By substitution,

$$T = G\alpha a^4\left[\frac{\pi}{2} - \frac{4}{\pi}\right]$$

exactly, and the numerical expression is

$$T = 0.297{,}556{,}782\ G\alpha a^4$$

which differ slightly from St. Venant's result of 0.296 or 0.2966.[2]

[2]The value 0.296 is from the table in S. Timoshenko's *Theory of elasticity*, p. 250, 1st ed., 2nd and 8th impression (1933 and 1934), McGraw-Hill; while from the datum given in I. Todhunter and K. Pearson's *A history of the theory of elasticity and of the strength of materials*, vol. II, part I, p. 193, we have

$$T = 0.3776M = 0.3776 \times G\alpha \times (\pi a^2)/2 \times a^2/2 = 0.2966G\alpha a^4.$$

A NOTE ON MY PAPER

ON STEADY LAMINAR TWO-DIMENSIONAL JETS IN COMPRESSIBLE VISCOUS GASES FAR BEHIND THE SLIT*

QUARTERLY OF APPLIED MATHEMATICS, 7, 313-323 (1949)

BY M. Z. KRZYWOBLOCKI (*University of Illinois*)

Determination of the constant of integration for the temperature distribution (p. 317, eq. 25) from the condition that that total flux of enthalpy across jet is alike at all cross-sections restricts solution to small Mach numbers (if the comparison cross-section is close to the slit, as pointed out by A. H. Shapiro, Appl. Mech. Rev. III (1950) p. 415, No. 2718) or to high Mach numbers (if the comparison section is far from the slit). To take into account all the relative cases, that constant may be determined from the condition that the total flux (enthalpy plus kinetic energy) is alike at all cross-sections:

$$2\int_0^\infty (Jc_pT_1 + u_1^2/2)u_1(\rho_\infty + \rho_1)\,dy = \text{const.}$$

*Received Feb. 22, 1951.

ON THE LEAST EIGENVALUE OF HILL'S EQUATION*

BY C. R. PUTNAM (*The Institute for Advanced Study, Princeton*)

The differential equation

$$x'' + [\lambda + f(t)]x = 0, \tag{1}$$

in which λ is a real parameter and $f(t)$, for $-\infty < t < \infty$, is a real-valued, continuous, periodic function ($\not\equiv 0$), arises in problems dealing with the propagation of waves in

*Received November 10, 1950.

periodic media; cf., e.g., [1], [5], [6]. For a proper choice of units on the t-axis it may be supposed that $f(t)$ has period 1 and hence possesses a Fourier series

$$f(t) \sim \sum_{n=-\infty}^{\infty} c_n e^{2\pi int}, \qquad (c_{-n} = \bar{c}_n). \tag{2}$$

There exists a sequence of finite intervals (the intervals of stability) $I_k : \lambda_k \leqq \lambda \leqq \lambda^k$, where $\lambda_k < \lambda^k < \lambda_{k+1}$ and $k = 1, 2, \cdots$, such that (1) possesses, or fails to possess, a solution x ($\neq 0$) which is bounded on $-\infty < t < \infty$ according as λ does, or does not, belong to the (closed) set $S = \sum I_k$; cf. [6], p. 16. It is known ([8], [3]) that the set S is identical with the invariant spectrum (Weyl [7], p. 251) associated with the differential equation (1), on either the full line $-\infty < t < \infty$ or a half-line, say, $0 \leqq t < \infty$.

Let λ_1 be denoted by μ so that μ is the least point of the set S. Alternatively, μ can be defined by the requirement that (1) be oscillatory, that is, every solution of (1) should possess an infinity of zeros on $0 \leqq t < \infty$ (or equivalently, in the present case, on $-\infty < t < \infty$) whenever $\lambda > \mu$, and non-oscillatory whenever $\lambda < \mu$. (This twofold characterization of μ holds also in the general case of (1) in which f need not be periodic; cf. [2].)

Wintner ([9], p. 116, and [10]) has obtained the following estimates of μ in terms of the Fourier coefficients of the function $f(t)$ defined by (2):

$$-c_0 - 2\sum_{n=1}^{\infty} |c_n|^2 \leqq \mu \leqq -c_0 . \tag{3}$$

The present note will be devoted to the problem of obtaining other such estimates. A formula for μ involving the Fourier coefficients of the given function $f(t)$ and those of arbitrary periodic functions, subject to certain specified conditions, will be deduced (cf. (12) below); furthermore, as a corollary of this formula, upper bounds for μ, involving not only c_0, as in (3), but also arbitrary coefficients c_n, will be obtained. Specifically, it will be shown that

$$\mu \leqq \pi^2 N^2 - c_0 + \Re(c_N), \qquad N = 1, 2, \cdots, \tag{4}$$

where $\Re(c)$ denotes the real part of a complex number c. In case $f(t)$ satisfies

$$-f(t) = f(t + c),$$

for some real number c, as, e.g., is the case if $f = \sin 2\pi t$, it is clear from the properties of μ that (4) can be refined to

$$\mu \leqq \pi^2 N^2 - c_0 - |\Re(c_N)|, \qquad N = 1, 2, \cdots .$$

2. It follows from [4] (cf. the Remark on p. 636) that μ satisfies

$$\mu = \lim_{T\to\infty} \left\{ \text{g.l.b.} \left[\int_T^{\infty} (x'^2 - fx^2)\, dt \Big/ \int_T^{\infty} x^2\, dt \right] \right\}, \tag{5}$$

where $x(t)$ belongs to the class of functions, Ω_T, which, on the half-line $T \leqq t < \infty$, are real-valued, continuous, and have piecewise continuous first derivatives (with respect to any finite subinterval of $T \leqq t < \infty$) and, furthermore, satisfy

$$x(T) = 0, \qquad 0 < \int_T^{\infty} x^2\, dt < \infty, \qquad \int_T^{\infty} x'^2\, dt < \infty. \tag{6}$$

Since $f(t)$ is periodic, it is clear that the expression $\{\cdots\}$ occurring in (5) is independent of T. Hence (5) remains valid if the limit sign is removed and the expression $[\cdots]$ of (5) is evaluated only for functions of class Ω_0.

Let $x(t)$ denote any function of class Ω_0. Clearly, it is possible to define a function $y(t)$ in Ω_0, such that $y(t) \equiv 0$ for sufficiently large t, and, in addition, is such that the expression $[\cdots]$ of (5) (for $T = 0$) evaluated for y differs from the corresponding expression for x by less than an arbitrarily preassigned positive number. (In fact, since $\int_0^\infty x^2\, dt < \infty$, there exists a sequence of points t_n such that $t_n \to \infty$ and $x(t_n) \to 0$, as $n \to \infty$. Let $z_n(t)$, where $n = 1, 2, \cdots$ and $t_n \leqq t \leqq t_n + 1$, be a sequence of continuous functions, with piecewise continuous first derivatives, such that $z_n(t_n) = x(t_n)$, $z_n(t_n + 1) = 0$, and

$$\int_{t_n}^{t_n+1} z^2\, dt \to 0, \qquad \int_{t_n}^{t_n+1} z_n'^2\, dt \to \infty, \qquad \text{as } t_n \to \infty.$$

If $y_n = x$ on $0 \leqq t \leqq t_n$, $y_n = z_n$ on $t_n \leqq t \leqq t_n + 1$ and $y_n \equiv 0$ for $t_n + 1 \leqq t < \infty$, it is clear that y_n satisfies the conditions claimed for the function y above, provided t_n is sufficiently large.) The corners of y can be "smoothed out" so that, in addition to the properties required above, y has a continuous first derivative on $0 \leqq t < \infty$.

It follows from the above discussion that μ can be defined by

$$\text{g.l.b.}\left[\int_0^Q (x'^2 - fx^2)\, dt \Big/ \int_0^Q x^2\, dt\right], \tag{7}$$

where x belongs to Γ_Q, the set of (real-valued) functions $x(t)$ ($\not\equiv 0$) which possess continuous first derivatives on $0 \leqq t \leqq Q$ and satisfy $x(0) = x(Q) = 0$, and Q is, for convenience, an arbitrary (variable) positive integer. Any function x of Γ_Q, and its derivative x', can be uniformly approximated on $0 \leqq t \leqq Q$ by a sequence of trigonometric polynomials of Γ_Q, and their derivatives, respectively. Hence, μ can be defined by (7) where, now, $x(t)$ is any function of the type

$$x(t) = \sum_{n=-N}^{N} a_n e^{2\pi i n(P/Q)t}, \qquad a_{-n} = \bar{a}_n, \qquad \sum_{n=-N}^{N} a_n = 0, \tag{8}$$

and where N, P and Q denote arbitrary positive integers. The remainder of this section will be devoted to obtaining a transcription of (7) in terms of the Fourier coefficients of the given f, defined by (2), and the Fourier coefficients of the variable function x, defined by (8).

Let P and Q denote relatively prime positive integers and note that (2) and the first relation of (8) can be rewritten as

$$f(t) \sim \sum C_n e^{2\pi i n t/Q} \qquad \text{and} \qquad x(t) = \sum A_n e^{2\pi i n t/Q}, \tag{9}$$

where

$$c_n = C_{Qn} \qquad \text{and} \qquad a_n = A_{Pn} \qquad \text{for } n = 0, \pm 1, \cdots, \pm N. \tag{10}$$

(By the uniqueness theorem for Fourier series, $C_n = 0$ if Q does not divide n and $A_n = 0$ if P does not divide n.) Since $x^2 = \sum B_n e^{2\pi i n t/Q}$, where $B_n = \sum_k A_k A_{n-k}$, it follows from the Parseval relation that

$$\int_0^Q x^2\, dt = Q \sum |A_n|^2, \qquad \int_0^Q x'^2\, dt = (4\pi^2/Q) \sum n^2 |A_n|^2$$

and

$$\int_0^Q fx^2\,dt = Q \sum \bar{C}_n B_n \,.$$

Thus, the expression $[\cdots]$ of (7) is equal to

$$\left[(4\pi^2/Q^2) \sum n^2 \mid A_n \mid^2 - Q \sum_n \bar{C}_n\Big(\sum_k A_k A_{n-k}\Big)\right] \Big/ Q \sum \mid A_n \mid^2. \tag{11}$$

From the definition (10) of C_n and A_n, it is clear that, in the numerator of (11), the n occurring in the first term may be replaced by Pn, while the n and k of the second term may be replaced by Qn and Pk respectively, so that the summation occurring in the second term of the numerator of (11) becomes $\sum_n \bar{C}_{Qn} \left(\sum_k A_{Pk} A_{Qn-Pk}\right)$. Since P and Q are relatively prime, $\sum_k A_{Pk} A_{Qn-Pk} = 0$ unless n is a multiple of P. But $A_{QPn-Pk} = A_{P(Qn-k)}$; hence, from (10), (11) and (7), there follows

$$\mu = \text{g.l.b.}\left\{\left[(4\pi^2P^2/Q^2) \sum n^2 \mid a_n \mid^2 - \sum_n \bar{c}_{Pn}\Big(\sum_k a_k a_{Qn-k}\Big)\right] \Big/ \sum \mid a_n \mid^2\right\}. \tag{12}$$

In the expression (12), the arbitrary positive integers P and Q and the finite set of complex numbers $a_{-N}, a_{-N+1}, \cdots, a_N$ are subject to

$$(P, Q) = 1, \qquad a_{-n} = \bar{a}_n \qquad \text{and} \qquad \sum_{n=-N}^{N} a_n = 0. \tag{13}$$

(It is to be noticed that each summation of (12) extends over only a finite range.) The formula (12), subject to (13), will be used in the next section to deduce (4).

3. Choose $a_1 = 1/2i$, $a_{-1} = -1/2i$, $a_n = 0$ if $n \neq \pm 1$; let $Q = 1$ and let P denote an arbitrary positive integer. (This selection corresponds to the choice $x = \sin 2\pi Pt$ in (8).) The expression $\{\cdots\}$ of (12) becomes $\pi^2(2P)^2 - c_0 + \Re(c_{2P})$ so that the equation (4) is proved for an arbitrary even positive integer $N = 2P$. Suppose now that N of (4) is odd and choose $P = N$ and $Q = 2$; let the a_n's be defined as before. Proceeding as above one sees that relation (12) implies $\mu \leqq \pi^2N^2 - c_0 + \Re(c_N)$ and the proof of (4) is complete.

Appendix

I. For computational purposes, one may easily verify that the formula (12) can be modified to

$$\mu = -c_0 + \text{g.l.b.}\left\{\left[(8\pi^2P^2/Q^2) \sum_{n=1}^{N} n^2 \mid a_n \mid^2 - 2\Re\Big(\sum_{n=1}^{N} \bar{c}_{Pn}\Big(\sum_k a_k a_{Qn-k}\Big)\Big)\right] \Big/ \Big(a_0^2 + 2 \sum_{n=1}^{N} \mid a_n \mid^2\Big)\right\}.$$

For appropriate choices of the sequences $\{a_j\}$ and the pairs of integers P and Q, subject to (13), various upper bounds for μ, in addition to those of (4), may be readily obtained from the above formula and (13). Non-trivial lower bounds for μ are not as easily obtained, in this manner, corresponding to the presence of the symbol "g.l.b." in the formulas for μ.

II. The question has been raised by Wintner [10] whether the constant 2 occurring as the coefficient in (3) is the least value of α for which

$$\mu \geq -c_0 - \alpha \sum_{n=1}^{\infty} | c_n |^2 \tag{14}$$

holds for an arbitrary periodic function $f(t)$ defined by (2). Although this question will remain unanswered, it can easily be shown, as a consequence of (4), that $\alpha \geqq 1/4\pi^2$. For, suppose (14) holds for all $f(t)$ defined by (2); then, by (4), $-\alpha \sum_{n=1}^{\infty} | c_n |^2 \leqq \pi^2 N^2 + \Re(c_N)$ holds for $N = 1, 2, \cdots$. If c_N is real, it follows that $\pi^2 N^2 + c_N + \alpha c_N^2 \geqq 0$; hence, by a consideration of the discriminant of this last quadratic expression, $1 - 4\alpha\pi^2 N^2 \leqq 0$. For $N = 1$, this implies $\alpha \geqq 1/4\pi^2$, which was to be shown.

References

1. L. Brillouin, *Wave propagation in periodic structures*, New York and London, 1946.
2. P. Hartman and C. R. Putnam, *The least cluster point of the spectrum of boundary value problems*, Am. J. Math. **70**, 849-855 (1948).
3. P. Hartman and A. Wintner, *On the location of spectra of wave equations*, Am. J. Math. **71**, 214-217 (1949).
4. C. R. Putnam, *An oscillation criterion involving a minimum principle*, Duke Math. J. **16**, 633-636 (1949).
5. F. Seitz, *The modern theory of solids*, New York, 1940.
6. M. J. O. Strutt, *Lamésche, Mathieusche und verwandte Funktionen in Physik und Technik*, Berlin, 1932.
7. H. Weyl, *Ueber gewöhnliche Differentialgleichungen mit Singularitäten und die zugehörigen Entwicklungen willkürlicher Funktionen*, Math. Ann. **68**, 222-269 (1910).
8. A. Wintner, *Stability and spectrum in the wave mechanics of lattices*, Phys. Rev. **72**, 81-82 (1947).
9. A. Wintner, *A criterion of oscillatory stability*, Q. Appl. Math. **7**, 115-119 (1949).
10. A. Wintner, *On the non-existence of conjugate points*, to be published.

A MODIFICATION OF SOUTHWELL'S METHOD*

By W. H. INGRAM (*New York*)

J. L. Synge[1] has given a geometrical interpretation of Southwell's method of solution of the problem $Ax = b$ when $A = (a_{ij})$ is symmetric and $\sum \sum a_{ij}x_i x_j$ is a positive definite form. A modification of the method having application to the more general case in which $xA_T Ax$ is a positive definite form makes use of the ellipsoids of the Gauss-Seidel process.

For any vector x, there is an error e defined by the equation

$$Ax - b = e, \tag{1}$$

therefore

$$(xA_T - b)W(Ax - b) = eWe; \tag{2}$$

*Received Jan. 2, 1951.

[1] J. L. Synge, *A geometrical interpretation of the relaxation method*, Q. Appl. Math., **2**, p. 87 (1944).