt, dx), \tag{5.9}$$

moreover for all $t > 0$, *and for all Borel sets* $B \in \mathcal{B}_{\mathbb{R}_+}$, *we have*

$$q_t^*(B) = t^{-1}\mathbb{E}\big(\ell(X_t)\mathbb{1}_{\{X_t \in B\}}\big). \tag{5.10}$$

(3) *For all* $t > 0$, *the measures* $q_t^*(dx)$ *and* $p_t^+(dx)$ *are equivalent on* $\mathbb{R}_+$.

PROOF. When 0 is regular for $(-\infty, 0)$, part (1) is proved in Theorem 3 of [1] and when 0 is regular for both $(-\infty, 0)$ and $(0, \infty)$, part (2) is proved in Theorem 5 of [1]. Although the proofs of parts (1) and (2) in the general case follow about the same scheme as in [1], it is necessary to check some details.

First recall the so-called Fristedt identity which is established in all the cases concerned by this lemma, in [14]; see Theorem 6.16. For all $\alpha \ge 0$ and $\beta \ge 0$, the characteristic exponent of the ladder process (τ, H) is given by

$$\kappa(\alpha, \beta) = \exp\left(\int_0^\infty dt \int_{[0,\infty)} (e^{-t} - e^{-\alpha t - \beta x})t^{-1}\mathbb{P}(X_t \in dx)\right). \tag{5.11}$$

Note that the constant k, which appears in this theorem, is equal to 1, according to our normalization; see Section 1. Then recall the definition of $\kappa(\alpha, \beta)$: $\mathbb{E}(e^{-\alpha\tau_u - \beta H_u}) = e^{-u\kappa(\alpha,\beta)}$. This expression is differentiable, in $\alpha > 0$ and in $u > 0$. Differentiating first both sides in α, we obtain

$$\mathbb{E}(\tau_u e^{-\alpha\tau_u - \beta H_u}) = u\mathbb{E}(e^{-\alpha\tau_u - \beta H_u})\frac{\partial}{\partial\alpha}\kappa(\alpha, \beta). \tag{5.12}$$

Then since

$$\begin{aligned}
\frac{\partial}{\partial\alpha}\int_0^\infty dt \int_{[0,\infty)} &(e^{-t} - e^{-\alpha t - \beta x})t^{-1}\mathbb{P}(X_t \in dx) \\
&= \int_0^\infty dt \int_{[0,\infty)} e^{-\alpha t - \beta x}\mathbb{P}(X_t \in dx) \\
&= \int_0^\infty e^{-\alpha t}\mathbb{E}\big(e^{-\beta X_t}\mathbb{1}_{\{X_t \ge 0\}}\big)\,dt,
\end{aligned}$$

we derive from (5.12) and (5.11) that

$$\begin{aligned}
\mathbb{E}(\tau_u e^{-\alpha\tau_u - \beta H_u}) &= -u\int_0^\infty e^{-\alpha t}\mathbb{E}\big(e^{-\beta X_t}\mathbb{1}_{\{X_t \ge 0\}}\big)\,dt\,\frac{\partial}{\partial u}\mathbb{E}(e^{-\alpha\tau_u - \beta H_u}) \\
&= -u\frac{\partial}{\partial u}\int_0^\infty \mathbb{E}\big(\mathbb{E}(e^{-\alpha\tau_u - \beta H_u})e^{-\alpha t}e^{-\beta X_t}\mathbb{1}_{\{X_t \ge 0\}}\big)\,dt.
\end{aligned}$$

Let $\tilde{X}$ be a copy of X which is independent of (τ_u, H_u). Then the above expression may be written as

$$\mathbb{E}(\tau_u e^{-\alpha\tau_u-\beta H_u}) = -u\frac{\partial}{\partial u}\mathbb{E}\biggl(\int_0^\infty \exp\bigl(-\alpha(t+\tau_u)-\beta(\tilde{X}_t+H_u)\bigr)\mathbb{1}_{\{\tilde{X}_t\geq 0\}}\,dt\biggr).$$

For $\tilde{X}$, we may take, for instance, $\tilde{X}=(X_{\tau_u+t}-X_{\tau_u}, t\geq 0)$, so that it follows from a change of variables and the definition of $(\ell_x, x\geq 0)$,

$$\begin{aligned}\mathbb{E}(\tau_u e^{-\alpha\tau_u-\beta H_u}) &= -u\frac{\partial}{\partial u}\mathbb{E}\biggl(\int_0^\infty \exp\bigl(-\alpha(t+\tau_u)-\beta X_{\tau_u+t}\bigr)\mathbb{1}_{\{X_{\tau_u+t}\geq H_u\}}\,dt\biggr)\\ &= -u\frac{\partial}{\partial u}\mathbb{E}\biggl(\int_0^\infty \exp(-\alpha t-\beta X_t)\mathbb{1}_{\{X_t\geq H_u,\tau_u\leq t\}}\,dt\biggr)\\ &= -u\frac{\partial}{\partial u}\int_0^\infty dt e^{-\alpha t}\int_{[0,\infty)} e^{-\beta x}\mathbb{P}(X_t\in dx, \ell_x>u),\end{aligned}$$

from which we deduce that

$$\begin{aligned}&\int_{[0,\infty)^2} e^{-\alpha t-\beta x} t\mathbb{P}(\tau_u\in dt, H_u\in dx)\,du\\ &\qquad = \int_{[0,\infty)^2} e^{-\alpha t-\beta x} u\mathbb{P}(X_t\in dx, \ell_x\in du)\,dt,\end{aligned}$$

and (5.8) follows by inverting the Laplace transforms.

Let $\mathbf{e}$ be an exponentially distributed random variable with parameter q, which is independent of X. From identity (6.18), page 159 in [14], we have

$$\mathbb{E}(\exp(-\beta\overline{X}_{\mathbf{e}})) = \kappa(q,0)\int_{[0,\infty)^2} e^{-qt-\beta x}\int_0^\infty \mathbb{P}(\tau_s\in dt, H_s\in dx)\,ds. \tag{5.13}$$

Suppose that X is of type 1 or 2. By taking the Laplace transforms in x and t of identity (4.7) in Corollary 3, we obtain

$$\mathbb{E}(\exp(-\beta\overline{X}_{\mathbf{e}})) = \bigl(q\mathtt{d}+n(1-e^{-q\zeta})\bigr)n^*\biggl(\int_0^\zeta e^{-qs}e^{-\beta\varepsilon_s}\,ds\biggr), \tag{5.14}$$

and by comparing (5.6), (5.13) and (5.14), it follows

$$n^*\biggl(\int_0^\zeta e^{-qs}e^{-\beta\varepsilon_s}\,ds\biggr) = \int_{[0,\infty)^2} e^{-qt-\beta x}\int_0^\infty \mathbb{P}(\tau_s\in dt, H_s\in dx)\,ds. \tag{5.15}$$

Then we derive part 2 from (5.15), (5.8) and uniqueness of the Laplace transform. If X is of type 3, then taking the Laplace transforms in x and t of identity (4.7), gives

$$\mathbb{E}(\exp(-\beta\overline{X}_{\mathbf{e}})) = n(1-e^{-q\zeta})\biggl(\mathtt{d}^*+n^*\biggl(\int_0^\zeta e^{-qt}e^{-\beta\varepsilon_t}\,dt\biggr)\biggr), \tag{5.16}$$

so that by comparing (5.6), (5.13) and (5.16), we obtain

$$
\begin{aligned}
&\mathrm{d}^* + n^*\left(\int_0^\zeta e^{-qt} e^{-\beta\varepsilon_t}\,dt\right) \\
\text{(5.17)}\qquad &\quad = \int_{[0,\infty)^2} e^{-qt-\beta x} \int_0^\infty \mathbb{P}(\tau_s \in dt, H_s \in dx)\,ds,
\end{aligned}
$$

and part 2 follows from (5.17) and (5.8) in this case.

Then we show the third assertion. First note that q_t^* is absolutely continuous with respect to p_t^+ for all $t > 0$, since from (5.10) we have for any Borel set $B \subset \mathbb{R}_+$ such that $\mathbb{P}(X_t \in B) = 0$,

$$q_t^*(B) = t^{-1}\mathbb{E}\big(\ell(X_t)\mathbb{1}_{\{X_t \in B\}}\big) = 0.$$

Conversely, take a Borel set $B \subset \mathbb{R}_+$, such that $\mathbb{P}(X_t \in B) > 0$. Then since $\mathbb{P}(X_t = 0) = 0$, there exists $y > 0$ such that $\mathbb{P}(X_t \in B, X_t > y) > 0$. As the right continuous inverse of a subordinator, (ℓ_x) is nondecreasing and we have for all $x > 0$, $\mathbb{P}(\ell_x > 0) = 1$. Therefore the result follows from the inequality

$$0 < \mathbb{E}\big(\ell_y \mathbb{1}_{\{X_t \in B, X_t > y\}}\big) \le \mathbb{E}\big(\ell(X_t)\mathbb{1}_{\{X_t \in B\}}\big),$$

together with identity (5.10). $\square$

Recall from Section 2 that π is the Lévy measure of the ladder time process τ and that $\overline{\pi}(t) = \pi(t, \infty)$.

LEMMA 2. *Under the assumption of Lemma 1, for all $t > 0$, the following measures of $\mathbb{R}_+$:*

$$\int_0^t \overline{\pi}(t-s) q_s^*(dx)\,ds \quad \textit{and} \quad \int_0^t q_s^*(dx)\,ds$$

are equivalent.

PROOF. For all Borel set $B \subset \mathbb{R}_+$, we have

$$\overline{\pi}(t) \int_0^t q_s^*(B)\,ds \le \int_0^t \overline{\pi}(t-s) q_s^*(B)\,ds,$$

hence $\int_0^t q_s^*(dx)\,ds$ is absolutely continuous with respect to $\int_0^t \overline{\pi}(t-s) q_s^*(dx)\,ds$. Moreover, for all $q \in (0, t)$ and all Borel set $B \subset \mathbb{R}_+$, we may write

$$\int_0^t \overline{\pi}(t-s) q_s^*(B)\,ds \le \overline{\pi}(q) \int_0^t q_s^*(B)\,ds + \int_{t-q}^t \overline{\pi}(t-s) q_s^*(B)\,ds < \infty.$$

Hence if $\int_0^t q_s^*(B)\,ds = 0$, then for all $q \in (0, t)$,

$$\int_0^t \overline{\pi}(t-s) q_s^*(B)\,ds \le \int_{t-q}^t \overline{\pi}(t-s) q_s^*(B)\,ds < \infty.$$

The finiteness of the right-hand side of the above inequality can be derived from relation (2.8) and Corollary 3. Hence this term tends to 0 as q tends to 0, so that the equivalence between the measures $\int_0^t \overline{\pi}(t-s)q_s^*(dx)\,ds$ and $\int_0^t q_s^*(dx)\,ds$ is proved. □

Now we are ready to prove all the results of Section 3.

PROOF OF THEOREM 1. When X is of type 1 or 2, it follows from Corollary 3, part 3 of Lemma 1, relation (2.8) and Lemma 2. When X is of type 3, the arguments are the same, except that one has to take account of the fact that the law of $\overline{X}_t$ has an atom at 0, as it is specified in Corollary 3. □

PROOF OF THEOREM 2. We first prove that part 2 implies part 1. To that aim, observe that

$$\overline{X}_{2t} = \max\Big\{\overline{X}_t, X_t + \sup_{0\le s\le t}(X_{t+s} - X_t)\Big\} = \max\Big\{\overline{X}_t, X_t + \sup_{0\le s\le t} X_s^{(1)}\Big\},$$

where $X^{(1)}$ is an independent copy of X. From this independence and the above expression, we easily deduce that if the law of $\overline{X}_t$ is absolutely continuous, then so is this of $\overline{X}_{2t}$. Therefore, from Theorem 1, the measure μ_{2t}^+ is absolutely continuous. This clearly implies that for all $s \in (0, 2t]$, the measure μ_s^+ is absolutely continuous. Applying Theorem 1 again, it follows that the law of $\overline{X}_s$ is absolutely continuous, for all $s \in (0, 2t]$. Then we show the desired result by reiterating this argument. So, part 1 is equivalent to part 2.

Let us assume that part 1 holds. Then for all $t > 0$, the law of $\overline{X}_t$ is absolutely continuous. Therefore the resolvent measure $U(dx)$ is absolutely continuous. Indeed, let $\mathbf{e}$ be an independent exponentially distributed random time with parameter 1, then the law of $\overline{X}_{\mathbf{e}}$ admits a density, hence the law of $X_{\mathbf{e}} = X_{\mathbf{e}} - \overline{X}_{\mathbf{e}} + \overline{X}_{\mathbf{e}}$ also admits a density, since the random variables $X_{\mathbf{e}} - \overline{X}_{\mathbf{e}}$ and $\overline{X}_{\mathbf{e}}$ are independent; see Chapter VI in [3]. Since the law of $X_{\mathbf{e}}$ is precisely the measure $U(dx)$, we have proved that part 1 implies part 4. Then part 4 implies part 3 and from Corollary 1, part 3 implies part 1.

It remains to show that part 5 is equivalent to part 1. To this aim, first observe that $V(dx)$ is absolutely continuous if and only if $\int_0^t q_s^*(dx)\,ds$ is absolutely continuous, for all $t > 0$. Indeed, from part 2 of Lemma 1, we have $V(dx) = \int_0^\infty q_s^*(dx)\,ds$, and hence if $V(dx)$ is absolutely continuous, then so are the measures $\int_0^t q_s^*(dx)\,ds$, for all $t > 0$. Conversely assume that the measures $\int_0^t q_s^*(dx)\,ds$ are absolutely continuous for all $t > 0$. Let A be a Borel set of $\mathbb{R}_+$ such that $\lambda_+(A) = 0$. From the assumption, $q_s^*(A) = 0$, for λ-almost every $s > 0$, hence $V(A) = \int_0^\infty q_s^*(A)\,ds = 0$, so that $V(dx)$ is absolutely continuous. Then from Lemma 2 and Corollary 3, for each t, the law of $\overline{X}_t$ is equivalent to the measure $\int_0^t q_s^*(dx)\,ds$. Therefore part 5 is equivalent to part 1. □

PROOF OF THEOREM 3. If $p_t^+ \ll \lambda^+$ for all $t > 0$, then from part 3 of Lemma 1, $q_t^* \ll \lambda^+$, for all $t > 0$. Suppose moreover that 0 is regular for $(0, \infty)$, and let A be a Borel subset of $\mathbb{R}$, such that $\lambda(A) = 0$. Then from Corollary 4 and Fubini theorem, we have

$$p_t(A) = \int_0^t ds q_s^* * \bar{q}_{t-s}(A) + \mathtt{d} q_t^*(A),$$

where $\bar{q}_s$ and q_s^* are extended on $\mathbb{R}$, as in this corollary. But from the assumptions, for all $0 < s < t$, $q_s^* * \bar{q}_{t-s}(A) = 0$ and $q_t^*(A) = 0$, hence $p_t(A) = 0$, for all $t > 0$ and p_t is absolutely continuous, for all $t > 0$. So part 1 implies part 2, and the converse is obvious.

Then it readily follows from part 3 of Lemma 1 and identity (5.9) that part 1 implies part 3 (recall that $\mathtt{d}^* = 0$ in the present case). Now suppose that $V(dt, dx)$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}_+^2$. Then we derive from identity (5.9) that the measures $q_t^*(dx)$ are absolutely continuous for λ-almost every $t > 0$. From Corollary 4, it means that p_t is absolutely continuous for λ-almost every $t > 0$. But if the semigroup p_t is absolutely continuous for some t, then p_s is absolutely continuous for all $s \ge t$. Hence p_t is actually absolutely continuous, for all $t > 0$, and part 3 implies part 2.

Then suppose that X is of type 1, and recall that $\mathtt{d} = \mathtt{d}^* = 0$ in this case. From Theorem 6 and part 2 of Lemma 1, we have

$$\mathbb{P}(g_t \in ds, \overline{X}_t \in dx) = n(t - s < \zeta) V(ds, dx). \tag{5.18}$$

Since $n(t - s < \zeta) > 0$, for all $s \in [0, t]$, we easily derive from identity (5.18) that part 3 and part 4 are equivalent.

Let us denote by p_t^- the restriction of p_t to $\mathbb{R}_-$. If part 2 is satisfied, then p_t^+ and p_t^- are absolutely continuous for all $t > 0$. Then from part 3 of Lemma 1 applied to X and its dual process $-X$, it follows that q_t and $\underline{q_t^*}$ are absolutely continuous for all $t > 0$, so that from Theorem 6, the triple $(g_t, \overline{X}_t, X_t)$ is absolutely continuous for all $t > 0$; hence part 2 implies part 5. Then part 5 clearly implies part 4. □

PROOF OF PROPOSITION 1. In this proof, it suffices to assume that Y is a deterministic process, that is, (T_n), (a_n) and (b_n) in (3.2) are deterministic sequences.

In order to prove part 1, let us first assume that $a_n = 0$, for all n. Then recall that from Theorem 2, the law of $\overline{X}_t$ is absolutely continuous, for all $t > 0$. Fix $t > 0$ and let n be such that $t \in [T_n, T_{n+1})$. Set $Z_k = Y_{T_k} + \sup_{T_k \le s < T_{k+1}} X_s$ and $Z = Y_{T_n} + \sup_{T_n \le s < t} X_s$, and then we have

$$\sup_{s \le t} X_s + Y_s = \max\{Z_1, Z_2, \ldots, Z_{n-1}, Z\}. \tag{5.19}$$

But we can write

$$Z_k = Y_{T_k} + X_{T_k} + \sup_{s \le T_{k+1} - T_k} X_s^{(k)} \quad \text{and} \quad Z = Y_{T_n} + X_{T_n} + \sup_{s \le t - T_n} X_s^{(n)}, \tag{5.20}$$

where $X^{(k)}$, $k = 1, \dots, n$ are copies of X such that X, $X^{(k)}$, $k = 1, \dots, n$ are independent. From Theorem 2, the laws of $\sup_{s \le T_{k+1} - T_k} X_s^{(k)}$ and $\sup_{s \le t - T_n} X_s^{(n)}$ are absolutely continuous. From the representation (5.20) and the independence hypothesis, we derive that the laws of $Z_1, Z_2, \dots, Z_{k-1}$ and Z are absolutely continuous. Since the maximum of any finite sequence of absolutely continuous random variables is itself absolutely continuous, we conclude that the law of $\sup_{s \le t} X_s + Y_s$ is absolutely continuous, and the first part is proved.

Now we assume that (a_n) is any deterministic sequence. Then we have (5.19) with

$$Z_k = b_k + X_{T_k} + \sup_{s \le T_{k+1} - T_k} X_s^{(k)} + a_k s \quad \text{and} \tag{5.21}$$
$$Z = b_n + X_{T_n} + \sup_{s \le t - T_n} X_s^{(n)} + a_n s,$$

where $X^{(k)}$, $k = 1, \dots, n$ are as above. If $p_t^+ \ll \lambda^+$ for all t, then this property also holds for the process X with any drift a, that is, $X_t + at$, so from Theorem 1 the laws of $\sup_{s \le T_{k+1} - T_k} X_s^{(k)} + a_k s$ and $\sup_{s \le t - T_n} X_s^{(n)} + a_n s$ are absolutely continuous, and we conclude that the law of $\sup_{s \le t} X_s + Y_s$ is absolutely continuous, in the same way as for the first part.

Finally, if X has unbounded variations, then it is of type 1. If moreover, for instance, the ladder height process at the supremum H has a positive drift, then from Theorem 2 and the remark thereafter, the law of $\overline{X}_t$ is absolutely continuous for all $t > 0$. Since X has unbounded variations, it follows from (iv) page 64 in [8] that for any $a \in \mathbb{R}$, the ladder height process at the supremum of the drifted Lévy process $X_t + at$ also has a positive drift, and since $X_t + at$ is also of type 1, the law of $\sup_{s \le t} X_s + as$ is absolutely continuous. Then from Theorem 2, the laws of $\sup_{s \le T_{k+1} - T_k} X_s^{(k)} + a_k s$ and $\sup_{s \le t - T_n} X_s^{(n)} + a_n s$ are absolutely continuous, and again we conclude that the law of $\sup_{s \le t} X_s + Y_s$ is absolutely continuous, in the same way as for the first part. $\square$

PROOF OF PROPOSITION 2. Recall that under the assumption of this proposition, we have $\mathrm{d}^* = 0$. So, we derive from Theorem 6, by integrating identity (4.4) over y and from part 2 of Lemma 1, that

$$\mathbb{P}(g_t \in ds, \overline{X}_t \in dx) = s^{-1} n(t - s < \zeta) \mathbb{E}\big(\ell(x) \mathbb{1}_{\{X_s \in dx\}}\big) \mathbb{1}_{(0,t]}(s)\, ds$$
$$+ \mathrm{d}\delta_{\{t\}}(ds) t^{-1} \mathbb{E}\big(\ell(x) \mathbb{1}_{\{X_s \in dx\}}\big).$$

Since X has no positive jumps, then $\overline{X}_t$ continuous. Moreover, it is an increasing additive functional of the reflected process $\overline{X}_t - X_t$, such that

$$\mathbb{E}\left(\int_0^\infty e^{-t}\, d\overline{X}_t\right) = \Phi(1)^{-1},$$

where Φ is the Laplace exponent of the subordinator $T_x = \inf\{t : X_t > x\}$. Hence we have $L_t = c\overline{X}_t$, with $c = \Phi(1)$. Then it follows from the definition of H and ℓ, that

$$H_u = c^{-1}u, \qquad \text{on } H_u < \infty \quad \text{and} \quad \ell_x = cx \qquad \text{on } \ell_x < \infty.$$

Besides, from part 1 of Lemma 1, we have by integrating (5.8) over $u \in [0, \infty)$,

$$cxp_t^+(dx)\,dt = ct\mathbb{P}(\tau_{cx} \in dt)\,dx, \tag{5.22}$$

as measures on $[0, \infty)^2$. This ends the proof of the proposition. □

Note that identity (5.22) may also be derived from Corollary VII.3, page 190 in [3] or from Theorem 3 in [1]. The constant c appearing in our expression is due to the choice of the normalization of the local time in (2.1).

PROOF OF COROLLARY 2. We may check that $\int_{(0,1)} x\Pi(dx) = \infty$ in both cases 1 and 2, so that X has unbounded variation and it is of type 1, from Rogozin's criterion; see [3], page 167.

On the one hand, in part 1, since X has no positive jumps, the ladder height process H is a pure drift, so it follows from Theorem 2 and the remark thereafter that the law of $\overline{X}_t$ is absolutely continuous for all $t > 0$. On the other hand, following [16], we see that $-\log|\psi(\lambda)|$ does not tend to $+\infty$ as $|\lambda| \to \infty$, so that from the Riemann–Lebesgue theorem, $p_t(dx)$ is not absolutely continuous. But since $\Pi(dx)$ is discrete with infinite mass, it follows from the Hartman–Wintner theorem (see Theorem 27.16 in [18]) that $p_t(dx)$ is continuous singular.

Then in part 2, since X is symmetric, it follows from the discussion which comes just before Theorem 3 that the resolvent measure $U(dx)$ of X is not absolutely continuous, so the result follows from Theorem 2. □

In order to prove Theorem 4, we need the following lemma. We say that a sequence of random variables $S_1, \dots, S_n, \dots,$ with $S_0 = 0$ is a cyclically exchangeable chain if for any $n \geq 1$, the increments $S_1, S_2 - S_1, \dots, S_n - S_{n-1}$ are cyclically exchangeable. We emphasize that any random walk satisfies this property. For $x \geq 0$ and $n \geq 1$, we define $\overline{S}_n = \max_{k \leq n} S_k$ and

$$\Lambda_n^x = \mathrm{Card}\{1 \leq k \leq n : S_k = \overline{S}_k, \overline{S}_n - x \leq \overline{S}_k \leq \overline{S}_n\}.$$

The random variable Λ_n^x may be considered as a counting measure of the times at which the chain reaches its past maximum between the levels $\overline{S}_n - x$ and $\overline{S}_n$. Note that $\Lambda_n^x \geq 1$, a.s., whenever $\mathbb{P}(S_1 < 0, S_2 < 0, \dots, S_n < 0) = 0$.

LEMMA 3. *Let $(S_n)_{n\geq 0}$ be any cyclically exchangeable chain such that $\mathbb{P}(S_1 < 0, S_2 < 0, \dots, S_n < 0) = 0$, then for any set $A \in \mathcal{B}_{\mathbb{R}_+}$ and $n \geq 1$,*

$$\mathbb{E}\big((\Lambda_n^{S_n})^{-1}\mathbb{1}_{\{S_n = \overline{S}_n \in A\}}\big) = \frac{1}{n}\mathbb{P}(S_n \in A).$$

PROOF. Fix $A \in \mathcal{B}_{\mathbb{R}_+}$, $n \geq 1$, and let $1 \leq k \leq n$ be such that $\mathbb{P}(\Lambda_n^{S_n} = k) > 0$. Then note that conditionally on $\Lambda_n^{S_n} = k$ and $S_n \in A$, the chain $S_0, S_1, \dots, S_n$ is cyclically exchangeable. Moreover, amongst the n cyclical permutations in the set of trajectories $\{S_0, \dots, S_n : \Lambda_n^{S_n} = k, S_n \in A\}$, there are exactly k trajectories which satisfy the condition $S_n = \overline{S}_n$. This proves the identity

$$\frac{1}{k}\mathbb{P}(S_n = \overline{S}_n \in A | \Lambda_n^{S_n} = k) = \frac{1}{n}\mathbb{P}(S_n \in A | \Lambda_n^{S_n} = k),$$

and the result is obtained by integrating over the law of $\Lambda_n^{S_n}$. □

Lemma 3 can be compared to Corollary 1 in [2]. The only difference is that in [2], only strict records of the chain are considered, whereas in our case, the "local time" $\Lambda_n^{S_n}$ counts all the records (i.e., weak records) between the levels $\overline{S}_n - S_n$ and $\overline{S}_n$.

PROOF OF THEOREM 4. Fix $t > 0$. Equivalence between the measures p_t^+, μ_t^+, U^+ and υ^+ simply follows from the following decompositions:

$$p_t^+(dx) = \sum_{n=0}^{\infty} \mathbb{P}(N_t = n)\mathbb{P}(S_n \in dx),$$

$$\mu_t^+(dx) = \sum_{n=0}^{\infty} \left(\int_0^t \mathbb{P}(N_s = n)\, ds\right)\mathbb{P}(S_n \in dx),$$

$$U^+(dx) = \sum_{n=0}^{\infty} \left(\int_0^{\infty} e^{-s}\mathbb{P}(N_s = n)\, ds\right)\mathbb{P}(S_n \in dx)$$

and the definition of υ^+.

It remains to prove the equivalence between $\mathbb{P}(\overline{X}_t \in dx)$ and $\upsilon^+(dx)$. To this aim, note that $\overline{X}_t = \overline{S}_{N_t}$, so that

$$\mathbb{P}(\overline{X}_t \in dx) = \sum_{n=0}^{\infty} \mathbb{P}(N_t = n)\mathbb{P}(\overline{S}_n \in dx). \tag{5.23}$$

Let $A \in \mathcal{B}_{\mathbb{R}_+}$ be such that $\upsilon^+(A) = 0$, then by definition of υ^+, $\mathbb{P}(S_n \in A) = 0$, for all $n \geq 0$. This implies that

$$\mathbb{P}(\overline{S}_n \in A) \leq \sum_{k=0}^{n} \mathbb{P}(S_k \in A, S_k = \overline{S}_k) = 0, \tag{5.24}$$

so that from (5.23), $\mathbb{P}(\overline{X}_t \in A) = 0$. Conversely, assume that $\mathbb{P}(\overline{X}_t \in A) = 0$. Then from (5.23), for any $n \geq 0$, $\mathbb{P}(\overline{S}_n \in A) = 0$. For $n = 0$, we have $S_0 = \overline{S}_0 = 0$, and for all $n \geq 1$, from Lemma 3, we have

$$\mathbb{E}\big((\Lambda_n^{S_n})^{-1}\mathbb{1}_{\{S_n=\overline{S}_n\in A\}}\big)=\frac{1}{n}\mathbb{P}(S_n\in A)=0.$$

We conclude that $\upsilon^+(A)=0$. $\square$

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