Annales Academiæ Scientiarum Fennicæ
Series A. I. Mathematica
Volumen 18, 1993, 369–375

ON THE LIMIT FUNCTIONS OF ITERATES IN WANDERING DOMAINS

Walter Bergweiler, Mako Haruta, Hartje Kriete, Hans-Günter Meier, and Norbert Terglane

RWTH Aachen, Lehrstuhl II für Mathematik, 5100 Aachen, Germany
Boston University, Department of Mathematics, 111 Cummington Street, Boston, MA 02215, U.S.A.
Ruhr-Universität Bochum, Fakultät und Institut für Mathematik, 4630 Bochum 1, Germany
Philips GmbH Forschungslaboratorien, Abt. III, Weißhausstraße 2, 5100 Aachen, Germany
RWTH Aachen, Lehrstuhl II für Mathematik, 5100 Aachen, Germany

Abstract. In this paper, it is shown that finite limit functions of iterates of entire functions in wandering domains are limit points of the forward orbits of the finite singularities of the inverse function. From this the absence of wandering domains for some classes of entire functions is deduced.

1. Introduction and results

Let f be a nonlinear entire function. The Fatou set F is the subset of the complex plane where the iterates f^n of f form a normal family. The complement of F is called the Julia set and denoted by J. If U is a component of F, then $f^n(U)$ is contained in some component of F which we denote by U_n. If $U_n \neq U_m$ for all $n \neq m$, then U is called wandering. Otherwise U is called preperiodic. In particular, if $U_n = U$ for some n, then U is called periodic.

Sullivan [25, 26] proved that rational functions do not have wandering domains. Transcendental entire functions, however, may have wandering domains, cf. [2, 3, 5, 10, 11, 17, 26]. On the other hand, certain classes of transcendental functions which do not have wandering domains are known, cf. [3, 8, 10, 13, 16, 24].

Denote by $\operatorname{sing}(f^{-1})$ the set of singularities of f^{-1}, that is, the set of critical and asymptotic values of f, and limit points of these values. Let S be the class of all entire functions for which $\operatorname{sing}(f^{-1})$ is finite. Eremenko and Lyubich [10, 13] and Goldberg and Keen [16] proved that if $f \in S$, then f does not have wandering domains. Baker [3] had proved this for a subclass of S. Other classes

1991 Mathematics Subject Classification: Primary 30D05; Secondary 58F08.

The authors thank the Mathematisches Forschungsinstitut Oberwolfach for hospitality and the Studienstiftung des Deutschen Volkes for financial support. Norbert Terglane is supported by a grant of the Deutsche Forschungsgemeinschaft.

of transcendental entire and meromorphic functions without wandering domains have been considered in [8, 24].

The proofs in [3, 8, 10, 13, 16, 24] use quasiconformal mappings, a tool introduced by Sullivan [25, 26] into this subject. We shall use an elementary method to prove that certain entire transcendental functions do not have wandering domains. Some of these functions are not contained in the classes considered in [3, 8, 10, 13, 16, 24].

We define $E = \cup_{n=0}^{\infty} f^n\big(\operatorname{sing}(f^{-1})\big)$ and denote by E' the derived set of E, that is, the set of finite limit points of E, and by $\overline{E}$ the closure of E.

It is well-known that all limit functions of $\{f^n|_U\}$ are constant if U is wandering, cf. [9, p. 317] and [14, Section 28]. Baker [1] proved that constant limit functions in (not necessarily wandering) domains of the Fatou set are in $\overline{E} \cup \{\infty\}$. Moreover, it is known (cf. e. g. [19, Theorem 6.6, Corollary 7.10] or [14, Section 30]) that constant limit functions in (pre)periodic domains are in $E' \cup \{\infty\}$, except possibly in (preimages of) superattracting domains.

Theorem. *Let f be an entire function and let U be a wandering domain of f. Then all limit functions of $\{f^n|_U\}$ are contained in $E' \cup \{\infty\}$.*

This answers a question by Baker [1, p. 5, Remark 1].

Obviously, limit functions of $\{f^n|_U\}$ are in J if U is a wandering domain. Denote by A the class of all entire functions with $J \cap E' = \emptyset$. It follows from the theorem that if $f \in A$, then $f^n \to \infty$ in all wandering domains of f. Eremenko and Lyubich [13] considered the class B of all entire functions f for which $\operatorname{sing}(f^{-1})$ is bounded and, using a logarithmic change of variable, proved that if $f \in B$, then there does not exist a component U of F such that $f^n \to \infty$ in U. Hence we obtain the following result.

Corollary. *If $f \in A \cap B$, then f does not have wandering domains.*

As an introduction to iteration theory, we recommend Beardon's book [7] and Milnor's lecture notes [19] for rational functions and the survey articles of Baker [6] and Eremenko and Lyubich [12] for transcendental entire (as well as rational) functions. The classical references are Fatou [14] and Julia [18] for rational and Fatou [15] for transcendental entire functions.

2. Proof of the theorem

Suppose that U is a wandering domain of f and that $a \in \mathbf{C} \setminus E'$ is a limit function of $\{f^n|_U\}$, say $f^{n_k} \to a$ in U.

It is not difficult to prove that f^n tends to ∞ in multiply-connected components of F, cf. [27, p. 67]. Hence U and all U_n are simply connected. By hypothesis, $U \cap E = \emptyset$ and $U_n \cap E = \emptyset$ for all $n \in \mathbf{N}$ so that f^{-n} exists locally on all U_n and can be continued analytically in U_n to a univalent function. Hence $f^n|_U$ is univalent.

We choose $z_0 \in \mathbf{C}$, $R > 0$ such that $\overline{D(z_0, R)} \subset U$. (Here and in the sequel $D(z, r)$ denotes the disc around z with radius r.) Without loss of generality, we shall assume that $a = 0$. We choose $r > 0$ such that $D(0, r) \cap E \setminus \{0\} = \emptyset$. We may assume $f^{n_k}\bigl(D(z_0, R)\bigr) \subset D(0, r) \setminus \{0\}$.

From Koebe's 1/4-theorem we obtain

$$\bigl|(f^{n_k})'(z_0)\bigr| \le 4\bigl|f^{n_k}(z_0)\bigr|/R.$$

Define $H = \{z : \operatorname{Re} z < \log r\}$ and g_k: $D(z_0, R) \to H$ by $g_k(z) = \log f^{n_k}(z)$, for some branch of the logarithm. Then

$$\bigl|g_k'(z_0)\bigr| = \frac{\bigl|(f^{n_k})'(z_0)\bigr|}{\bigl|f^{n_k}(z_0)\bigr|} \le \frac{4}{R}.$$

Since H is simply connected, the inverse function of g_k can be continued analytically to a single-valued function h_k in H, that is, h_k: $H \to \mathbf{C}$ and $h_k\bigl(g_k(z)\bigr) = z$ for $z \in D(z_0, R)$.

First we assume that h_k is univalent in H. Define $w_k = g_k(z_0)$. Then $w_k \in H$, that is, $\operatorname{Re} w_k < \log r$. By Koebe's 1/4-theorem we have

$$h_k(H) \supset h_k\bigl(D(w_k, \log r - \operatorname{Re} w_k)\bigr) \supset D\bigl(z_0, \tfrac{1}{4}\bigl|h_k'(w_k)\bigr|(\log r - \operatorname{Re} w_k)\bigr).$$

For arbitrary entire transcendental f, there are infinitely many periodic cycles of every order $n \ge 2$, see [23]. This means, there are infinitely many cycles $\{p_0, \dots, p_{n-1}\}$ such that $f^j(p_0) = p_j$ and $f^n(p_0) = p_0$. Now let $\{p, q\}$ be a periodic cycle of order 2 with $D(0, r) \cap \{p, q\} = \emptyset$. Then

$$h_k(H) \cap \{p, q\} = \emptyset.$$

It follows that

$$\tfrac{1}{4}\bigl|h_k'(w_k)\bigr|(\log r - \operatorname{Re} w_k) \le M$$

where $M = \min\bigl\{|z_0 - p|, |z_0 - q|\bigr\}$. Since $\operatorname{Re} w_k \to -\infty$ we conclude that $h_k'(w_k) \to 0$. But $h_k\bigl(g_k(z)\bigr) = z$, so $h_k'(w_k) g_k'(z_0) = 1$ and this gives a contradiction.

Now we assume that h_k is not univalent in H. As R. Nevanlinna [21, p. 283] we deduce that there exists $l_k \in \mathbf{N}$ such that h_k is periodic with period $2\pi l_k i$ and h_k is univalent in the half strip $\{z : \operatorname{Re} z < \log r; c < \operatorname{Im} z < c + 2\pi l_k i\}$ if c is real.

If $l_k \to \infty$, we obtain a contradiction as before. Hence we may assume that $l_k \not\to \infty$ and, restricting to a subsequence if necessary, we may suppose that $l_k = l$ for all k.

We now consider $G_k = \exp(g_k/l)$, $r' = r^{1/l}$ and the function $H_k\colon D(0,r') \setminus \{0\} \to \mathbf{C}$ defined by $H_k(z) = h_k(l \log z)$. Clearly, $H_k\bigl(G_k(z)\bigr) = z$ for $z \in D(z_0, R)$ and using Koebe's 1/4-theorem again we obtain

$$\bigl|G_k'(z_0)\bigr| = \bigl|G_k(z_0)\bigr| \frac{1}{l} \bigl|g_k'(z_0)\bigr| \le \frac{4\bigl|G_k(z_0)\bigr|}{lR}$$

so that $\bigl|G_k'(z_0)\bigr| \to 0$.

Since H_k is univalent, 0 is not an essential singularity of H_k. Suppose that 0 is a (simple) pole of H_k. Then $H_k\bigl(D(0,r') \setminus \{0\}\bigr)$ contains a neighborhood of infinity. But every neighborhood of infinity contains periodic cycles of f, which, as noted above, cannot be contained in $H_k\bigl(D(0,r') \setminus \{0\}\bigr)$ as soon as they have an empty intersection with $D(0,r)$. This is a contradiction, hence H_k has an analytic (and univalent) continuation to $D(0,r')$. Define $z_k = G_k(z_0)$. As before, we deduce from the Koebe 1/4-theorem that

$$\bigl|H_k'(z_k)\bigr| \le \frac{4M}{r' - |z_k|}.$$

Since $|z_k| \to 0$, we have $\bigl|H_k'(z_k)\bigr| \le 8M/r'$ for sufficiently large k. This contradicts $H_k'(z_k) G_k'(z_0) = 1$ and therefore completes the proof.

3. Examples

Eremenko and Lyubich [13] remarked that $f(z) = \sin z/z \in B \setminus S$. It is easy to see that $f \in A$. Hence $f(z) = \sin z/z$ has no wandering domains. This can also be seen by using Baker's result [1, Theorem 2] instead of our theorem.

As a second example we consider

$$f_\alpha(z) = \pi^2 - \alpha \frac{\sin\sqrt{z}}{\sqrt{z}}$$

where $\pi^2 < \alpha < 2\pi^2$. It is easy to check that all critical points of f_α are real and positive. We denote them by z_j, $0 < z_1 < z_2 < \cdots$. The critical values of f_α are denoted by c_k, i.e., $c_k = f_\alpha(z_k)$. Obviously, $c_k \to \pi^2$ as $k \to \infty$. We also note that π^2 is the only asymptotic value of f_α. Hence $f_\alpha \in B \setminus S$.

Clearly, π^2 is an attracting fixed point of f_α and one can show that f_α has precisely one more real fixed point which we denote by x_α. This fixed point satisfies $0 < x_\alpha < \pi^2$ and $x_\alpha \to \pi^2$ as $\alpha \to 2\pi^2$. Also, $f_\alpha(x) < x$ if $x < x_\alpha$ or $x > \pi^2$ and $x < f_\alpha(x) < \pi^2$ if $x_\alpha < x < \pi^2$.

We now consider the sequences $f_\alpha^m(c_k)$ as $m \to \infty$ and distinguish four cases:

(i) $f_\alpha^n(c_k) \ge \pi^2$ for all n:
Then $f_\alpha^{m+1}(c_k) \le f_\alpha^m(c_k)$ for all m and hence $f_\alpha^m(c_k) \to \pi^2$.

(ii) $x_\alpha < f_\alpha^n(c_k) < \pi^2$ for some n:
Then $f_\alpha^{m+1}(c_k) > f_\alpha^m(c_k)$ for all $m \geq n$ and hence $f_\alpha^m(c_k) \to \pi^2$.

(iii) $f_\alpha^n(c_k) < x_\alpha$ for some n:
Then $f_\alpha^{m+1}(c_k) < f_\alpha^m(c_k)$ for all $m \geq n$ and hence $f_\alpha^m(c_k) \to -\infty$.

(iv) $f_\alpha^n(c_k) = x_\alpha$ for some n:
Then $f_\alpha^m(c_k) = x_\alpha$ for all $m \geq n$.

Because $c_k \to \pi^2$ and π^2 is an attracting fixed point of f_α, cases (iii) and (iv) occur for at most finitely many k so that $f_\alpha \in A$. Note, however, that cases (iii) and (iv) do occur for certain values of α. It follows from our corollary that f_α does not have wandering domains. It seems impossible to deduce this result by using Baker's result [1, Theorem 2] instead of our theorem, if cases (iii) or (iv) occur.

The third example we mention is

$$g(z) = \frac{\pi^2}{\pi^2 - z^2} \sin z.$$

One can show that $|g(x)| < |x|$ for $x \in \mathbf{R} \setminus \{0\}$ so that $g^n(x) \to 0$ for all $x \in \mathbf{R}$. Moreover, it is not difficult to prove that all critical values of g are real and that 0 is the only asymptotic value of g. We deduce that $g \in B \setminus S$ and that $E' = \{0\}$. But 0 is also a fixed point of multiplier 1, hence $0 \in J$ so that $E' \cap J = \{0\} \neq \emptyset$. Hence $g \notin A$ and our corollary is not applicable. On the other hand, the behaviour of the iterates in the neighborhood of fixed points of multiplier 1 is well understood. In particular, it is known that a fixed point of multiplier 1 is a limit function in certain invariant domains attached to it, and preimages thereof, but not limit function in any other domain of the Fatou set [14, Section 13]. This implies that g does not have wandering domains.

More generally, using these ideas one can prove that an entire transcendental function f does not have wandering domains if $f \in B$ and if $E' \cap J$ is finite and consists only of rationally indifferent or repelling periodic points and preimages of such points.

Finally, we mention that our corollary implies that the exponential function does not have wandering domains. Once this is known, it is not difficult to prove that the Julia set of the exponential function is $\mathbf{C}$. This was proved first by Misiurewicz [20], confirming a conjecture of Fatou. More generally, define $g_\lambda(z) = \lambda \exp z$. We deduce from our corollary that if $g_\lambda^n(0) \to \infty$, then g_λ does not have wandering domains and hence $J = \mathbf{C}$. This was proved first by Baker and Rippon [4, Corollary 1]. We note that Misiurewicz's method was elementary while Baker and Rippon used quasiconformal mappings.

Remark. Our method can also be applied to rational functions. We conclude that constant limit functions are in E' except possibly in (preimages of) superattracting domains. This result was stated by Baker [1, p. 5] without proof. Thus we obtain an elementary proof that rational functions do not have wandering domains if $J \cap E' = \emptyset$ or, more generally, if $J \cap E'$ is finite and consists of rationally indifferent or repelling periodic points or preimages of such points. Elementary proofs that rational functions satisfying $J \cap \overline{E} = \emptyset$ do not have wandering domains have been given by Beardon [7, p. 202] and Norton [22, p. 182].

References

[1] Baker, I.N.: Limit functions and sets of non-normality in iteration theory. - Ann. Acad. Sci. Fenn. Ser. A I 467, 1970, 2–11.

[2] Baker, I.N.: An entire function which has wandering domains. - J. Australian Math. Soc. Ser. A 22, 1976, 173–176.

[3] Baker, I.N.: Wandering domains in the iteration of entire functions. - Proc. London Math. Soc. (3) 49, 1984, 563–576.

[4] Baker, I.N., and P.J. Rippon: Iteration of expontial functions. - Ann. Acad. Sci. Fenn. Ser. A I 9, 1984, 49–77.

[5] Baker, I.N.: Some entire functions with multiply-connected wandering domains. - Ergodic Theory Dynamical Systems 5, 1985, 163–169.

[6] Baker, I.N.: Iteration of entire functions: an introductory survey. - In Lectures on complex analysis, World Scientific, Singapore–New Jersey–London–Hong Kong, 1987, 1–17.

[7] Beardon, A.F.: Iteration of rational functions. - Springer-Verlag, New York–Berlin–Heidelberg, 1991.

[8] Bergweiler, W.: Newton's method and a class of meromorphic functions without wandering domains. - Preprint, 1992.

[9] Cremer, H.: Über die Schrödersche Funktionalgleichung und das Schwarzsche Eckenabbildungsproblem. - Sitzungsber. Math.-phys. Klasse, Ber. Verh. Sächs. Akad. Wiss. Leipzig 84, 1932, 291–324.

[10] Eremenko, A.E., and M.Yu. Lyubich: Iterates of entire functions. - Soviet Math. Dokl. 30, 1984, 592–594.

[11] Eremenko, A.E., and M.Yu. Lyubich: Examples of entire functions with pathological dynamics. - J. London Math. Soc. (2), 36, 1987, 458–468.

[12] Eremenko, A.E., and M.Yu. Lyubich: The dynamics of analytic transforms. - Leningrad Math. J. 1, 1990, 563–634.

[13] Eremenko, A.E., and M.Yu. Lyubich: Dynamical properties of some classes of entire functions. - Stony Brook Institute for Mathematical Sciences, Preprint 1990/4.

[14] Fatou, P.: Sur les équations fonctionelles. - Bull. Soc. Math. France 47, 1919, 161–271; 48, 1920, 33–94, 208–314.

[15] Fatou, P.: Sur l'itération des fonctions transcendantes entières. - Acta Math. 47, 1926, 337–360.

[16] Goldberg, L.R., and L. Keen: A finiteness theorem for a dynamical class of entire functions. - Ergodic Theory Dynamical Systems 6, 1986, 183–192.

[17] Herman, M.: Exemples de fractions rationelles ayant une orbite dense sur la sphere de Riemann. - Bull. Soc. Math. France 112, 1984, 93–142.

[18] Julia, G.: Sur l'itération des fonctions rationelles. - J. Math. Pures Appl. (7) 4, 1918, 47–245.

[19] Milnor, J.: Dynamics in one complex variable: Introductory lectures. - Stony Brook Institute for Mathematical Sciences, Preprint 1990/5.

[20] Misiurewicz, M.: On iterates of e^z. - Ergodic Theory Dynamical Systems 1, 1981, 103–106.

[21] Nevanlinna, R.: Analytic functions. - Springer-Verlag, Berlin–Heidelberg–New York, 1970.

[22] Norton, A.: An area approach to wandering domains for smooth surface endomorphisms. - Ergodic Theory Dynamical Systems 11, 1991, 181–187.

[23] Rosenbloom, P.C.: L'itération des fonctions entières.- C. R. Acad. Sci. Paris 227, 1948, 382–383.

[24] Stallard, G.W.: A class of meromorphic functions with no wandering domains. - Ann. Acad. Sci. Fenn. Ser. A I 16, 1991, 211–226.

[25] Sullivan, D.: Itération des fonctions analytiques complexes. - C. R. Acad. Sci. Paris 294, 1982, 301–303.

[26] Sullivan, D.: Quasiconformal homeomorphisms and dynamics I: Solution of the Fatou–Julia problem on wandering domains. - Ann. Math. 122, 1985, 401–418.

[27] Töpfer, H.: Über die Iteration der ganzen transzendenten Funktionen, insbesondere von $\sin z$ und $\cos z$. - Math. Ann. 117, 1939, 65–84.

Received 17 June 1992