ON THE LIMIT BEHAVIOUR OF EXTREME ORDER STATISTICS[1]

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0. Introduction. This paper is concerned with some recent developments in the theory of limit behaviour of extreme order statistics. Most of the paper is devoted to the discussion of limit distributions and of stability properties for order statistics of independent random variables. While the situation here is already well explored, very little is known in the case of dependent random variables. Those results that are known for the dependent case have been obtained in the last few years.

In Section 1 we introduce notations and definitions and state a few elementary facts concerning the distributions of the set of order statistics corresponding to a set of independent, identically distributed random variables. Throughout Sections 2, 3 and 4 we assume independence of the basic variables. In Section 2 we deal with limit distributions, while in Sections 3 and 4 we deal with stability in probability and stability almost surely. Finally, in Section 5 we turn to the dependence case, summarizing some recent results due mainly to Berman [3], [4] and [5].

Our principal result is contained in Section 4; it is a proof of the sufficiency of a simple condition for stability almost surely of the maximal order statistic. That condition was introduced and studied by Geffroy [7].

The aim in writing this paper has been twofold: to give a brief summary of the current state of research for the topic in question and to indicate certain recent contributions to this topic due to the author. A number of contributions are not mentioned; these are noted in the references.

I want to thank Professor Glen Baxter for the stimulating interest he has shown in this work.

1. Preliminaries. Let $X_1, X_2, \cdots, X_n, \cdots$, be a sequence of random variables defined on a probability field $(\Omega, \mathcal{A}, P)$. To any set $X_1, X_2, \cdots, X_n$ $(n = 1, 2, \cdots)$ let $X_{n1}, X_{n2}, \cdots, X_{nn}$ denote the corresponding set of order statistics, where for all points $\omega \varepsilon \Omega$, $X_{nk}(\omega)$ is equal to the kth largest of the values $X_1(\omega), X_2(\omega), \cdots, X_n(\omega)$. Thus $X_{n1} \geqq X_{n2} \geqq \cdots \geqq X_{nn}$. Throughout the paper k will denote a fixed positive integer. We will study the limit behaviour of sequences $\{X_{nk}\}$ of extreme upper order statistics. Obviously the results we mention hold with trivial modifications for sequences $\{X_{n,n-k}\}$ of extreme lower order statistics.

In the sequel, unless otherwise explicitly stated, we assume the variables

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X_n to be independent and identically distributed with common distribution function (d.f.) F, where $F(x) = P\{X_n \leqq x\}$, $-\infty < x < \infty$.

Letting $F_{nk}(x) = P\{X_{nk} \leqq x\}$ we have

$$(1.1)\quad \begin{aligned} F_{nk}(x) &= \sum_{i=0}^{k-1} \binom{n}{i} F^{n-i}(x)(1 - F(x))^i \\ &= (n - k + 1) \binom{n}{k-1} \int_0^{F(x)} t^{n-k}(1 - t)^{k-1}\, dt; \end{aligned}$$

the second equality follows by partial integration. Also we note, that

$$(1.2)\quad P\{X_{nk} \leqq x, X_{n+1,k} > y\} = \binom{n}{k-1} F^{n-k+1}(x)(1 - F(y))^k \quad \text{for } x \leqq y.$$

Suppose F is continuous. The values of $X_1, X_2, \cdots, X_n$ are then almost surely distinct and the following definition makes sense. We say that the rank of X_j in the set $X_1, X_2, \cdots, X_n$ is r if $X_j = X_{nr}$. We have

THEOREM 1.1. *Suppose F is continuous and let R_j be the rank of X_j in the set $X_1, X_2, \cdots, X_j$. The random variables $R_1, R_2, \cdots, R_n, \cdots$ are independent* and $P\{R_n = r\} = 1/n$, $r = 1, 2, \cdots, n$.

REMARKS. For a number of applications of this result see [14]. It follows for instance, that if A_n is the event $\{X_n \geqq \max(X_1, \cdots, X_{n-1})\} = \{X_n = X_{n1}\}$, then $A_1, A_2, \cdots, A_n, \cdots$ are independent and $PA_n = P\{R_n = 1\} = 1/n$.

PROOF. Let $x_1, x_2, \cdots, x_n$ be n distinct real numbers. We define a mapping φ from the set $\{(x_{i_1}, x_{i_2}, \cdots, x_{i_n})\}$ consisting of the $n!$ vectors obtained by permuting the coordinates of $(x_1, x_2, \cdots, x_n)$, into the set $\{(r_1, r_2, \cdots, r_n): r_1 = 1; r_2 = 1, 2; \cdots r_n = 1, 2, \cdots n\}$. The jth coordinate of $\varphi(x_{i_1}, x_{i_2}, \cdots, x_{i_n})$ is the rank of x_{i_j} in the set $x_{i_1}, x_{i_2}, \cdots, x_{i_j}$ i.e. the jth coordinate is r if x_{i_j} is the rth largest among $x_{i_1}, x_{i_2}, \cdots, x_{i_j}$. The mapping φ is one-to-one and onto and therefore we have

$$(1.3)\quad \begin{aligned} &P\{R_1 = r_1, R_2 = r_2, \cdots, R_n = r_n\} = 1/n!, \\ &\qquad r_1 = 1; r_2 = 1, 2; \cdots; r_n = 1, 2, \cdots, n. \end{aligned}$$

Consequently

$$(1.4)\quad P\{R_i = r_i\} = 1/i, \qquad r_i = 1, 2, \cdots, i; i = 1, 2, \cdots, n$$

and

$$(1.5)\quad \begin{aligned} &P\{R_1 = r_1, R_2 = r_2, \cdots, R_n = r_n\} \\ &\qquad = P\{R_1 = r_1\}P\{R_2 = r_2\} \cdots P\{R_n = r_n\}, \end{aligned}$$

q.e.d.

2. Limit distributions. The problem of finding the family $\mathfrak{L}_k$ of all possible (nondegenerate) limit distributions for sequences of the form $b_n^{-1}(X_{nk} - a_n)$ where a_n and b_n $(b_n > 0)$ are constants was attacked by several authors. For the case $k = 1$ a complete solution with specification of domains of attraction were given by Gnedenko [8]. His results were generalized by Smirnov [17] to arbitrary values of k. We state their main results.

THEOREM 2.1. *The family $\mathfrak{L}_1$ is given by*

$$\begin{aligned} \Lambda_1(x) &= 0 && x \leqq 0, \alpha > 0 \\ &= \exp(-x^{-\alpha}) && x > 0, \alpha > 0 \end{aligned} \tag{2.1}$$

$$\begin{aligned} \Lambda_2(x) &= \exp(-(-x)^{\alpha}) && x \leqq 0, \alpha > 0 \\ &= 1 && x > 0, \alpha > 0 \end{aligned} \tag{2.2}$$

and

$$\Lambda_3(x) = \exp(-e^{-x}) \qquad -\infty < x < \infty. \tag{2.3}$$

REMARK. The clue to the proof of this theorem is the following simple observation.

Suppose that F, $\{a_n\}$ and $\{b_n\}$ are such that $F_{n1}(b_n \cdot + a_n)$ converges weakly to some d.f. Λ, i.e.

$$F^n(b_n x + a_n) \to \Lambda(x) \qquad \forall\, x \in C_\Lambda, \tag{2.4}$$

where C_Λ denotes the set of continuity points for Λ. Then for any positive integer i we have

$$F^{ni}(b_{ni} x + a_{ni}) \to \Lambda(x) \qquad \forall\, x \in C_\Lambda \tag{2.5}$$

or

$$F^n(b_{ni} x + a_{ni}) \to \Lambda^{1/i}(x) \qquad \forall\, x \in C_\Lambda. \tag{2.6}$$

It follows from a well-known theorem of Khintchine (see e.g. the book by Gnedenko and Kolmogorov [9] pp. 40–42) that to every i there exists constants α_i and β_i such that $\Lambda^i(\beta_i x + \alpha_i) = \Lambda(x)$, $-\infty < x < \infty$.

THEOREM 2.2. *The family $\mathfrak{L}_k$ is given by*

$$\begin{aligned} \Lambda_1^{(k)}(x) &= 0 && x \leqq 0, \alpha > 0, \\ &= [1/(k-1)!] \int_{x^{-\alpha}}^{\infty} e^{-t} t^{k-1}\, dt && x > 0, \alpha > 0, \end{aligned} \tag{2.7}$$

$$\begin{aligned} \Lambda_2^{(k)}(x) &= [1/(k-1)!] \int_{(-x)^{\alpha}}^{\infty} e^{-t} t^{k-1}\, dt && x \leqq 0, \alpha > 0 \\ &= 1 && x > 0, \alpha > 0 \end{aligned} \tag{2.8}$$

and

$$\Lambda_3^{(k)}(x) = [1/(k-1)!] \int_{e^{-x}}^{\infty} e^{-t} t^{k-1}\, dt \qquad -\infty < x < \infty. \tag{2.9}$$

Various generalizations of Theorems 2.1 and 2.2 have been obtained.

Berman [5] has shown that the limit distribution for the maximal order statistic of a random number of independent and identically distributed variables under certain general conditions is a mixture of distributions from $\mathfrak{L}_1$.

As is well-known, in order to give a satisfactory treatment of the central limit problem for sums of independent random variables one is led to consider limit distributions of row sums from triangular arrays

$$\begin{array}{c} Y_{11}, Y_{12}, \cdots, Y_{1r_1} \\ Y_{21}, Y_{22}, \cdots, Y_{2r_2} \\ \cdots \\ Y_{n1}, Y_{n2}, \cdots, Y_{nr_n} \\ \cdots \\ \cdots \end{array}$$

where $r_n \to \infty$ as $n \to \infty$ and where in each row the variables are assumed to be independent. For a discussion of asymptotic distributions of the row maxima from such arrays, see Loève [11], p. 333 and Maslov [12]. Rowsums and rowmaxima are special instances of the general type of functionals of $(Y_{n1}, Y_{n2}, \cdots, Y_{nr_n})$ considered in [12].

3. Stability in probability. We say that a sequence $\{U_n\}$ of random variables is stable in probability, if there exists a sequence of constants $\{a_n\}$ such that $U_n - a_n \to 0$ in probability (i.p.). Also we say that $\{U_n\}$ is majorized in probability by a sequence $\{a_n\}$, symbolically $U_n \ll a_n$ i.p., if $P\{U_n > a_n\} \to 0$ as $n \to \infty$ and that $\{U_n\}$ is minorized in probability by $\{a_n\}$, symbolically $a_n \ll U_n$ i.p., if $P\{U_n \leqq a_n\} \to 0$ as $n \to \infty$. Clearly

$$U_n - a_n \to 0 \text{ i.p.} \Leftrightarrow a_n - \epsilon \ll U_n \ll a_n + \epsilon \quad \text{i.p. } \forall \epsilon > 0. \tag{3.1}$$

Elementary computations show (see [7]) that $\{X_{nk}\}$ is stable in probability if and only if $\{X_{n1}\}$ is stable in probability and that in this case $X_{n1} - X_{nk} \to 0$ i.p. as $n \to \infty$. Furthermore, from (3.1) we find

$$X_{n1} - a_n \to 0 \text{ i.p.} \Leftrightarrow \begin{array}{ll} \text{(i)} & n[1 - F(a_n + \epsilon)] \to 0 \\ \text{(ii)} & n[1 - F(a_n - \epsilon)] \to \infty \end{array} \quad \forall \epsilon > 0. \tag{3.2}$$

Hence, if there is a number a such that $F(a) = 1$ and $F(a - \epsilon) < 1$, $\forall \epsilon > 0$, then trivially $X_{nk} - a \to 0$ i.p. In the rest of this section we shall therefore assume $F(x) < 1$, $\forall x$.

THEOREM 3.1. *The sequence* $\{X_{nk}\}$ *is stable* i.p. *if and only if*

$$[1 - F(x + \epsilon)]/[1 - F(x)] \to 0 \quad \text{as} \quad x \to \infty, \qquad \forall \epsilon > 0. \tag{3.3}$$

REMARK. Theorem 3.1 is due to Gnedenko [8]. As could be expected the criterion (3.3) is a growth condition on F. Simple estimates show (see Geffroy [7], p. 77) that (3.3) implies $\int_0^\infty x^r \, dF(x) < \infty$, $\forall r > 0$. The converse proposition does not hold.

PROOF. It suffices to prove the theorem for $k = 1$.

Suppose that $\{X_{n1}\}$ is stable i.p. and that $X_{n1} - a_n \to 0$ i.p. Without loss of generality we may assume $\{a_n\}$ strictly increasing. To any $x \geqq a_1$ and any $\epsilon > 0$ there is then a uniquely determined n such that $a_n - \epsilon/2 \leqq x < a_{n+1} - \epsilon/2$. Clearly

$$(3.4) \quad [1 - F(x + \epsilon)]/[1 - F(x)] \leqq [1 - F(a_n + \epsilon/2)]/[1 - F(a_{n+1} - \epsilon/2)]$$

where, on account of (3.2), the right hand side tends to 0 as x and hence n tends to ∞.

In order to show that, conversely, stability is implied by (3.3) let us associate with any d.f. F a function F^{-1} defined by $F^{-1}(y) = \inf\{x:F(x) \geqq y\}, 0 \leqq y \leqq 1$, where $\inf\{x:F(x) \geqq 1\}$ is to be interpreted as $+\infty$ if $F(x) < 1$, $\forall x$ and where $\inf\{x:F(x) \geqq 0\}$ is to be interpreted as $-\infty$.

Letting $\alpha_n = F^{-1}(1 - 1/n)$, for $n = 1, 2, \cdots$, we have $F(\alpha_n) \geqq 1 - 1/n$, $F(\alpha_n - 0) \leqq 1 - 1/n$ and hence

$$(3.5) \quad \begin{aligned} &\text{(i) } n[1 - F(\alpha_n - \epsilon)] \geqq [1 - F(\alpha_n - \epsilon)]/[1 - F(\alpha_n - 0)] \\ &\text{(ii) } n[1 - F(\alpha_n + \epsilon)] \leqq [1 - F(\alpha_n + \epsilon)]/[1 - F(\alpha_n)] \end{aligned} \qquad \forall \epsilon > 0.$$

It follows that (3.3) implies (3.2) (i) and (ii) with $a_n = \alpha_n$; thus $X_{n1} - \alpha_n \to 0$ i.p.

From the proof we see

THEOREM 3.2. *$\{X_{nk}\}$ is stable* i.p. *if and only if $X_{nk} - \alpha_n \to 0$* i.p., *(where $\alpha_n = F^{-1}(1 - 1/n)$) and this condition is equivalent to*

$$(3.6) \quad X_{nk} \ll \alpha_n + \epsilon \text{ i.p.} \qquad \forall \epsilon > 0.$$

(The sufficiency of (3.6) follows from the inequalities

$$(3.7) \quad \begin{aligned} \frac{1 - F(x + \epsilon)}{1 - F(x)} &\leqq \frac{1 - F(\alpha_n + \epsilon/2)}{1 - F(\alpha_{n+1} - \epsilon/2)} \\ &\leqq \frac{1 - F(\alpha_n + \epsilon/2)}{1 - F(\alpha_{n+1} - 0)} \leqq (n + 1)[1 - F(\alpha_n + \epsilon/2)], \end{aligned}$$

holding for $\alpha_n - \epsilon/2 \leqq x \leqq \alpha_{n+1} - \epsilon/2$.)

4. Stability almost surely. We say that a sequence $\{U_n\}$ of random variables is stable almost surely if there exists a sequence of constants $\{a_n\}$ such that $U_n - a_n \to 0$ almost surely (a.s.). Also we say that $\{U_n\}$ is majorized a.s. by a sequence $\{a_n\}$, symbolically $U_n \ll a_n$ a.s., if $P\{U_n > a_n \text{ i.o.}\} = 0$ (i.o. being an abbreviation for infinitely often) and that $\{U_n\}$ is minorized a.s. by $\{a_n\}$, symbolically $a_n \ll U_n$ a.s., if $P\{U_n \leqq a_n \text{ i.o.}\} = 0$. Clearly

$$(4.1) \quad U_n - a_n \to 0 \text{ a.s.} \Leftrightarrow a_n - \epsilon \ll U_n \ll a_n + \epsilon \text{ a.s.} \qquad \forall \epsilon > 0.$$

In order to analyze these concepts, as applied to order statistics, we shall invoke the following lemma, which is a slight strengthening of Lemma 1* in [2].

LEMMA 4.1. *Let $A_1, A_2, \cdots, A_n, \cdots$ be a sequence of events and let A_n^c denote the complement of A_n. If*

$$\sum_{n=1}^{\infty} P(A_n \cap A_{n+1}^c) < \infty \tag{4.2}$$

or

$$\sum_{n=1}^{\infty} P(A_n^c \cap A_{n+1}) < \infty \tag{4.3}$$

then $P(\limsup A_n - \liminf A_n) = 0$.

REMARK. The convergence part of the Borel-Cantelli is an immediate consequence of this result.

PROOF. Let I_n be the indicator function for A_n. Then $\{I_n : n = 1, 2, \cdots\}$ is a stochastic process with state space $\{0, 1\}$. The event $\limsup A_n - \liminf A_n$ is the event that infinitely many transitions take place between state 0 and state 1, which under (4.2) or (4.3) has probability 0.

Now, suppose that $\{\lambda_n\}$ is a nondecreasing sequence of real numbers and let $A_{nk} = \{X_{nk} \leqq \lambda_n\}$. Then

$$P(A_{nk} \cap A_{n+1,k}^c) = \binom{n}{k-1} (F(\lambda_n))^{n-k+1}(1 - F(\lambda_{n+1}))^k. \tag{4.4}$$

From Lemma 4.1 we have

THEOREM 4.1. *If $P\{X_{nk} \leqq \lambda_n\} \to 0$ as $n \to \infty$ and*

$$\sum_{n=1}^{\infty} n^{k-1}(F(\lambda_n))^{n-k+1}(1 - F(\lambda_{n+1}))^k < \infty \tag{4.5}$$

then $\lambda_n \ll X_{nk}$ a.s.

A partial converse of Theorem 4.1 is given by

THEOREM 4.2. *If the sequence $\{(F(\lambda_n))^n\}$ is nonincreasing and*

$$\sum_{n=3}^{\infty} (F(\lambda_n))^n (\log \log n)/n = \infty \tag{4.6}$$

then $P\{X_{n1} \leqq \lambda_n \text{ i.o.}\} = 1$.

REMARK. It can be proved that convergence of the series (4.6) always entails $P\{X_{n1} \leqq \lambda_n \text{ i.o.}\} = 0$ or (equivalently) $\lambda_n \ll X_{n1}$ a.s., whether $\{(F(\lambda_n))^n\}$ is nonincreasing or not, cf. [2], p. 392.

For a proof of Theorem 4.2, see [2].

Using the fact (see [7], p. 49) that $P\{X_{nk} > \lambda_n\} \to 0$ an $n \to \infty$ implies $P\{X_{n1} > \lambda_n\} \to 0$ and thus $F^n(\lambda_n) \to 1$, from Lemma 4.1 we also find

THEOREM 4.3. *If $P\{X_{nk} > \lambda_n\} \to 0$ as $n \to \infty$ and $\sum_{n=1}^{\infty} n^{k-1}(1 - F(\lambda_n))^k < \infty$ then* $X_{nk} \ll \lambda_n$ a.s.

Since $\{X_{n1} > \lambda_n \text{ i.o.}\} = \{X_n > \lambda_n \text{ i.o.}\}$, in view of the Borel-Cantelli lemma, we have the following partial converse of Theorem 4.3.

THEOREM 4.4. *If $\sum_{n=1}^{\infty} (1 - F(\lambda_n)) = \infty$ then* $P\{X_{n1} > \lambda_n \text{ i.o.}\} = 1$.

REMARK. Theorem 4.4 and its converse, which also follows immediately from the Borel-Cantelli lemma, are due to Geffroy [7].

Let $\{a_n\}$ be a nondecreasing sequence of real numbers. From (4.1) and Theorems 4.1 and 4.3 it follows, that if for all $\epsilon > 0$, $F^n(a_n - \epsilon) \to 0$ as $n \to \infty$, and if the following two series converge

$$\sum_{n=1}^{\infty} n^{k-1}F^n(a_n - \epsilon)[1 - F(a_{n+1} - \epsilon)]^k < \infty \tag{4.7}$$

and

$$\sum_{n=1}^{\infty} n^{k-1}[1 - F(a_n + \epsilon)]^k < \infty \tag{4.8}$$

then $X_{nk} - a_n \to 0$ a.s. Hence, if there is a number a such that $F(a) = 1$ and $F(a - \epsilon) < 1$, $\forall\, \epsilon > 0$, then trivially $X_{nk} - a \to 0$ a.s. In the rest of this section we shall therefore assume $F(x) < 1\ \forall\, x$.

As in Section 3, let $\alpha_n = F^{-1}(1 - 1/n)$. We note

LEMMA 4.2. *Suppose that F is continuous and strictly increasing and let $\epsilon > 0$.*

(i) *Convergence of the series*

$$\sum_{n=1}^{\infty} n^{k-1}F^n(\alpha_n - \epsilon)[1 - F(\alpha_{n+1} - \epsilon)]^k \tag{4.9}$$

is implied by convergence of the integral

$$I = \int_{-\infty}^{\infty} \frac{[1 - F(x - \epsilon)]^k}{(1 - F(x))^{k+1}} \exp\left(-\frac{1 - F(x - \epsilon)}{1 - F(x)}\right) dF(x). \tag{4.10}$$

(ii) *Convergence of the series*

$$\sum_{n=1}^{\infty} n^{k-1}[1 - F(\alpha_n + \epsilon)]^k \tag{4.11}$$

is equivalent to convergence of the integral

$$J = \int_{-\infty}^{+\infty} \frac{(1 - F(x))^{k-1}}{[1 - F(x - \epsilon)]^k} dF(x). \tag{4.12}$$

PROOF. Let $\alpha_t = F^{-1}(1 - 1/t)$, $1 < t < \infty$ and consider the integral

$$I^* = \int_1^{\infty} t^{k-1}F^t(\alpha_t - \epsilon)[1 - F(\alpha_t - \epsilon)]^k\, dt. \tag{4.13}$$

We have

$$\int_n^{n+1} t^{k-1}F^t(\alpha_t - \epsilon)[1 - F(\alpha_t - \epsilon)]^k\, dt \geqq n^{k-1}F^{n+1}(\alpha_n - \epsilon)[1 - F(\alpha_{n+1} - \epsilon)]^k;$$

thus, if I^* is finite, so is the series (4.9). Substituting $(1 - F(x))^{-1}$ for t in (4.13) we obtain

$$I^* = \int_{-\infty}^{+\infty} \frac{[1 - F(x - \epsilon)]^k}{(1 - F(x))^{k+1}} \exp\left(\frac{\log F(x - \epsilon)}{1 - F(x)}\right) dF(x) \leqq I. \tag{4.14}$$

The first assertion of the lemma is established.

Next, let $J^* = \int_1^\infty t^{k-1}[1 - F(\alpha_t + \epsilon)]^k \, dt$. Convergence of J^* is clearly equivalent to convergence of the series (4.11). The substitution $t = (1 - F(x))^{-1}$ yields

$$J^* = \int_{-\infty}^{\infty} \{[1 - F(x + \epsilon)]^k/(1 - F(x))^{k+1}\} \, dF(x). \tag{4.15}$$

Let $-\infty < c < \infty$. We find

$$\begin{aligned} &\int_{-\infty}^{c} \frac{[1 - F(x + \epsilon)]^k}{(1 - F(x))^{k+1}} dF(x) \\ &= \frac{1}{k}\left[\left(\frac{1 - F(c + \epsilon)}{1 - F(c)}\right)^k - 1\right] + \int_{-\infty}^{c} \frac{[1 - F(x + \epsilon)]^{k-1}}{(1 - F(x))^k} dF(x + \epsilon). \end{aligned} \tag{4.16}$$

$J^* < \infty$ implies convergence of (4.11) and hence $n[1 - F(\alpha_n + \epsilon)] \to 0$; consequently we have (cf. Section 3, Formula (3.7))

$$J^* < \infty \Rightarrow J = \int_{-\infty}^{+\infty} \frac{[1 - F(x + \epsilon)]^{k-1}}{(1 - F(x))^k} \, dF(x + \epsilon) < \infty. \tag{4.17}$$

A simple calculation shows that, conversely, $J < \infty \Rightarrow J^* < \infty$, which proves (ii).

Lemma 4.2 enables us to derive

THEOREM 4.5. *The condition*

$$\int_{-\infty}^{\infty} \frac{(1 - F(x))^{k-1}}{[1 - F(x - \epsilon)]^k} dF(x) < \infty \qquad \forall \epsilon > 0 \tag{4.18}$$

is sufficient for stability a.s. *of* $\{X_{nk}\}$. *For* $k = 1$ *it is also necessary.*

REMARKS. The integral in (4.18) is to be interpreted as a Lebesgue-Stieltjes integral. The necessity assertion of Theorem 4.5 is due to Geffroy [7].

PROOF. Let $\epsilon' > 0$ be such that the functions $F(x)$ and $1 - F(x - \epsilon')$ have no discontinuity points in common and such that $\alpha_n + \epsilon' \neq \alpha_{n+k}$, $\forall\, n$ and $k = 1, 2, \cdots$. Then, by some rather tedious and complicated considerations we can show, that there exists a continuous, strictly increasing d.f. F^* with the following properties: $\forall\, n$, $\alpha_n^* = F^{*-1}(1 - 1/n) \geqq \alpha_n$, $F(\alpha_n + \epsilon') \leqq F^*(\alpha_n^* + \epsilon')$,

$$n^{k-1}[1 - F(\alpha_n + \epsilon')]^k \leqq n^{k-1}[1 - F^*(\alpha_n^* + \epsilon')]^k + 2^{-n} \tag{4.19}$$

and

$$\begin{aligned} \int_{-\infty}^{+\infty} \frac{(1 - F^*(x))^{k-1}}{[1 - F^*(x - \epsilon')]^k} dF^*(x) - 1 &\leqq \int_{-\infty}^{+\infty} \frac{(1 - F(x))^{k-1}}{[1 - F(x - \epsilon')]^k} dF(x) \\ &\leqq \int_{-\infty}^{+\infty} \frac{(1 - F^*(x))^{k-1}}{[1 - F^*(x - \epsilon')]^k} dF^*(x) + 1. \end{aligned} \tag{4.20}$$

Thus, from Lemma 4.2 we see that the Condition (4.18) is equivalent to

$$\sum_{n=1}^{\infty} n^{k-1}[1 - F(\alpha_n + \epsilon)]^k < \infty \qquad \forall\, \epsilon > 0. \tag{4.21}$$

If this condition is satisfied we have $n[1 - F(\alpha_n + \epsilon)] \to 0\ \forall\, \epsilon > 0$ and hence $[1 - F(x - \epsilon)]/(1 - F(x)) \to \infty$, as $x \to \infty$, $\forall\, \epsilon > 0$. Therefore (4.18) implies

$$\int_{-\infty}^{+\infty} \frac{[1 - F(x - \epsilon)]^k}{(1 - F(x))^{k+1}} \exp\left(-\frac{1 - F(x - \epsilon)}{1 - F(x)}\right) dF(x) < \infty \qquad \forall\, \epsilon > 0. \tag{4.22}$$

Proceeding analogously to above one finds, invoking again Lemma 4.2, that (4.22) in turn implies

$$\sum_{n=1}^{\infty} n^{k-1} F^n(\alpha_n - \epsilon)[1 - F(\alpha_{n+1} - \epsilon)]^k < \infty \qquad \forall\, \epsilon > 0. \tag{4.23}$$

To sum up, (4.18) is equivalent to (4.21) and implies (4.23) and this together with Theorems 4.1, 4.3 and 4.4 show the validity of Theorem 4.5.

Clearly

THEOREM 4.6. *$\{X_{n1}\}$ is stable* a.s. *if and only if $X_{n1} - \alpha_n \to 0$* a.s., *and this condition is equivalent to $X_{n1} \ll \alpha_n + \epsilon$* a.s., $\forall\, \epsilon > 0$.

Theorems 3.2 and 4.6 reveal a peculiar "skewness" in the distribution of the extreme order statistics.

5. Dependent basic random variables. Berman has considered the question of limit distributions for $\{X_{n1}\}$ when the variables X_n are exchangeable (see [5]). A basic tool in his investigation was the de Finetti representation formula for the finite-dimensional distributions of $X_1, X_2, \cdots, X_n, \cdots$ (see [11], pp. 364–365). The following theorem exemplifies the type of his results.

Suppose the variables X_n are exchangeable, so that we have a representation formula for $P\{X_1 \leqq x_1, X_2 \leqq x_2, \cdots, X_n \leqq x_n\}$ of the form

$$P\{X_1 \leqq x_1, X_2 \leqq x_2, \cdots, X_n \leqq x_n\} = \int_\Omega H_\omega(x_1)H_\omega(x_2) \cdots H_\omega(x_n)\, dP(\omega) \tag{5.1}$$

where for each fixed ω in the sample space Ω, H_ω is a (one-dimensional) d.f. Let $\{a_n\}$, $\{b_n\}$ and the d.f. G be such that $G(x) < 1$, $\forall\, x$, and $G^n(b_n x + a_n) \to \Lambda_3(x)$, $\forall\, x \in (-\infty, \infty)$.

THEOREM 5.1. *We have $P\{b_n^{-1}(X_{n1} - a_n) \leqq x\} \to \Lambda(x)$, $\forall\, x \in C_\Lambda$ for some* d.f. Λ *if and only if there exists a* d.f. W *such that $W(0) = 0$ and*

$$\lim_{u\to\infty} P\{\log H_\omega(u)/\log G(u) \leqq x\} \to W(x) \qquad \forall\, x \in C_W.$$

In this case $\Lambda(x) = \int_0^\infty [\Lambda_3(x)]^s\, dW(s)$, $\forall\, x$.

A related result was obtained, also by Berman, in [4]. Specifically, if the sequence $\{X_n\}$ is a stationary Gaussian process with $EX_n = 0$, $EX_n^2 = 1$ and

$EX_nX_m = \rho, 0 < \rho < 1$ for $m \neq n$, then

$$P\{X_{n1} - [2(1-\rho)\log n]^{\frac{1}{2}} \leqq x\} \to (2\pi\rho)^{-\frac{1}{2}} \int_{-\infty}^{x} e^{-t^2/2\rho}\, dt.$$

The proof runs as follows. The process $\{X_n\}$ is representable in the form $X_n = Y + Z_n$ where Y and $Z_1, Z_2, \cdots, Z_n, \cdots$ are independent and Gaussian with $EY = EZ_n = 0$, $EY^2 = \rho$, $EZ_n^2 = 1 - \rho$ $(n = 1, 2, \cdots)$. Clearly $X_{n1} = Y + \max(Z_1, \cdots, Z_n)$ and from (3.2) we obtain $\max(Z_1, \cdots, Z_n) - [2(1-\rho)\log n]^{\frac{1}{2}} \to 0$ i.p. As Professor Berman has kindly pointed out to me, this last result is not a special case of Theorem 5.1, cf. [5], p. 903.

A sufficient condition for stability i.p. of $\{X_{n1}\}$ when $\{X_n\}$ is stationary is given by (Berman [3]).

Lemma 5.1. *If the sequence* $\{a_n\}$ *satisfies*

(i) $n[1 - F(a_n + \epsilon)] \to 0$, (ii) $n[1 - F(a_n - \epsilon)] \to \infty$ *and*

$$\frac{2}{n^2}\sum_{i=2}^{n}(n - i + 1)\frac{P\{X_1 > a_n - \epsilon, X_i > a_n - \epsilon\}}{P^2\{X_1 > a_n - \epsilon\}} \to 1 \tag{5.2}$$

$\forall\, \epsilon > 0$ *and* $n \to \infty$, *then* $X_{n1} - a_n \to 0$, i.p.

Corollary 5.1. *Let* $\{X_n\}$ *be stationary Gaussian with* $EX_n = 0$, $EX_n^2 = 1$ *and let* $r_n = EX_1X_n$, $(n = 1, 2, \cdots)$. *If* $nr_n \to 0$ *then* $X_{n1} - (2\log n)^{\frac{1}{2}} \to 0$, i.p.

For a proof see [3]. A similar result for continuous time stationary Gaussian processes is derived in [6].

We conclude with a proposition due to Robbins [15].

Theorem 5.2. *Let the random variables* X_n *be nonnegative and identically distributed.*

If $EX_1^r < \infty$ *for some* $r > 0$, *then* $n^{-1/r}X_{n1} \to 0$, a.s. *If* $r \geqq 1$, *then, moreover* $n^{-1/r}EX_{n1} \to 0$.

Proof. First, let us consider the case $r = 1$. $EX_1 < \infty$ implies

$$\sum_{n=1}^{\infty} P\{X_n > n\epsilon\} < \infty, \qquad \forall\, \epsilon > 0$$

and hence

$$P\{\limsup n^{-1}X_{n1} \leqq \epsilon\} = 1, \qquad \forall\, \epsilon > 0,$$

or

$$n^{-1}X_{n1} \to 0 \qquad \text{a.s.} \tag{5.3}$$

For any $A \in \mathfrak{A}$ we have

$$0 \leqq \int_A \frac{X_{n1}}{n}\, dP \leqq \frac{1}{n}\sum_{i=1}^{n}\int_A X_i\, dP = \int_A X_1\, dP.$$

Thus the set functions $\int n^{-1}X_{n1}\, dP$ are uniformly absolutely continuous and this together with (5.3) imply $n^{-1}EX_{n1} \to 0$.

The transformation $X_n^* = X_n^r$ and the inequality $(EX_n)^r \leqq EX_n^*$, $(r \geqq 1)$ reduces the general case to the one treated above.

COROLLARY 5.2. *If the random variables* X_n *are identically distributed and* $Ee^{X_1} < \infty$, *then* $X_{n1} - \log n \to -\infty$, a.s.

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