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Abstract

In most cases 2D (or bivariate) wavelets are constructed as a tensor product of 1D wavelets. Such wavelets are called separable. However, there are various applications, e.g. in image processing, for which non-separable 2D wavelets are preferable. In this paper, we are concerned with the class of compactly supported 2D wavelets that was introduced by Belogay and Wang [2]. A characteristic feature of this class of wavelets is that the support of the corresponding filter comprises only two rows. As a result, the 2D wavelets in this class are intimately related to some underlying 1D wavelet. We explore this relation in detail, and we explain how the 2D decompositions can be realized by a lifting scheme, and hence allow an efficient implementation. We also desrcibe an easy way to construct wavelets with more rows and shorter columns.

1. Introduction

Wavelets do not need any introduction. They have become part of the basic toolbox of any applied mathematician or electrical engineer [8] in the same way as the Fourier transform has belonged to this toolbox for many decades. There are some major differences between both tools, however. Basically, there exists only one Fourier transform. Wavelets, however, occur in many different tastes. They may be orthogonal [5] or biorthogonal [4], they may have compact support or not, they may have different accuracies, they may show different degrees of regularity, they may be separable or not (in higher dimensions), they may be integer-, real-, or complex-valued, and they may even be nonlinear. It is this diversity that has made wavelets into such a practical and flexible tool. It is also this diversity that makes wavelets into a fascinating research area.

In this paper, we will be concerned with 2D (or bivariate) wavelets. Wavelets in two and higher dimensions are often constructed as tensor products of 1D wavelets, resulting in so-called separable wavelets. But the tensor product approach has several drawbacks. In fact, this approach is only suited for basic square grids, and cannot cope with arbitrary sampling lattices offering more degrees of freedom. On the other hand, construction of non-separable 2D wavelets is far from trivial [7]. The spectrum factorization method that has been used with great success in the 1D case, is hard to extend to two dimensions. In [2], Belogay and Wang have constructed a family of 2D non-separable orthogonal wavelets related to the dilation matrix $\left(\begin{smallmatrix} 0 & 2 \\ 1 & 0 \end{smallmatrix}\right)$ in which the spectrum factorization method can be used thanks to the special structure of the wavelet filters. Furthermore, the members of this wavelet family can have any prescribed accuracy.

In this paper, which presents some generalizations to the results in [2], we investigate the relation between 2D orthonormal wavelets with the aforementioned dilation matrix and 1D wavelets with scaling factor 2. We also show how to design lifting schemes for this family of 2D wavelets. Then we develop an easy way to construct wavelets with more rows and shorter columns. More precisely, for a given accuracy r, the support of the two-row filters constructed in [2] lie within the range $[0, 4r-1] \times [0, 1]$ whereas the support of the filters with more rows lie within the range $[0, 2r-1] \times [0, r]$.

2. Subband schemes and wavelet transforms

2.1. Subsampling with dilation matrices

Wavelets are known to have a tight relation with subband schemes, also known as filter banks. From now on, we are mainly concerned with filter banks for 2D signals with two bands.

A square matrix D with integer entries is said to be a *dilation matrix* if the absolute values of its eigenvalues are larger than 1. In subband schemes, dilation matrices are used to specify the subsampling lattice. Note that for the two-band system, we have $|\det D| = 2$.

In the sequel, any element in $\mathbb{Z}^2$, denoted as a column vector, indicates a point, a vector or an index. Let $\mathbf{e}_1 = (1,0)^T$ and $\mathbf{e}_2 = (0,1)^T$, where T stands for the vector transpose.

Let $D = (d_{ij})$ be a 2×2 dilation matrix with $|\det D| = 2$. Suppose $\operatorname{adj} D$ is the *adjugate* of D, defined by $D^{-1} = \operatorname{adj} D / (\det D)$. One can easily show that D, and hence

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$\text{adj}\, D$, must have an odd entry. We give the following result without proof.

Lemma 1 *Suppose that D is a dilation matrix and that $\text{adj}\, D$ has an odd entry at position (l,k). Take $\mathbf{u} = \mathbf{e}_k$ and $\mathbf{v} = (\mathbf{e}_l^T \text{adj}\, D)^T$, i.e., $\mathbf{v}^T$ is the l-th row of $\text{adj}\, D$. Then*

1. *The set $\mathbb{Z}^2$ can be divided into two disjoint sets $D\mathbb{Z}^2$ and $D\mathbb{Z}^2 + \mathbf{u}$;*
2. *$\mathbf{v}^T\mathbf{u}$ is odd and $\mathbf{v}^T D\mathbf{n}$ is even for any $\mathbf{n} \in \mathbb{Z}^2$.*

We introduce some further notation. For a signal $x \in \ell_2(\mathbb{Z}^2)$ we define the z-transform by

$$X(\mathbf{z}) = \sum_{\mathbf{n}\in\mathbb{Z}^2} x_\mathbf{n} \mathbf{z}^\mathbf{n} \,,$$

where $\mathbf{z}^\mathbf{n} = z^{n_1} w^{n_2}$ for $\mathbf{z} = (z, w)$ and $\mathbf{n} = (n_1, n_2)^T$. It's conjugate is $\bar{X}(\mathbf{z}) = \sum x_\mathbf{n} \mathbf{z}^{-\mathbf{n}}$.

The output y after downsampling x with dilation matrix D is $y = D_\downarrow(x)$, i.e., $y_\mathbf{n} = x_{D\mathbf{n}}$. The output y after upsampling x with dilation matrix D is $y = D^\uparrow(x)$, i.e., $y_\mathbf{n} = x_\mathbf{k}$, if $\mathbf{n} = D\mathbf{k}$ and $y_\mathbf{n} = 0$ otherwise. Define

$$\underline{C}(\mathbf{z}) = C((-1)^{v_1} z, (-1)^{v_2} w)$$

for a z-form $C(\mathbf{z})$, where $\mathbf{v} = (v_1, v_2)^T$ is defined in Lemma 1. Then the composition $D^\uparrow D_\downarrow$ has the following compact formulation:

$$D^\uparrow D_\downarrow : X \mapsto (X + \underline{X})/2$$

Obviously, $\bar{z} = z^{-1}$ if $|z| = 1$. Therefore, we will use $\bar{z}$ and z^{-1} interchangeably.

2.2. Wavelets and subband schemes

A *wavelet* is a function $\psi \in L_2(\mathbb{R}^2)$ so that $\{\psi_{j,\mathbf{n}} : j \in \mathbb{Z}, \mathbf{n} \in \mathbb{Z}^2\}$ is an orthonormal basis of $L_2(\mathbb{R}^2)$, where $f_{j,\mathbf{n}}(\mathbf{x}) = 2^{j/2} f(D^j\mathbf{x} - \mathbf{n})$. In a multiresolution analysis (MRA) setting, the wavelet ψ is uniquely related to a so-called *scaling function* ϕ satisfying the dilation relation

$$\phi(\mathbf{x}) = 2^{1/2} \sum_{\mathbf{n}\in\mathbb{Z}^2} h_\mathbf{n} \phi(D\mathbf{x} - \mathbf{n}) \,.$$

Under some properties, the wavelet is then defined by

$$\psi(\mathbf{x}) = 2^{1/2} \sum_{\mathbf{n}\in\mathbb{Z}^2} g_\mathbf{n} \phi(D\mathbf{x} - \mathbf{n}) \,.$$

Let $H(\mathbf{z}) = \sum_{\mathbf{n}\in\mathbb{Z}^2} h_\mathbf{n} \mathbf{z}^\mathbf{n}$ and $G(\mathbf{z}) = \sum_{\mathbf{n}\in\mathbb{Z}^2} g_\mathbf{n} \mathbf{z}^\mathbf{n}$. Then

$$G = \alpha \underline{\bar{H}}, \tag{1}$$

where α is a monomial with $\alpha = -\underline{\alpha}$. For simplicity, we can assume that $\alpha(z) = z^\mu$ where μ is an odd number.

The orthogonality of H, and hence also G, is specified by

$$\bar{H} H + \underline{\bar{H}}\, \underline{H} = 2. \tag{2}$$

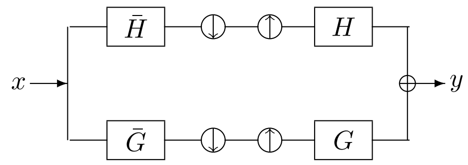

Figure 1. Subband schemes with two bands. Here ↓ denotes down-sampling and ↑ denotes up-sampling.

These results are captured by the 2-band filter bank depicted in Figure 2.2, where H and G are the low-pass and high-pass filters respectively. Define the *modulation matrix*

$$M = \begin{pmatrix} H & \underline{H} \\ G & \underline{G} \end{pmatrix} .$$

Now the perfect reconstruction condition can be formulated as

$$M^T \bar{M} = 2I, \tag{3}$$

where I is the identity matrix.

3. Non-separable orthonormal wavelets

In the sequel all filters are assumed to have finite supports. Thus their z-forms are Laurent polynomials, i.e., they have finitely many terms. The construction in §3.1 and §3.2 below is close in spirit to the one in [2], but our approach is slightly different.

3.1. Construction

From now on, we will restrict ourselves to the dilation matrix

$$D = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} . \tag{4}$$

Thus $\det D = -2$ and $\text{adj}\, D = \left(\begin{smallmatrix} 0 & -2 \\ -1 & 0 \end{smallmatrix}\right)$ has an odd entry -1 at location $(2,1)$. By Lemma 1, $\mathbf{u} = \mathbf{e}_1 = (1,0)^T$ and $\mathbf{v} = (-1,0)^T$. Let $S(\mathbf{z})$ be an arbitrary Laurent polynomial. For the dilation matrix D given by (4) we have

$$\underline{S}(z, w) = S(-z, w) \,. \tag{5}$$

Therefore, if $S(\mathbf{z}) = S(z)$ then $\underline{S}(\mathbf{z}) = S(-z)$, and if $S(\mathbf{z}) = S(w)$ then $\underline{S}(\mathbf{z}) = S(w)$.

In [2], Belogay and Wang construct 2D orthonormal wavelets whose main restriction concerns the support of the underlying filters. In fact, they assume that the low-pass filter H has only two rows, i.e., it has a z-transform

$$H(\mathbf{z}) = a(z) + (w-1) b(z) \,. \tag{6}$$

By substituting (6) into (2) and using (5), we get

$$|a|^2 + |\underline{a}|^2 = 2 \tag{7}$$
$$(\bar{a} - \bar{b})b + (\bar{\underline{a}} - \bar{\underline{b}})\underline{b} = 0. \tag{8}$$

Assume that we have a univariate Laurent polynomial $P(z) = \sum_{n=N_1}^{N_2} p_n z^n$ with degree $\deg P = N_2 - N_1$. If P is nontrivial, i.e., $\deg P > 0$, then P can be factorized as

$$P(z) = p_{N_2} z^{N_1} \prod_{n=1}^{r_1} (z^2 - \gamma_n^2) \prod_{n=1}^{r_2} (z - \eta_n)\,,$$

where both γ_n and $-\gamma_n$ are roots of P for $n = 1, \cdots, r_1$, and η_n, $n = 1, \cdots, r_1$, are the other roots of P. Therefore it is easy to see that P can be written as $P(z) = s(z)q(z)$, where the factor $s(z)$ is even in z and has maximal degree, and where $q(z)$ and $q(-z)$ have no common nontrivial factors.

Now consider condition (8). Assume

$$a(z) - b(z) = s(z^2)l(z), \quad b(z) = q(z^2)g(z)\,,$$

where $l(z)$ and $g(z)$ are Laurent polynomials such that neither $l(z)$ and $l(-z)$ nor $g(z)$ and $g(-z)$ have common nontrivial factors. Combining this with (8), we obtain $g(z) = \beta(z)\bar{\underline{l}}(z)$ where $\beta(z)$ is a monomial in z such that $\beta(-z) = -\beta(z)$. For simplicity, we can assume that $\beta(z) = z^\nu$ where ν is an odd number.

Therefore, condition (8) is equivalent to

$$a(z) - b(z) = s(z^2)l(z), \quad b(z) = z^\nu q(z^2)\bar{\underline{l}}(z)\,, \tag{9}$$

where ν is odd.

Substituting (9) into (7), we have

$$(|l(z)|^2 + |l(-z)|^2)(|q(z^2)|^2 + |s(z^2)|^2) = 2$$

in which both factors are necessarily monomials. Without loss of generality, we may assume

$$|l(z)|^2 + |l(-z)|^2 = 2 \tag{10}$$
$$|q(z^2)|^2 + |s(z^2)|^2 = 1. \tag{11}$$

To compute l from (10), we may simply use the results from the one-dimensional case.

3.2. Accuracy of H

A scaling function ϕ is said to have *accuracy* r if the space spanned by integer translates $\phi(\mathbf{x} - \mathbf{k})$, where $\mathbf{k} \in \mathbb{Z}^2$, contains all polynomials of degree $r - 1$ or less. There exist several equivalent formulations of this property. For us, the "sum rules" on the filter coefficients [2, 3] is the most interesting. It says that the filter H has accuracy r if

$$\frac{\partial^{p+q}}{\partial z^p \partial w^q} H(-1, 1) = 0\,, \text{ for all } p, q \geq 0 \text{ with } p + q < r.$$

where $(-1, 1)$ comes from $(-1)^{\mathbf{v}} = ((-1)^{v_1}, (-1)^{v_2})$.

Lemma 2 *If H is of the form given in (6) and has accuracy $r+1$, then a and b can be written as*

$$a = \left(\frac{1+z}{2}\right)^{r+1} a_0, \quad b = \left(\frac{1+z}{2}\right)^{r} b_0, \tag{12}$$

where a_0, b_0 are Laurent polynomials.

The r-accuracy of a, b in combination with (9) yields

$$q(z^2) = \left(\frac{1+z}{2}\right)^r \left(\frac{1-z}{2}\right)^r q_0(z^2), \tag{13}$$

or alternatively

$$q(z) = \left(\frac{1-z}{4}\right)^r q_0\,, \tag{14}$$

and

$$l(z) = \left(\frac{1+z}{2}\right)^r l_0(z)\,, \tag{15}$$

where q_0, l_0 are Laurent polynomials.

Further, for a to have accuracy $r + 1$, we need

$$(-1)^\nu q_0(1) l_0(1) + s(1) l_0(-1) = 0.$$

Notice that $q(1) = 0$ and hence $s(1) = \pm 1$ by (11). Thus we arrive at the following result.

Theorem 1 *Let l given by (15), where l_0 is a Laurent polynomial, be a 1D low-pass filter of accuracy r satisfying (10) and $l_0(1) = \sqrt{2}$. Let q be given by (14) where q_0 is a Laurent polynomial with $q_0(1) = l_0(-1)/\sqrt{2}$, and let s be a Laurent polynomial such that (11) holds and $s(1) = 1$. Then H given by (6), where a, b are given by (9), defines a 2D low-pass filter with accuracy $r + 1$.*

If we use the canonical definition of l given in [5], then

$$|l_0(z)|^2 = 2 \sum_{k=0}^{r-1} \binom{r+k-1}{k} \left(\frac{2 - z - z^{-1}}{4}\right)^k, \tag{16}$$

then we have

$$[q_0(1)]^2 = \sum_{k=0}^{r-1} \binom{r+k-1}{k}.$$

3.3. Modulation matrices

In this subsection we derive a new factorization for the modulation matrix. Later we will see how this factorization can be used to design a lifting scheme.

From the above deduction, we have

$$\begin{aligned} M &= \begin{pmatrix} H & \underline{H} \\ G & \underline{G} \end{pmatrix} \\ &= \begin{pmatrix} sl + \beta q \bar{\underline{l}} w & s\underline{l} - \beta q \bar{l} w \\ \alpha[\bar{s}\bar{\underline{l}} - \bar{\beta}\bar{q} l \bar{w}] & -\alpha[\bar{s}\bar{l} + \bar{\beta}\bar{q}\underline{l}\bar{w}] \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & \bar{w} \end{pmatrix} S(z^2) \begin{pmatrix} 1 & 0 \\ 0 & w \end{pmatrix} L(z)\,, \end{aligned} \tag{17}$$

where $\alpha(z) = z^\mu$, $\beta(z) = z^\nu$,

$$Q(z) = z^{(\nu-\mu)/2} q(z), \tag{18}$$

$$S(z) = \begin{pmatrix} s(z) & Q(z) \\ -\bar{Q}(z) & \bar{s}(z) \end{pmatrix}, \tag{19}$$

$$L(z) = \begin{pmatrix} l(z) & l(-z) \\ \alpha(z) l(-z^{-1}) & -\alpha(z) l(z^{-1}) \end{pmatrix}. \tag{20}$$

Recall that both μ and ν are odd, and therefore $(\nu - \mu)/2$ is an integer. Interestingly, the matrix in (20) is nothing but the modulation matrix of a 1D subband scheme. Therefore, we can follow Daubechies' construction in [5] to obtain l. We will not give any further details here. Note also that the matrix S in (19) depends only on the variable z. Thus (17) means that the filters with 2-row support can be factorized as a composition of one-dimensional filters.

3.4. Non-separability in a 2×2 sampling lattice

2D wavelets are a popular tool in image processing applications such as denoising and compression. As observed before, the sampling lattice is $2\mathbb{Z} \times 2\mathbb{Z}$ in most cases, corresponding with the dilation matrix $2I$, where I is the identity matrix. This gives rise to four different subbands, one approximation and three details. It's not easy to construct filters with dilation matrix $2I$ directly; instead, tensor product of 1D filters are being used in most cases. In the analysis stage, the input image is filtered first row by row and then column by column. These two steps correspond with matrices $D_1 = \left(\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $D_2 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right)$ respectively. Note that these two matrices are not 2D dilation matrices according to the definition in §2.1; each of them has only one sampling direction. The resulting filters are therefore separable.

Interchanging the two columns of D_1 or, alternatively, the two rows of D_2, one gets the dilation matrix

$$D = D_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} D_2 = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}. \tag{21}$$

Notice that $D^2 = 2I$. This shows that we may use dilation matrix D twice to obtain 2D filters with dilation matrix $2I$. However, the filters derived in this way can be non-separable in contrast to those derived from the tensor product method, which are always separable. See Figure 2 for an illustration, and Figure 3 for a decomposition of the Lenna image.

The two-row filter H discussed before is non-separable if and only if a is not a divisor of b. The only separable filter devised in this context corresponds with the case $r = 0$.

4. Lifting scheme

4.1. Polyphase representation

Recall that the monomial $\alpha(z) = z^\mu$ in (1) is odd in z. We may assume without loss of generality (and for the sake of simplicity) that $\alpha(z) = z^{-1}$, i.e., $\mu = -1$.

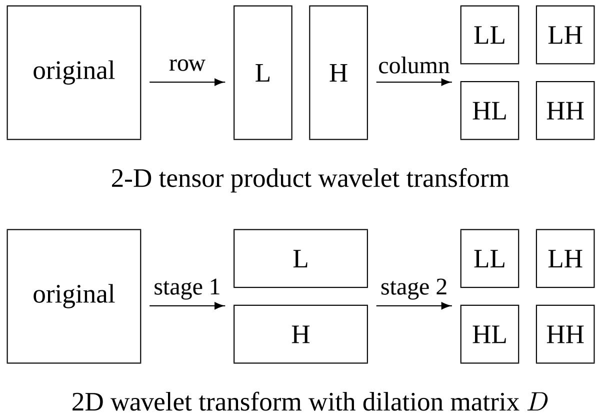

Figure 2. 2D wavelet transforms using a tensor product (top) and a dilation matrix D (bottom). Here L stands for the low-pass band and H for the high-pass band.

We split the univariate z-form $l(z)$ into two parts:

$$l(z) = l_e(z^2) + z^{-1} l_o(z^2),$$

where l_e contains the even coefficients and l_o the odd. If $p(z) = z^{-1} l(-z^{-1})$, then

$$p_e(z) = -p_o(z^{-1}), \quad p_o(z) = l_e(z^{-1}).$$

Therefore the modulation matrix L in (20) can be written as

$$L = P_1 \begin{pmatrix} 1 & 1 \\ z & -z \end{pmatrix} \text{ with } P_1 = \begin{pmatrix} l_e & l_o \\ -\bar{l}_o & \bar{l}_e \end{pmatrix}.$$

Note that P_1 is the 1D polyphase matrix.

Analogously to the 1D case, the bivariate Laurent polynomial $H(z, w) = a(z) + wb(z)$ can be split into the odd part $H_o(z, w) = a_o(z) + wb_o(z)$ and the even part $H_e(z, w) = a_e(z) + wb_e(z)$. Now the following relation holds:

$$H(z, w) = H_o(z^2, w) + z^{-1} H_o(z^2, w).$$

Suppose H is the two-row filter defined in §3. It has polyphase matrix

$$P(\mathbf{z}) = \begin{pmatrix} a_e + wb_e & a_o + wb_o \\ -\bar{a}_o - \bar{w}\bar{b}_o & \bar{a}_e + \bar{w}\bar{b}_e \end{pmatrix} \tag{22}$$

It is easy to show that

$$M = P \begin{pmatrix} 1 & 1 \\ z^{-1} & -z^{-1} \end{pmatrix}.$$

Using that $M^T \bar{M} = 2I$ (see (3)) we find that P is unitary. We obtain from (17) that

$$P(z, w) = \begin{pmatrix} 1 & 0 \\ 0 & w \end{pmatrix} S(z) \begin{pmatrix} 1 & 0 \\ 0 & \bar{w} \end{pmatrix} P_1(z). \tag{23}$$

Thanks to this factorization, we can design an algorithm for the 2D transform (x, y) of a signal x_0 that is based on the underlying 1D wavelet transforms. Below the algorithm is given only for the forward transform, in which case we must use $\bar{P}$ rather than P. In the following, '$\star_{\mathsf{R}}$' denotes row-wise convolution.

1. Let (x_1, y_1) be the row-wise wavelet transform of x_0 with a 1D wavelet of given accuray r; this corresponds with the matrix $P_1(z)$ in (23).

2. Apply forward vertical shift to y_1; this corresponds with the diagonal matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & \bar{w} \end{smallmatrix}\right)$

3a. Compute $x_2 = x_1 \star_{\mathsf{R}} \bar{s} + y_1 \star_{\mathsf{R}} \bar{Q}$;

3b. Compute $y_2 = -x_1 \star_{\mathsf{R}} Q + y_1 \star_{\mathsf{R}} s$;
Note that these two expressions correspond with the multiplication with matrix S in (19).

4. Apply backward vertical shift to y_2.

5. Define $x = x_2^T$ and $y = y_2^T$. This step is necessary because of the transpose in dilation matrix D.

Figure 3. Wavelet transforms of the Lenna image in two stages with dilation matrix D.

4.2. Lifting schemes

Any 1D wavelet transform using finite impulse response (FIR) filters can be factorized into lifting steps by means of the Euclidean algorithm [6]. One of the advantages of the lifting scheme factorization is that it enables a fast and efficient implementation. Unfortunately, the factorization results for the 1D case do not have a straightforward generalization to the general nonseparable 2D case(see [9]). However, for the two-row filters explored in this paper, we can use the factorization in (23) to design a lifting scheme factorization of the matrices P_1 and S. We consider the case $r = 1$ as an example. Here,

$$P_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \text{ and } S = \frac{1}{4} \begin{pmatrix} \gamma + \eta z & -1 + z \\ 1 - z^{-1} & \gamma + \eta z^{-1} \end{pmatrix}$$

with $\gamma = 2 - \sqrt{3}$ and $\eta = \gamma^{-1} = 2 + \sqrt{3}$. Then, after some manipulations, we derive

$$P = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & w \end{pmatrix} \begin{pmatrix} 1 & -2\eta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1-z}{2} & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 2\eta z^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & w^{-1} \end{pmatrix} \begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

For an input x, split it, according to dilation matrix D, into low-band L and high-band H, force the predict, update, shift and scaling operations with respect to matrices in (23) from right to left. Then the lifting scheme based on P is realized.

5. Shorter filters with more rows

6. Filters having more than two rows

In general we choose q_0 in (14) constant, for simplicity. Therefore, for given accuracy r, we have $\deg q = \deg s = r$. If we use the canonical definition of l given by (15) with (16), we have $\deg l = 2r - 1$, and hence $\deg a = \deg b = 4r - 1$. Thus in this case the filters have a support stretched along the horizontal direction. Here we will given an alternative factorization with filters that are less stretched. Towards this goal we replace $S(z)$ in factorization (23) by $S(w)$. Thus we get a polyphase matrix

$$P(z, w) = \begin{pmatrix} 1 & 0 \\ 0 & \bar{w} \end{pmatrix} S(w) \begin{pmatrix} 1 & 0 \\ 0 & w \end{pmatrix} P_1(z). \qquad (24)$$

This matrix corresponds to the 2D low-pass filter

$$H(z, w) = A(z, w) + wB(z, w) \qquad (25)$$

where A and B are 2D Laurent polynomials defined similarly as in (9):

$$A(z, w) = s(w)l(z), \quad B(z, w) = z^\nu q(w)\underline{\bar{l}}(z)\,. \qquad (26)$$

Here l satisfies (10), q is the same as in (14), but with z replaced by w, and s is given by the following modification of (11):

$$|q(w)|^2 + |s(w)|^2 = 1. \qquad (27)$$

The factors $\left(\frac{1+z}{2}\right)^r$ of $l(z)$ and $\left(\frac{1-w}{2}\right)^r$ of $q(w)$ ensure that the filter H as defined in (25) is of accuracy r. But $\frac{\partial^r}{\partial z^r} H(-1, 1) = 0$ results in $s(1) = 0$, which is not possible. This means that, unlike the two row case, the filter (25) connot have accuracy $r + 1$.

We can prove the following analogue of Theorem 1.

Theorem 2 *Let l given by (15), where l_0 is a Laurent polynomial, be a 1D low-pass filter of accuracy r satisfying (10) and $l_0(1) = \sqrt{2}$. Let q be given by (14) where q_0 is a Laurent polynomial with $q_0(1) = l_0(-1)/\sqrt{2}$, and let s be a Laurent polynomial such that (27) holds and $s(1) = 1$. Then H given by (25), where A, B are given by (26), defines a 2D low-pass filter with accuracy r.*

For a 2D Laurent polynomial

$$A(z, w) = \sum_{I_1 \le i \le I_2} \sum_{J_1 \le j \le J_2} a_{ij} z^i w^j,$$

we can define the degree of A as

$$\deg A = (I_2 - I_1, J_2 - J_1).$$

It is trivial that under the assumptions in Theorem 2,

$$\deg A = \deg B = (2r - 1, r).$$

If we compare the expressions for A, B in (26) with those for a, b in (9), we see that $s(z^2)$ has been replaced by $s(w)$. A similar substitution should be used when we compute the modified modulation matrix. However, in the polyphase matrix in (24) we encounter the matrix $S(w)$ whereas in (23), the matrix $S(z)$ occurs. The algorithm at the end of §4.1 remains unchanged except for steps 3a and 3b which have to be changed into

3a'. Compute $x_2 = x_1 \star_{\mathsf{C}} \bar{s} + y_1 \star_{\mathsf{C}} \bar{Q}$;

3b'. Compute $y_2 = -x_1 \star_{\mathsf{C}} Q + y_1 \star_{\mathsf{C}} s$;

here '$\star_{\mathsf{C}}$' denotes column-wise convolution.

7. Conclusion

We have investigated the class of 2D filters that was introduced by Belogay and Wang in [2], and we have derived a new factorization of the corresponding modulation matrix. We have shown that, for any accuracy, the two-row orthonormal filters can be realized from 1D filters and allows an efficient implementation based on the lifting scheme. We have also given a modification of the Belogay-Wang approach that uses $r + 1$ rows and $2r$ columns for a decomposition with accuracy r.

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